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Asymptotic Behavior of a Hyperbolic-parabolic Coupled System Arising in Fluid-structure Interaction

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Abstract. In this paper we summarize some recent results on the asymptotic behavior of a linearized model arising in fluid-structure interaction, where a wave and a heat equation evolve in two bounded domains, with natural transmission conditions at the interface. These conditions couple, in particular, the heat unknown with the velocity of the wave solution. First, we show the strong asymptotic stability of solutions. Next, based on the construction of ray-like solutions by means of Geometric Optics expansions and a careful analysis of the transfer of the energy at the interface, we show the lack of uniform decay of solutions in general domains. Finally, we obtain a polynomial decay result for smooth solutions under a suitable geometric assumption guaranteeing that the heat domain envelopes the wave one. The system under consideration may be viewed as an approximate model for the motion of an elastic body immersed in a fluid, which, in its most rigorous modeling should be a nonlinear free boundary problem, with the free boundary being the moving interface between the fluid and the elastic body.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain with C^2 boundary $\Gamma = \partial\Omega$. Let Ω_1 be a sub-domain of Ω and set $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. We denote by γ the interface, $\Gamma_j = \partial\Omega_j \setminus \overline{\gamma}$

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($j = 1, 2$), and ν_j the unit outward normal vector of Ω_j ($j = 1, 2$). We assume $\gamma \neq \emptyset$ and γ is of class C^1 (unless otherwise stated). Denote by $\Box$ the d'Alembert operator $\partial_{tt} - \Delta$. Consider the following hyperbolic-parabolic coupled system:

$$\begin{cases} y_t - \Delta y = 0 & \text{in } (0,\infty) \times \Omega_1, \\ \Box z = 0 & \text{in } (0,\infty) \times \Omega_2, \\ y = 0 & \text{on } (0,\infty) \times \Gamma_1, \\ z = 0 & \text{on } (0,\infty) \times \Gamma_2, \\ y = z_t, \quad \frac{\partial y}{\partial \nu_1} = -\frac{\partial z}{\partial \nu_2} & \text{on } (0,\infty) \times \gamma, \\ y(0) = y_0 & \text{in } \Omega_1, \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega_2. \end{cases} \tag{1.1}$$

This is a simplified and linearized model for fluid-structure interaction. In system (1.1), y may be viewed as the velocity of the fluid; while z and z_t represent respectively the displacement and velocity of the structure. This system consists of a wave and a heat equation coupled through an interface with transmission conditions. More realistic models should involve the Stokes (*resp.* the elasticity) equations instead of the heat (*resp.* the wave) ones. In [7] and [11], the same system was considered but for the transmission condition $y = z$ on the interface instead of $y = z_t$. From the point of view of fluid-structure interaction, the transmission condition $y = z_t$ in (1.1) is more natural since y and z_t represent velocities of the fluid and the elastic body, respectively. On the other hand, in the most rigorous formulation the model should consist on a free boundary problem, with the free boundary being the moving interface between the fluid and the elastic body. After linearization around the trivial solution the interface is kept fixed in time. Our analysis concerns this later linearized formulation.

Put $H^1_{\Gamma_1}(\Omega_1) \stackrel{\triangle}{=} \left\{ h|_{\Omega_1} \,\middle|\, h \in H^1_0(\Omega) \right\}$ and $H^1_{\Gamma_2}(\Omega_2) \stackrel{\triangle}{=} \left\{ h|_{\Omega_2} \,\middle|\, h \in H^1_0(\Omega) \right\}$. As we shall see, system (1.1) is well posed in the Hilbert space

$$H \stackrel{\triangle}{=} L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2).$$

The space H is asymmetric with respect to the wave and heat components since the regularity differs in one derivative from one side to the other.

When Γ_2 is a non-empty open subset of the boundary, the following is an equivalent norm on H:

$$|f|_H = \sqrt{|f_1|^2_{L^2(\Omega_1)} + |\nabla f_2|^2_{(L^2(\Omega_2))^n} + |f_3|^2_{L^2(\Omega_2)}}, \qquad \forall\, f = (f_1, f_2, f_3) \in H.$$

This simplifies the dynamical properties of the system in the sense that the only stationary solution is the trivial one. The analysis is simpler as well. The same can be said when Γ_2 has positive capacity since, then, the Poincaré inequality holds. Note that when $\Gamma_2 = \emptyset$ or, more generally, when Cap Γ_2, the capacity of Γ_2, vanishes, $|\cdot|_H$ is no longer a norm on H. In this case, there are non-trivial stationary solutions of the system. Thus, the asymptotic behavior is more complex and one should rather expect the convergence of each individual trajectory to a specific stationary solution. Therefore, to simplify the presentation of this paper, we shall assume Cap $\Gamma_2 \neq 0$ in what follows.

Define the energy of system (1.1) by

$$E(t) \triangleq E(y, z, z_t)(t) = \frac{1}{2}|(y(t), z(t), z_t(t))|_H^2.$$

By means of the classical energy method, it is easy to check that

$$\frac{d}{dt}E(t) = -\int_{\Omega_1} |\nabla y|^2 dx. \tag{1.2}$$

Therefore, the energy of (1.1) is decreasing as $t \to \infty$. First of all, we show that $E(t) \to 0$ as $t \to \infty$, without any geometric conditions on the domains Ω_1 and Ω_2. Note however that, due to the lack of compactness of the resolvent of the generator of the underlying semigroup of system (1.1) for $n \geq 2$, one can not use directly the LaSalle's invariance principle to prove this result. Instead, using the "relaxed invariance principle" ([9]), we conclude first that the first and third components of every solution (y, z, z_t) of (1.1), y and z_t, tend to zero strongly in $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively; while its second component z tends to zero weakly in $H^1_{\Gamma_2}(\Omega_2)$ as $t \to \infty$. Then, we use the special structure of (1.1) and the key energy dissipation law (1.2) to "recover" the desired strong convergence of z in $H^1_{\Gamma_2}(\Omega_2)$.

The main goal of this paper is to summarize the results we have obtained in the analysis of the longtime behavior of $E(t)$. Especially, we study whether or not the energy of solutions of system (1.1) tends to zero uniformly as $t \to \infty$, i.e., whether there exist two positive constants C and α such that

$$E(t) \leq CE(0)e^{-\alpha t}, \qquad \forall\, t \geq 0 \tag{1.3}$$

for every solution of (1.1).

According to the energy dissipation law (1.2), the uniform decay problem (1.3) is equivalent to showing that: there exists $T > 0$ and $C > 0$ such that every solution of (1.1) satisfies

$$|(y_0, z_0, z_1)|_H^2 \leq C \int_0^T \int_{\Omega_1} |\nabla y|^2 dx dt, \qquad \forall\, (y_0, z_0, z_1) \in H. \tag{1.4}$$

Inequality (1.4) can be viewed as an observability estimate for equation (1.1) with observation on the heat subdomain.

Note however that, as indicated in [10], there is no uniform decay for solutions of (1.1) even in one space dimension. The analysis in [10] exhibits the existence of a hyperbolic-like spectral branch such that the energy of the eigenvectors is concentrated in the wave domain and the eigenvalues have an asymptotically vanishing real part. This is obviously incompatible with the exponential decay rate. The approach in [10], based on spectral analysis, does not apply to multidimensional situations. But the $1-d$ result in [10] is a warning in the sense that one may not expect (1.4) to hold.

Exponential decay property also fails in several space dimensions, as the $1-d$ spectral analysis suggests. For this purpose, following [7], we analyze carefully the interaction of the wave and heat-like solutions on the interface for general geometries. The main idea is to use Gaussian Beams ([6] and [5]) to construct approximate solutions of (1.1) which are highly concentrated along the generalized rays of the d'Alembert operator $\Box$ in the wave domain Ω_2 and are almost completely reflected on the interface γ. Due to the asymmetry of the energy space H, the same construction in [7] does not give the desired estimate. One has to compute higher order corrector terms on the phases and amplitudes of the wave-like solutions to recover an accurate description.

In view of the above analysis, it is easy to see that, one can only expect a polynomial stability property of smooth solutions of (1.1) even under the Geometric Control Condition (GCC for short, *see* [1]), i.e., when the heat domain where the damping of the system is active is such that all rays of Geometric Optics propagating in the wave domain touch the interface in an uniform time. To verify this, we need to derive a weakened observability inequality by viewing the whole system as a perturbation of the wave equation in the whole domain Ω. This technique was applied in the simpler model analyzed in [7]. However, as before, some efforts are necessary to treat the asymmetric structure of the energy space H.

We refer to [12] for the details of the proofs of the results in this paper and other results in this context (especially for the analysis without the technical assumption Cap $\Gamma_2 \neq 0$).

2. Some preliminary results

In this section, we shall present some preliminary results.

Define an unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ by $\mathcal{A}Y = (\Delta Y_1, Y_3, \Delta Y_2)$, where $Y = (Y_1, Y_2, Y_3) \in D(\mathcal{A})$, and

$$D(\mathcal{A}) = \Big\{(Y_1, Y_2, Y_3) \in H \;\Big|\; \Delta Y_1 \in L^2(\Omega_1),\ \Delta Y_2 \in L^2(\Omega_2),\ Y_3 \in H^1(\Omega_2),$$
$$Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0,\ Y_1|_\gamma = Y_3|_\gamma,\ \frac{\partial Y_1}{\partial \nu_1}\Big|_\gamma = -\frac{\partial Y_2}{\partial \nu_2}\Big|_\gamma\Big\}.$$

Remark 1. Obviously, in one space dimension, i.e., $n = 1$, we have $D(\mathcal{A}) = \Big\{(Y_1, Y_2, Y_3) \in H \;\Big|\; Y_1 \in H^2(\Omega_1),\ Y_2 \in H^2(\Omega_2),\ Y_3 \in H^1(\Omega_2),\ Y_1|_{\Gamma_1} = Y_3|_{\Gamma_2} = 0,\ Y_1|_\gamma = Y_3|_\gamma,\ \frac{\partial Y_1}{\partial \nu_1}\Big|_\gamma = -\frac{\partial Y_2}{\partial \nu_2}\Big|_\gamma\Big\} \subset H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$. But this is not longer true in several space dimensions.

It is easy to see that system (1.1) can be re-written as an abstract Cauchy problem in H: $X_t = \mathcal{A}X$ for $t > 0$ with $X(0) = X_0$, where $X = (y, z, z_t)$ and $X_0 = (y_0, z_0, z_1)$. We have the following result.

Theorem 1. *The operator $\mathcal{A}$ is the generator of a contractive C_0-semigroup in H, and $0 \in \rho(\mathcal{A})$, the resolvent of $\mathcal{A}$.*

Remark 2. When $n = 1$, in view of the embedding in Remark 1, it is easy to check that $\mathcal{A}^{-1}$ is compact. However, $\mathcal{A}^{-1}$ is not guaranteed to be compact in several space dimensions, i.e., $n \geq 2$. Indeed, for any $F = (F_1, F_2, F_3) \in H$, the second component Y_2 of $\mathcal{A}^{-1}F$ belongs to $H^1_{\Gamma_2}(\Omega_2)$, which has the same regularity as the second component F_2 of F. (According to the regularity theory of elliptic equations, this regularity property for Y_2 is sharp as we shall see.)

The following result shows that $\mathcal{A}^{-1}$ is not compact.

Proposition 1. *In dimensions $n \geq 2$, the domain $D(\mathcal{A})$ is noncompact in H.*

The proof of Proposition 1 is due to Thomas Duyckaerts. The main idea is as follows: It suffices to show that there exists a sequence of $\{(Y_1^k, Y_2^k, Y_3^k)\}_{k=1}^{\infty} \subset D(\mathcal{A})$ such that $(Y_1^k, Y_2^k, Y_3^k) \rightharpoonup 0$ in $D(\mathcal{A})$ as $k \to \infty$ and $\inf\limits_{k \in \mathbb{N}} |(Y_1^k, Y_2^k, Y_3^k)|_H \geq c$ for some constant $c > 0$. For this purpose, for any nonempty open subset Γ_0 of Γ, we denote by $H_{\Gamma_0}^{-1/2}(\Gamma)$ the completion of $C(\overline{\Gamma_0})$ with respect to the norm:

$$|u|_{H_{\Gamma_0}^{-1/2}(\Gamma)} = \sup \left\{ \frac{|\int_\Gamma uf d\Gamma|}{|f|_{H^{1/2}(\Gamma)}} \;\middle|\; f \in H^{1/2}(\Gamma) \setminus \{0\} \text{ and } f = 0 \text{ on } \Gamma \setminus \overline{\Gamma_0} \right\}.$$

Since $H_\gamma^{-1/2}(\partial\Omega_1)$ can be identified with $H_\gamma^{-1/2}(\partial\Omega_2)$ (algebraically and topologically), we denote them simply by $H_\gamma^{-1/2}$. It is easy to see that $H_\gamma^{-1/2}$ is an infinite-dimensional separable Hilbert space whenever $n \geq 2$. Hence there is a sequence $\{\beta^k\}_{k=1}^{\infty} \subset H_\gamma^{-1/2}$ such that $|\beta^k|_{H_\gamma^{-1/2}} = 1$ for each k and $\beta^k \rightharpoonup 0$ in $H_\gamma^{-1/2}$ as $k \to \infty$.

We solve the following two systems

$$\begin{cases} \Delta Y_1^k = 0 & \text{in } \Omega_1, \\ Y_1^k = 0 & \text{on } \Gamma_1, \\ \frac{\partial Y_1^k}{\partial \nu_1} = -\beta^k & \text{on } \gamma, \end{cases} \qquad \begin{cases} \Delta Y_2^k = 0 & \text{in } \Omega_2, \\ Y_2^k = 0 & \text{on } \Gamma_2, \\ \frac{\partial Y_2^k}{\partial \nu_2} = \beta^k & \text{on } \gamma \end{cases}$$

to get $Y_i^k \in H^1_{\Gamma_i}(\Omega_i)$ $(i = 1, 2)$, and then solve

$$\begin{cases} \Delta Y_3^k = 0 & \text{in } \Omega_2, \\ Y_3^k = 0 & \text{on } \Gamma_2, \\ Y_3^k = Y_1^k & \text{on } \gamma \end{cases}$$

to get $Y_3^k \in H^1_{\Gamma_2}(\Omega_2)$. This produces the desired $\{(Y_1^k, Y_2^k, Y_3^k)\}_{k=1}^{\infty}$.

Remark 3. Noting the structure of $D(\mathcal{A})$, it is easy to see that

$$D(\mathcal{A}) \subset H^1_{\Gamma_1}(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times H^1_{\Gamma_2}(\Omega_2). \tag{2.1}$$

This, at least, produces H^1-regularity for the heat and wave components of system (1.1) whenever its initial datum belongs to $D(\mathcal{A})$. One may need the H^2-regularity for the heat and wave components of system (1.1) when the initial data are smooth. For this to be true it is not sufficient to take the initial data in $D(\mathcal{A})$ since generally $D(\mathcal{A}) \not\subset (H^2(\Omega_1) \cap H^1_{\Gamma_1}(\Omega_1)) \times (H^2(\Omega_2) \cap H^1_{\Gamma_2}(\Omega_2)) \times H^1_{\Gamma_2}(\Omega_2)$ unless $n = 1$.

In order to prove the existence of smooth solutions of (1.1), we introduce the following Hilbert space:

$$V=\Big\{(y_0,z_0,z_1)\in D(\mathcal{A})\;\Big|\;y_0\in H^2(\Omega_1),z_0\in H^2(\Omega_2)\Big\}\subset D(\mathcal{A}),$$

with the canonical norm. Note however that, according to Proposition 1, $D(\mathcal{A}^k)$ is not necessarily a subspace of V even if $k\in\mathbb{N}$ is sufficiently large.

We have the following regularity result:

Theorem 2. *Let $\Gamma\cap\gamma=\emptyset$ and $\gamma\in C^2$. Then for any $(y_0,z_0,z_1)\in V$, the solution of (1.1) satisfies $(y,z,z_t)\in C([0,\infty);V)$, and for any $T\in(0,\infty)$, there is a constant $C_T>0$ such that*

$$|(y,z,z_t)|_{C([0,T];V)}\le C_T|(y_0,z_0,z_1)|_V.$$

The main idea to show Theorem 2 is as follows: We first take the tangential derivative of the system and show that the tangential derivative of the solution is of finite energy and then by using the original equation, one obtains the regularity of the other derivatives.

3. Asymptotic behavior

First of all, we show the strong asymptotic stability of (1.1) without the GCC.

Theorem 3. *For any given $(y_0,z_0,z_1)\in H$, the solution (y,z,z_t) of (1.1) tends to 0 strongly in H as $t\to\infty$.*

To prove Theorem 3, by density, it suffices to assume $(y_0,z_0,z_1)\in D(\mathcal{A})$. As we said above, we apply the relaxed invariance principle, using the energy as Lyapunov function. This yields the strong convergence to zero of the components y and z_t of the solution in the corresponding spaces. But this argument fails to give strong convergence to zero of z in $H^1_{\Gamma_2}(\Omega_2)$, because of the lack of compactness of the embedding from $D(A)$ into H. This argument, in principle, only yields the weak convergence of z. We need a further argument to show that the convergence of z holds in the strong topology of $H^1_{\Gamma_2}(\Omega_2)$. The key point is that, in view of the energy dissipation law (1.2), one has

$$\nabla y\in L^2(0,\infty;(L^2(\Omega_1))^n). \tag{3.1}$$

Also, by the standard semigroup theory and (2.1) in Remark 3, we see that $\nabla y\in C([0,\infty);(L^2(\Omega_1))^n)$. Therefore, (3.1) implies that there is a sequence $\{s_n\}_{n=1}^\infty$ which tends to ∞ such that

$$\nabla y(s_n)\to 0\ \text{ strongly in }\ (L^2(\Omega_1))^n\ \text{ as }\ n\to\infty.$$

With this, we can deduce that

$$\nabla z(s_n)\to 0\ \text{ strongly in }\ (H^1(\Omega_2))^n\ \text{ as }\ n\to\infty,$$

and, using the decreasing character of the energy of the system, we may conclude that the convergence holds along all the continuous one parameter family $z(s)$ as s tends to infinity.

Next, we analyze the non-uniform decay of solutions to (1.1). For this purpose, we recall that a *null bicharacteristic* for $\Box$ in $\mathbb{R}^n$ is defined to be a solution of the ODE:

$$\begin{cases} \dot{x}(t) = 2\xi(t), & \dot{\xi}(t) = 0, \\ x(0) = x^0, & \xi(0) = \xi^0, \end{cases}$$

where the initial data ξ^0 are chosen such that $|\xi^0| = 1/2$. Clearly, $(t, x(t))$, the projection of the null bicharacteristic to the physical time-space, traces a line in $\mathbb{R}^{1+n}$ (starting from $(0, x^0)$), which is called a *ray* for $\Box$ in the sequel. Sometimes, we also refer to $(t, x(t), \xi(t))$ as the ray. Obviously, rays for $\Box$ in $\mathbb{R}^n$ are simply straight lines.

In the presence of boundaries, rays, when reaching the boundary, are reflected following the usual rules of Geometric Optics. More precisely, for a $T > 0$ and a bounded domain $M \subset \mathbb{R}^n$ with piecewise C^1 boundary ∂M, the singular set being localized on a closed (topological) sub-manifold S with $\dim S \leq n-2$, we introduce the following definition of multiply reflected rays.

Definition 1. A continuous parametric curve: $[0,T] \ni t \mapsto (t, x(t), \xi(t)) \in C([0,T] \times \overline{M} \times \mathbb{R}^n)$, with $x(0) \in M$ and $x(T) \in M$, is called a multiply reflected ray for the operator $\Box$ in $[0,T] \times \overline{M}$ if there exist $m \in \mathbb{N}$, $0 < t_0 < t_1 < \cdots < t_m = T$ such that each $(t, x(t), \xi(t))|_{t_i < t < t_{i+1}}$ is a ray for $\Box$ $(i = 0, 1, 2, \ldots, m-1)$, which arrives at $\partial M \setminus S$ at time $t = t_{i+1}$, and is reflected by $(t, x(t), \xi(t))|_{t_{i+1} < t < t_{i+2}}$ by the usual geometric optics law whenever $i < m - 1$.

In view of [7, Lemma 2.2 and Remark 2.4], we have the following geometric lemma.

Lemma 1. *For each $T > 0$, there is a multiply reflected ray for the operator $\Box$ in M which meets $\partial M \setminus S$ transversally and non-normally.*

We have the following key result.

Theorem 4. *Let the boundary $\partial\Omega_2$ of the wave domain Ω_2 be of class C^4. For any $T > 0$, let $[0,T] \ni t \mapsto (t, x(t), \xi(t)) \in C([0,T] \times \overline{\Omega_2} \times \mathbb{R}^n)$ be a multiply reflected ray for the operator $\Box$ in Ω_2, which meets the boundary $\partial\Omega_2$ transversally and non-normally. Then there is a family of solutions $\{(y_\varepsilon, z_\varepsilon)\}_{\varepsilon>0}$ of system* (1.1) *in* $(0,T)$ (*the initial conditions being excepted*)*, such that*

$$|\nabla y_\varepsilon|^2_{(L^2((0,T)\times\Omega_1))^n} = O(\varepsilon), \qquad E_\varepsilon(0) = E(y_\varepsilon, z_\varepsilon, \partial_t z_\varepsilon)(0) \geq c_0,$$

where $c_0 > 0$ is a constant, independent of ε.

Now, combining Lemma 1 and Theorem 4, one obtains the following non-uniform decay result:

Theorem 5. *Let the boundary $\partial\Omega_2$ of the wave domain Ω_2 be of class C^4. Then*

- i) *For any given $T > 0$, there is no constant $C > 0$ such that* (1.4) *holds for all solutions of* (1.1)*;*
- ii) *The energy $E(t)$ of solutions of system* (1.1) *does not decay exponentially as $t \to \infty$.*

Finally, we analyze the long time behavior of solutions of system (1.1) in several space dimensions under suitable geometric assumptions.

We introduce the following internal observability assumption for the wave equation in Ω:

(H) There exist $T_0 > 0$ such that for some constant $C > 0$, all solutions of the following system

$$\begin{cases} \Box\zeta = 0 & \text{in} \quad (0,T_0)\times\Omega, \\ \zeta = 0 & \text{on} \quad (0,T_0)\times\Gamma, \\ \zeta(0)=\zeta_0, \quad \zeta_t(0)=\zeta_1 & \text{in} \quad \Omega \end{cases}$$

satisfy

$$|\zeta_0|^2_{H_0^1(\Omega)} + |\zeta_1|^2_{L^2(\Omega)} \le C\int_0^{T_0}\int_{\Omega_1}|\zeta_t|^2dxdt, \ \ \forall\,(\zeta_0,\zeta_1)\in H_0^1(\Omega)\times L^2(\Omega).$$

It is well known that assumption (H) holds when T_0 and Ω_1 satisfy the *Geometric Optics Condition* (GCC) introduced in [1]. This condition asserts that all rays of Geometric Optics propagating in Ω and bouncing on the boundary enter the control domain Ω_1 in a uniform time $T_0 > 0$. A relevant particular case in which the GCC is satisfied is when the heat domain Ω_1 envelopes the wave domain Ω_2. This simple case can be handled by the multiplier method ([4]).

Now, we may state our polynomial decay result for system (1.1) as follows.

Theorem 6. *Let T_0 and Ω_1 satisfy (H). Then there is a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$, the solution of (1.1) satisfies*

$$|(y(t), z(t), z_t(t))|_H \le \frac{C}{t^{1/6}}|(y_0, z_0, z_1)|_{D(\mathcal{A})}, \quad \forall\, t > 0.$$

Remark 4. Theorem 6 is not sharp for $n = 1$ since in [10] we have proved that the decay rate is $1/t^2$. However, similar to [7], the WKB asymptotic expansion for the flat interface allows to show that it is impossible to expect the same decay rate for several space dimensions. This suggests that the rate of decay in the multidimensional case is slower than in the one dimensional one. According to Remark 5 and the possible sharp weakened observability inequality (3.4) below, it seems reasonable to expect $1/t$ to be the sharp polynomial decay rate for smooth solutions of (1.1) with initial data in $D(\mathcal{A})$. But this is an open problem. We refer to [3] for an interesting partial solution to this problem with a decay rate of the order of $1/t^{1-\delta}$ for all $\delta > 0$ but under stronger assumptions on the geometry that Ω is of C^∞ and $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$.

The proof of Theorem 6 is based on the following key weakened observability inequality for equation (1.1):

Theorem 7. *Let T_0 and Ω_1 satisfy (H). Then there exist two constants T_0 and $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A}^3)$, and any $T \ge T_0$, the solution of (1.1) satisfies*

$$|(y_0, z_0, z_1)|_H \le C|\nabla y|_{H^3(0,T;(L^2(\Omega_1))^n)}. \tag{3.2}$$

The main idea to prove Theorem 7 is as follows: Setting $w = y\chi_{\Omega_1} + z_t\chi_{\Omega_2}$, noting (1.1) and recalling that $\partial z_t/\partial\nu_2 = -\partial y_t/\partial\nu_1$ on $(0,T)\times\gamma$, and by $(y_0, z_0, z_1) \in D(\mathcal{A}^2)$, one sees that $w \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ satisfies

$$\begin{cases} \Box\, w = (y_{tt} - y_t)\chi_{\Omega_1} + \left(\frac{\partial y}{\partial\nu_1} - \frac{\partial y_t}{\partial\nu_1}\right)\delta_\gamma & \text{in } (0,T)\times\Omega, \\ w = 0 & \text{on } (0,T)\times\Gamma, \\ w(0) = y_0\chi_{\Omega_1} + z_1\chi_{\Omega_2},\ w_t(0) = (\Delta y_0)\chi_{\Omega_1} + (\Delta z_0)\chi_{\Omega_2} & \text{in } \Omega. \end{cases} \tag{3.3}$$

Then, by means of the energy method and assumption (H), one concludes Theorem 7.

Remark 5. Note that (3.2) is, indeed, a weakened version of (1.4), in which we do not only use the norm of ∇y on $(L^2((0,T)\times\Omega_1))^n$ to bound the total energy of solutions but the stronger one on $H^3(0,T;(L^2(\Omega_1))^n)$. Nevertheless, inequality (3.2) is very likely not sharp. One can expect, under assumption (H), the following stronger inequality to hold:

$$|(y_0, z_0, z_1)|_H \le C|\nabla y|_{H^{1/2}(0,T;(L^2(\Omega_1))^n)}. \tag{3.4}$$

This is also an open problem.

4. Open problems

This subject is full of open problems. Some of them seem to be particularly relevant and could need important new ideas and further developments:

• *Logarithmic decay without the GCC.* Inspired on [8], it seems natural to expect a logarithmic decay result for system (1.1) without the GCC. However, there is a difficulty to do this. In [7] we show this decay property for system (1.1) but with the interface condition $y = z_t$ replaced by $y = z$. The key point is to apply the known very weak observability inequalities for the wave equations without the GCC ([8]) to a perturbed wave equation similar to (3.3), and use the crucial fact that the generator of the underlying semigroup has compact resolvent. It is precisely the lack of compactness for (1.1) in multi-dimensions that prevents us from showing the logarithmic decay result in the present case.

• *More complex and realistic models.* In the context of fluid-structure interaction, it is more physical to replace the wave equation in system (1.1) by the system of elasticity and the heat equation by the Stokes system, and the fluid-solid interface γ by a free boundary. It would be interesting to extend the present analysis to these situations. But this remains to be done.

• *Nonlinear models.* A more realistic model for fluid-structure interaction would be to replace the heat and wave equations in system (1.1) by the Navier-Stokes and elasticity systems coupled through a moving boundary. To the best of our knowledge, very little is known about the well-posedness and the long time behavior for the solutions to the corresponding equations (We refer to [2] for some existence results of weak solutions in two space dimensions).

• *Control problems.* In [10], we analyze the null controllability problem for system (1.1) in one space dimension by means of spectral methods. It is found that the controllability results depend strongly on whether the control enters the system through the wave component or the heat one. This problem is completely open in several space dimensions.

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