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Localized solutions for the finite difference semi-discretization of the wave equation

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Abstract

We study the propagation properties of the solutions to the finite difference space semi-discrete wave equation on an uniform grid of the whole Euclidean space. We provide several constructions of high frequency wave packets that propagate along the corresponding bi-characteristic rays of Geometric Optics traveling with a velocity arbitrarily close to zero. Our analysis is motivated by control theoretical issues. In particular, the continuous wave equation has the so-called observability property: for a sufficiently large time, the energy of its solutions can be estimated in terms of the energy concentrated in the exterior of a compact set. Our results show that this fails to be true, uniformly on the mesh-size parameter, for the semi-discrete scheme, whatever the observability time is. Our constructions show that the observability constant blows-up at an arbitrarily large polynomial order. The techniques of proof combine Fourier analysis and Geometric Optics tools and can be adapted to other classes of numerical schemes as classical finite elements or discontinuous Galerkin methods.

Key words: Wave equation, finite difference schemes, high frequency wave packets, localized waves, lack of uniform observability and controllability.
Field: control.
Presentation: oral communication.

1 Introduction

For a finite time horizon $T > 0$, consider the Cauchy problem associated to the d-dimensional conservative wave equation:

$$\begin{cases} \partial_t^2 \phi(x,t) - \Delta \phi(x,t) = 0, & x \in \mathbb{R}^d, t \in (0,T] \\ \phi(x,T) = \phi^0(x), \partial_t \phi(x,T) = \phi^1(x), & x \in \mathbb{R}^d. \end{cases} \tag{1}$$

The equation (1) is well posed in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and its energy,

$$E(\phi^0,\phi^1) = \frac{1}{2}\big(\|\phi(t)\|^2_{\dot{H}^1(\mathbb{R}^d)} + \|\partial_t \phi(t)\|^2_{L^2(\mathbb{R}^d)}\big), \tag{2}$$

is conserved in time. Here $\dot{H}^1(\mathbb{R}^d)$ is the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|^2_{\dot{H}^1(\mathbb{R}^d)} = \|\nabla \cdot\|^2_{L^2(\mathbb{R}^d)}$.

For all time $T > 2$, there exists a constant $C(T) > 0$ such that, for all $(\phi^0,\phi^1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, the following observability inequality holds:

$$E(\phi^0,\phi^1) \leqslant C(T) \int_0^T \int_{\Omega^d} |\partial_t \phi(x,t)|^2 \, dx \, dt, \tag{3}$$

where the observation region $\Omega^d = \mathbb{R}^d \setminus B^d(0,1)$ is the complement of the d-dimensional unit ball. Actually, the inequality (3) can be proved for any exterior domain of the form $\Omega^d = \mathbb{R}^d \setminus U^d$, provided $T > T^* := \mathrm{diam}(U^d)$, where $U^d \subset \mathbb{R}^d$ is any bounded open set. In the following, without loss of generality, we will focus only on the case $U^d = B^d(0,1)$.

The observability problem is motivated by controllability issues. More precisely, by means of the Hilbert Uniqueness Method (HUM) (see [3]), (3) is equivalent to the fact that, for all $T > T^* = 2$ and all initial data $(u^0,u^1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, there exists a control function $f \in L^2(\Omega^d \times (0,T))$ such that the solution of the inhomogeneous Cauchy problem

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) = f(x,t)\chi_{\Omega^d}(x), & x \in \mathbb{R}^d, \ t \in (0,T] \\ u(x,0) = u^0(x), \ \partial_t u(x,0) = u^1(x), \ x \in \mathbb{R}^d \end{cases} \tag{4}$$

satisfies $u(x,T) = \partial_t u(x,T) = 0$ for all $x \in \mathbb{R}^d$. Here χ_{Ω^d} denotes the characteristic function of the set Ω^d.

These issues are by now well understood for the continuous wave equation [9]. In particular, it is well known that observability holds under the sharp Geometric Control Condition (GCC) [1] requiring that all rays of Geometric Optics enter the observation set during the observability time. When the GCC is not satisfied, the observability property fails because of the existence of Gaussian beam solutions localized around a bi-characteristic ray that escapes the observation region during the time interval $[0,T]$ [6].

Given a mesh size $h > 0$, we define an uniform grid of the whole Euclidean space by $x_{\mathbf{j}} = h\mathbf{j}$, $\mathbf{j} \in \mathbb{Z}^d$. We also introduce two discrete operators: the gradient $\nabla_h^+ = (\partial_{h,k}^+)_{k=1,\cdots,d}$ and the Laplacian Δ_h, where

$$\partial_{h,k}^+ \overrightarrow{f} = \frac{\overrightarrow{f}_{\cdot+\mathbf{e}_k} - \overrightarrow{f}}{h}, \quad \Delta_h \overrightarrow{f} = \frac{1}{h^2}\sum_{l=1}^{d}(\overrightarrow{f}_{\cdot+\mathbf{e}_l} - 2\overrightarrow{f} + \overrightarrow{f}_{\cdot-\mathbf{e}_l}), \tag{5}$$

$\overrightarrow{f} = (f_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$ is any sequence and $(\mathbf{e}_l)_{l=1}^d$ is the canonical basis in $\mathbb{R}^d$.

We consider the finite difference space semi-discretization of wave equation (1):

$$\begin{cases} \partial_t^2\phi_{\mathbf{j}}^h(t) - \Delta_h\phi_{\mathbf{j}}^h(t) = 0, \quad \mathbf{j}\in\mathbb{Z}^d, t\in(0,T] \\ \phi_{\mathbf{j}}^h(T) = \phi_{\mathbf{j}}^{h,0}, \quad \partial_t\phi_{\mathbf{j}}^h(T) = \phi_{\mathbf{j}}^{h,1}, \quad \mathbf{j}\in\mathbb{Z}^d. \end{cases} \tag{6}$$

Define the following two discrete Sobolev spaces

$$\ell^2(h\mathbb{Z}^d) = \{\overrightarrow{f} \text{ s.t. } \|\overrightarrow{f}\|^2_{\ell^2(h\mathbb{Z}^d)} := h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}|f_{\mathbf{j}}|^2 < \infty\}, \tag{7}$$

$$\dot{\hbar}^1(h\mathbb{Z}^d) = \{\overrightarrow{f} \text{ s.t. } \|\overrightarrow{f}\|^2_{\dot{\hbar}^1(h\mathbb{Z}^d)} := \sum_{k=1}^{d}\|\partial_{h,k}^+\overrightarrow{f}\|^2_{\ell^2(h\mathbb{Z}^d)} < \infty\}. \tag{8}$$

The problem (6) is well-posed in $\dot{\hbar}^1(h\mathbb{Z}^d)\times\ell^2(h\mathbb{Z}^d)$. The energy associated to its solution is conserved in time and is defined by

$$E_h(\overrightarrow{\phi}^{h,0}, \overrightarrow{\phi}^{h,1}) = \frac{1}{2}\big(\|\overrightarrow{\phi}^h(t)\|^2_{\dot{\hbar}^1(h\mathbb{Z}^d)} + \|\partial_t\overrightarrow{\phi}^h(t)\|^2_{\ell^2(h\mathbb{Z}^d)}\big). \tag{9}$$

For a fixed $T>0$, consider the semi-discrete version of (3)

$$E_h(\overrightarrow{\phi}^{h,0}, \overrightarrow{\phi}^{h,1}) \le C_h(T)h^d\sum_{x_{\mathbf{j}}\in\Omega^d}\int_0^T|\partial_t\phi_{\mathbf{j}}(t)|^2\,dt. \tag{10}$$

In what follows, we analyze the dependence on the mesh size parameter h of the observability constant $C_h(T)$. For the finite difference and P_1-classical finite element semi-discretizations of the wave equation on particular bounded domains like d-dimensional cubes, the corresponding observability constant blows-up as $h\to 0$ because of the pathological behavior of the spurious high frequency numerical solutions [10]. Similar pathological high frequency phenomena have been observed in the context of the Strichartz dispersive estimates for the finite difference approximations of the Schrödinger equation [2].

Our constructions of highly concentrated wave packets allow to obtain polynomial lower bounds on $C_h(T)$ and make rigorous the argument leading to the introduction of the notion of group velocity in [8]. Set $\Pi_h^d := [-\pi/h,\pi/h]^d$. A precise mathematical definition of the group velocity follows by applying the semi-discrete Fourier transform (SDFT) at the scale h [7]

on (6). Denote by $\widehat{\phi}^h(\xi,t)$ the SDFT of the solution $\overrightarrow{\phi}^h(t)$ of (6). It satisfies the following second-order ODE depending on the parameter ξ:

$$\begin{cases} \partial_t^2\widehat{\phi}^h(\xi,t)+\omega_{d,h}^2(\xi)\widehat{\phi}^h(\xi,t)=0, & \xi\in\Pi_h^d, t\in(0,T] \\ \widehat{\phi}^h(\xi,0)=\widehat{\phi}^{h,0}(\xi), \quad \partial_t\widehat{\phi}^h(\xi,0)=\widehat{\phi}^{h,1}(\xi) & \xi\in\Pi_h^d, \end{cases} \tag{11}$$

where $\omega_{d,h}(\xi)$ is the multi-dimensional dispersion relation associated to (6),

$$\omega_{d,h}^2(\xi)=\frac{4}{h^2}\sum_{k=1}^d\sin^2\Big(\frac{\xi_k h}{2}\Big), \quad \xi=(\xi_k)_{1\le k\le d}\in\Pi_h^d. \tag{12}$$

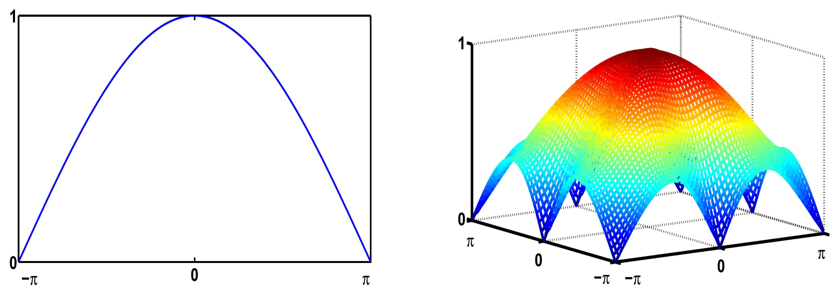

Figure 1: The velocity of propagation $|\nabla\omega_{d,1}(\xi)|$ for $d=1,2$ (left/right).

The semi-discrete rays of Geometric Optics corresponding to the semi-discrete problem (6) are of the form $x_h^{\pm}(t)=x\pm\nabla\omega_{d,h}(\xi)t$, $x\in\mathbb{R}^d$, $\xi\in\Pi_h$. The gradient $\nabla\omega_{d,h}(\xi)$ is precisely the group velocity corresponding to the approximation (6). Observe that the velocity of propagation along the rays,

$$|\nabla\omega_{d,h}(\xi)|=\sqrt{\sum_{k=1}^d\sin^2(\xi_k h)}\left(\sqrt{4\sum_{k=1}^d\sin^2\Big(\frac{\xi_k h}{2}\Big)}\right)^{-1},$$

$(x^{-1}:=1/x)$ vanishes for all $\xi\in\{\pm\pi/h,0\}^d\backslash\{0\}$, whereas, in the continuous case (1), the dispersion relation is $\omega_d(\xi)=|\xi|$ and the velocity of propagation along the ray is $|\nabla\omega_d(\xi)|=1$ for all $\xi\in\mathbb{R}^d$.

In the context of the finite difference semi-discretization of the $1-d$ wave equation on a bounded interval, the corresponding observability constant blows-up exponentially as $h\to 0$ for all $T>0$ [5]. Although the constructions in this paper provide slightly weaker results, they have several advantages with respect to the previously existing ones, in the sense that they are based on the analysis of the symbol of the operator and do not require an explicit knowledge of the spectrum of the system. Therefore, they can be applied to both Cauchy problems on the whole Euclidian space and boundary-value problems with various boundary conditions.

2 Main results

Theorem 1 *For a fixed $T > 0$, consider a wave number $\eta_0 = h\xi_0 \in \Pi_1^d \setminus \{0\}$ such that the corresponding semi-discrete ray does not enter the observation region in the time T, i.e. $2\delta := 2 - T|\nabla\omega_{d,1}(\eta_0)| > 0$.*

Set $x \in B^d(0,1)$ such that $d(x, \partial B^d(0,1)) = \delta$. For $\alpha \in (0,1)$, define $\gamma := 1/(\delta^2 h^\alpha)$ and

$$\sigma_\gamma(z) = \exp\Big(- \frac{\gamma|z|^2}{2}\Big)$$

whose continuous Fourier transform is denoted by $\widehat{\sigma}_\gamma(\xi)$.

Consider initial data $(\overrightarrow{\phi}^{h,0}, \overrightarrow{\phi}^{h,1})$ in (6) such that

$$\widehat{\phi}^{h,0}(\xi) = \widehat{\sigma}^d_\gamma(\xi - \xi_0)\exp(-ix(\xi - \xi_0)) \text{ and } \widehat{\phi}^{h,1}(\xi) = i\omega_{d,h}(\xi)\widehat{\phi}^{h,1}(\xi).$$

Then, for all $N \in \mathbb{N}$, there exist two constants $C_N, c_N(\alpha) > 0$ such that, for fixed α, $c_N(\alpha) \to \infty$ as $N \to \infty$ and the constant $C_h(T)$ in (10) satisfies

$$C_h(T) \geqslant C_N h^{-c_N(\alpha)}.$$

In Figure 2, we plot the solution of the wave equation (1) with initial data $\phi^1 = \phi^0_z$ and $\phi^0(z) = \sigma_\gamma(z)\exp(i\xi_0 z)$ for $d = 1$, $h = 0.001$, $\xi_0 h = 19\pi/20$, $\alpha = 0.75$ (left) and $\alpha = 0.9$ (right). In red, the Gaussian wave packet corresponding to both continuous and discrete cases at time $t = 0$. In blue, the solution of the continuous wave equation (1) and in black the one corresponding to the semi-discrete wave equation (6) at time $t = 1$. The same kind of results can be obtained by adapting the construction

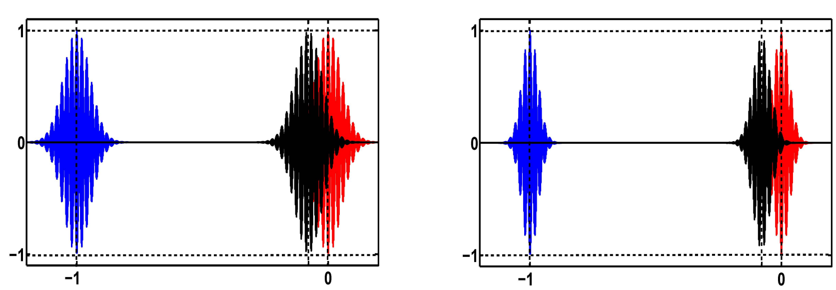

Figure 2: Transport of continuous versus discrete high frequency wave packets.

of the so-called Gaussian beams [6] to the semi-discrete case. They verify asymptotically the hyperbolic PDEs or their numerical approximations. For a finite $t > 0$, their energy concentrated outside a small ball centered at the ray $x_h^+(t)$ is exponentially small with respect to the mesh size h.

In view of the divergence of the discrete observability constant with respect to h, one can show the existence of finite energy data for which the controls of the semi-discrete system diverge polynomially at an arbitrary fast order as the mesh size tends to zero. This confirms the well known pathology ensuring that the controls of the semi-discrete system do not necessarily converge to those of the continuous wave equation.

By means of filtering mechanisms like Fourier filtering of high frequencies, bi-grid algorithms or vanishing viscosity, such pathological solutions are avoided.

The construction of high frequency wave packets can be adapted to more sophisticated numerical schemes like those arising from discontinuous Galerkin or higher order classical finite element methods, for which the behavior is more complex, in the sense that they generate several dispersion relations, each of them having points where the corresponding group velocity vanishes [4].

Some open problems: • the exponential blow-up with respect to the mesh size of the discrete observability constant; • the analysis of the propagation properties of the numerical methods on discrete heterogeneous media represented by irregular meshes or variable coefficients.

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