Explicit Exponential Decay Rates for Solutions of von Kármán's System of Thermoelastic Plates

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Abstract – We consider the von Kármán system for a bounded smooth thermoelastic plate clamped on its boundary. We prove explicit exponential decay rates showing that the energy of solutions corresponding to initial data of energy equal to R decays as $t \to \infty$ like exp $(-\omega t/(1+R^2))$ for some universal positive constant ω.

Estimations Explicites sur le Taux de Decroissance Exponentielle pour le Système de von Kármán de Plaques Thermoélastiques

Résumé – On considère le système de von Kármán pour une plaque thermoélastique bornée et régulière. On suppose que la plaque est encastrée. On démontre des estimations explicites sur le taux de décroissance des solutions montrant que l'énergie des solutions d'énergie initiale égale à R décroit comme exp $(-\omega t/(1+R^2))$ lorsque $t \to \infty$ où $\omega > 0$ est une constante indépendante de la donnée initiale.

Version Française Abrégée – Soit Ω un ouvert borné régulier de $I\!R^2$. Considérons le système de von Kármán pour une plaque thermoélastique encastrée occupant le domaine Ω:

$$
(1) \qquad \begin{cases} u_{tt} + \Delta^2 u - h\Delta u_{tt} + \Delta\theta = [u,v] & \text{dans } \ \Omega \times (0,\infty) \\ \Delta^2 v = -[u,u] & \text{dans } \ \Omega \times (0,\infty) \\ \theta_t - \Delta\theta - \Delta u_t = 0 & \text{dans } \ \Omega \times (0,\infty) \\ u = \dfrac{\partial u}{\partial \nu} = 0, \qquad v = \dfrac{\partial v}{\partial \nu} = 0, \ \ \theta = 0 & \text{sur } \ \partial\Omega \times (0,\infty) \\ u(x,0) = u^0(x), \ \ u_t(x,0) = u^1(x), \ \ \theta(x,0) = \theta^0(x) & \text{dans } \ \Omega. \end{cases}
$$

Dans (1), le crochet $[\,,\,]$ dénote

$$
[u,v] = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - 2\,\frac{\partial^2 u}{\partial x_1 \partial x_2}\,\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}
$$

et ν la normale éxterieure unitaire. Le paramètre h est supposé non-négatif et losque $h > 0$ le terme $-h\Delta u_{tt}$ corréspond à l'inértie de rotation de la plaque.

Le système est bien posé dans $H_+ = H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ lorsque $h > 0$ et dans $H = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ lorsque $h = 0$ (voir [2]). Plus précisement, pour tout $(u^0, u^1, \theta^0) \in H_+$ (resp. H) il existe une unique solution de (1) telle que $(u, u_t, \theta) \in C([0,\infty);\ H_+)$ (resp. $C([0,\infty);\ H)$) lorsque $h > 0$ (resp. $h = 0$).

L'énergie du système est donnée par

$$
(2) \quad E(t) = \frac{1}{2}\int_\Omega \left[u_t^2(x,t) + |\Delta u(x,t)|^2 + h|\nabla u_t(x,t)|^2 + \theta^2(x,t)\right]\,dx + \frac{1}{4}\int_\Omega |\Delta v(x,t)|^2 dx
$$

et décroit le long des trajectories. En effet,

$$\frac{dE}{dt}(t) = -\int_\Omega |\nabla\theta(x,t)|^2 dx. \tag{3}$$

Dans [2] on a montré que l'énergie de toute solution décroit avec un taux exponentiel localement uniforme. Plus précisement, on a montré que pour tout $R > 0$ il existe $C(R)$ et $\omega(R) > 0$ tels que

$$E(t) \le C(R)\exp\left(-\omega(R)t\right)E(0), \qquad \forall t \ge 0 \tag{4}$$

pour toute solution de (1) avec $E(0) \le R$. Le démonstration de [2] est basée sur la compacité de la non-linéarité dans l'espace de l'énergie. Le problème est ainsi réduit au système linéaire des plaques thermoélastiques pour lequel on a demontré la décroissance uniforme en utilisant, en particulier, des méthodes de découplage inspirées de [4].

Dans cette Note on construit une fonctionnelle de Lyapunov qui équivaut à l'énergie et pour laquelle on peut exhiber des taux de décroissance uniformes. La construction de cette fonctionnelle est basée dans l'usage de multiplicateurs qui permettent d'exhiber le couplage entre la composente élastique et thermique de l'énergie du système.

On obtient ainsi le résultat suivant:

Théorème. – *Il existe C et $\omega > 0$ tels que*

$$E(t) \le C\exp\left(-\omega t/(1+R^2)\right)E(0), \qquad \forall t > 0$$

pour toute solution de (1) *telle que* $E(0) \le R$.

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1. Introduction and Main Result. – Let Ω be a bounded smooth domain in $\mathbb{R}^2$. Consider the von Kármán system for a coupled thermoelastic plate occupying Ω:

$$\begin{cases} u_{tt} + \Delta^2 u - h\Delta u_{tt} + \Delta\theta = [u,v] & \text{in } \Omega\times(0,\infty) \\ \Delta^2 v = -[u,u] & \text{in } \Omega\times(0,\infty) \\ \theta_t - \Delta\theta - \Delta u_t = 0 & \text{in } \Omega\times(0,\infty) \\ u = \dfrac{\partial u}{\partial\nu} = 0, \qquad v = \dfrac{\partial v}{\partial \nu} = 0, \ \ \theta = 0 & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0) = u^0(x), \ \ u_t(x,0) = u^1(x), \ \ \theta(x,0) = \theta^0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

In (1), the bracket $[\,,\,]$ denotes

$$[u,v] = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} - 2\frac{\partial^2 u}{\partial x_1\partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2} \tag{2}$$

and ν the outward unit normal to Ω. The parameter h is assumed to be non-negative. When $h > 0$ the term $-h\Delta u_{tt}$ represents the rotational inertia of the plate. We have chosen to assume all the other parameters of the system (mass density, flexural rigidity, coupling parameters, etc) to be equal to one to simplify the notations, although the results of this Note apply in general.

System (1) is well posed in $H_+ = H_0^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$ when $h > 0$ and in $H = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ when $h = 0$ (see [2]). More precisely, for any $(u^0, u^1, \theta^0) \in H_+$ (resp. H) there exists an unique solution $(u, u_t, \theta) \in C([0,\infty);\ H_+)$ (resp. $C([0,\infty);\ H)$) of (1) when $h > 0$ (resp. $h = 0$).

The energy of the system is given by

$$(3)\quad E(t) = \frac{1}{2}\int_\Omega \left[u_t^2(x,t) + |\Delta u(x,t)|^2 + h|\nabla u_t(x,t)|^2 + \theta^2(x,t)\right] dx + \frac{1}{4}\int_\Omega |\Delta v(x,t)|^2 dx$$

and decreases along trajectories. Indeed,

$$(4)\qquad \frac{dE}{dt}(t) = -\int_\Omega |\nabla\theta(x,t)|^2 dx \leq 0.$$

In [2] we have proved that the energy decays exponentially uniformly in bounded sets of the energy space. More precisely, we have shown that for any $R > 0$ there exist $C(R)$, $\omega(R) > 0$ such that

$$(5)\qquad E(t) \leq C(R)\exp\left(-\omega(R)t\right) E(0), \qquad \forall t > 0$$

for any solution of (1) such that $E(0) \leq R$. The proof of [2] is based on the compactness of the non-linear term in the energy space. This allows to reduce the problem to the decay of solutions of the underlying linear system. In [2] the decay of the linear system was proved by using a decoupling method introduced in [4].

The decay property (5) does not provide any explicit decay rate in terms of the initial energy of the solution R.

In this Note we construct a Lyapunov function which is equivalent to the energy of solutions and that allows to obtain explicit rates of decay.

Our main result is as follows:

THEOREM 1. – *For any $h \geq 0$ there exist positive constants C, $\omega > 0$ such that*

$$(6)\qquad E(t) \leq C\exp\left(-\omega t/(1+R^2)\right) E(0), \qquad \forall t \geq 0$$

for any solution of (1) *such that* $E(0) \leq R$.

Remark . –

(a) (6) provides an explicit decay rate that vanishes quadratically as $R \to \infty$. Whether (6) is sharp or not is an open problem.

(b) In the absence of thermal effects the exponential rate of decay is known to be uniform (see [1]).

(c) The methods of this paper can be applied for other systems of thermoelasticity. In particular, they can be adapted when the von Kármán thermoelastic plate satisfies other boundary conditions. For instance when the boundary conditions in (1) are replaced by

$$u = \Delta u = 0, \quad v = \Delta v = 0, \quad \theta = 0 \qquad \text{on } \partial\Omega \times (0,\infty)$$

the proof is simpler since some of the boundary integrals vanish (see formula (16) below). □

Our construction of the Lyapunov functional is based on the use of multipliers that allow to exhibit explicit relations between the energy of the elastic and thermal components of the system.

2. Sketch of the Proof. – We focus on the case $h>0$. The case $h=0$ can be treated in a similar way.

By density it is sufficient to consider smooth solutions of (1). Thus, in the sequel, we will deal with smooth solutions and all the integrations by parts we will perform will be justified.

First of all we need the following result of "hidden regularity".

Lemma 1. – *Assume that $h>0$. Then there exists a positive constant $C>0$ such that*

$$\int_0^t \int_{\partial\Omega} |\Delta u|^2 d\sigma\, ds \le C\left[E(0) + \int_0^t \left[E(s) + E^{\frac{3}{2}}(s)\right] ds\right], \quad \forall t>0 \tag{7}$$

for any solution of (1).

Proof of Lemma 1. – Let $q=q(x)\in (C^1(\overline{\Omega}))^2$ be such that $q=\nu$ on $\partial\Omega$. Multiplying in (1) by $q\cdot\nabla u$ and integrating by parts it follows that (we use the convention of summation of repeated indexes)

$$\begin{aligned}
\frac{1}{2}\int_0^t \int_{\partial\Omega} |\Delta u|^2 d\sigma\, ds &= \frac{1}{2}\int_0^t \int_{\partial\Omega} (q\cdot\nu)|\Delta u|^2 d\sigma\, ds = \\
&= \int_\Omega \left[q\cdot\nabla u u_t + h\nabla u_t\cdot\nabla(q\cdot\nabla u)\right] dx \Big|_0^t + \\
&+ \int_0^t \int_{\partial\Omega} \left[\frac{\text{div }(q)}{2}\left(|u_t|^2 + h|\nabla u_t|^2 - |\Delta u|^2\right) - h\partial_j u_t \partial_j q_i \partial_i u_t + \Delta u \Delta q\cdot\nabla u + \right. \\
&\left. + 2\Delta u \partial_j q_i \partial_j \partial_i u - \nabla\theta\cdot\nabla(q\cdot\nabla u) - [u,v]\, q\cdot\nabla u\right] dxdt.
\end{aligned} \tag{8}$$

All the terms on the right hand side of (8) make sense in the energy space. In particular, concerning the last two terms, one has to take into account that, in view of (4), $\nabla\theta\in L^2(\Omega\times(0,t))$ and, on the other hand, that $[u,v]\in L^\infty(0,t;\ H^{-1}(\Omega))$.

From (8) the estimate (7) follows easily using that (see [3]).

$$||[u,v]||_{H^{-1}(\Omega)} \le C||\Delta u||_{L^2(\Omega)}||\Delta v||_{L^2(\Omega)}. \qquad \square$$

Let us introduce now the functional:

$$H(t) = E(t) + \varepsilon\int_\Omega \left(hu_t\theta - \frac{h}{2}\,\theta^2 + u_t(-\Delta)^{-1}\theta\right) dx + \frac{\varepsilon}{2}\int_\Omega \left[uu_t + h\nabla u\cdot\nabla u_t\right] dx \tag{9}$$

with $\varepsilon>0$ small enough that will be fixed later on. In (9), $(-\Delta)^{-1}$ denotes the inverse of the Dirichlet laplacian. It is clear that for $0<\varepsilon<\varepsilon_0$ small enough independent of the initial data it follows that

$$\frac{1}{2}\,H(t) \le E(t) \le 2H(t), \qquad \forall t\ge 0. \tag{10}$$

We also have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left[hu_t\theta - \frac{h}{2}\,\theta^2 + u_t(-\Delta)^{-1}\theta\right] dx &= \\
= \int_\Omega \left[h|\nabla\theta|^2 - h|\nabla u_t|^2 - \nabla u\cdot\nabla\theta + \theta^2 + [u,v](-\Delta)^{-1}\theta - \theta u_t - u_t^2\right] dx + \\
+ \int_{\partial\Omega} \Delta u\,\frac{\partial}{\partial\nu}\,(-\Delta)^{-1}\theta\, d\sigma.
\end{aligned} \tag{11}$$

This identity follows combining the identities one gets when multiplying the heat equation in (1) by u_t, θ and $(-\Delta)^{-1}u_t$ and the plate equation satisfied by u by $(-\Delta)^{-1}\theta$.

On the other hand

$$(12)\qquad \frac{d}{dt}\int_\Omega [uu_t + h\nabla u\cdot\nabla u_t]\,dx = \int_\Omega \left[u_t^2 + h|\nabla u_t|^2 - |\Delta u|^2 - |\Delta v|^2 + \nabla\theta\cdot\nabla u\right]\,dx.$$

To prove (12) it is sufficient to multiply by u the equation satisfied by u and to take into account that $\int_\Omega [u,v]\,u\,dx = -\int_\Omega |\Delta v|^2 dx$.

Combining (4), (11) and (12) it follows that

$$\begin{aligned}(13)\frac{d}{dt}H(t) = &-\int_\Omega \left[(1-\varepsilon h)\,|\nabla\theta|^2 - \varepsilon\theta^2 + \frac{\varepsilon}{2}\,\left(u_t^2 + h|\nabla u_t|^2\right) + \frac{\varepsilon}{2}\,|\Delta u|^2 + \frac{\varepsilon}{2}\,|\Delta v|^2\right]\,dx \\ &+ \int_\Omega \left[\frac{\varepsilon}{2}\,\Delta u\theta - \varepsilon\theta u_t\right]\,dx + \varepsilon\int_\Omega [u,v](-\Delta)^{-1}\theta dx + \varepsilon\int_{\partial\Omega}\Delta u\,\frac{\partial}{\partial\nu}\,(-\Delta)^{-1}\theta\,d\sigma \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t).\end{aligned}$$

We have

$$\begin{aligned}(14)|I_3(t)| &\le C\varepsilon\,||\Delta u||_{L^2(\Omega)}||\Delta v||_{L^2(\Omega)}||(-\Delta)^{-1}\theta||_{L^\infty(\Omega)} \\ &\le \frac{\varepsilon}{4}\,\left[||\Delta u||^2_{L^2(\Omega)} + ||\Delta v||^2_{L^2(\Omega)}\right] + C\varepsilon\left[||\Delta u||^2_{L^2(\Omega)} + ||\Delta v||^2_{L^2(\Omega)}\right]\,||\nabla\theta||^2_{L^2(\Omega)} \\ &\le \frac{\varepsilon}{4}\,\left[||\Delta u||^2_{L^2(\Omega)} + ||\Delta v||^2_{L^2(\Omega)}\right] + C\varepsilon\,H(0)\,\int_\Omega |\nabla\theta|^2 dx\end{aligned}$$

since $||(-\Delta)^{-1}\theta||_{L^\infty(\Omega)} \le \frac{1}{8} + C||\nabla\theta||^2_{L^2(\Omega)}$.

Moreover

$$\begin{aligned}(15)\qquad |I_4(t)| &\le C\varepsilon\,||\frac{\partial}{\partial\nu}\,(-\Delta)^{-1}\theta||_{L^2(\partial\Omega)}||\Delta u||_{L^2(\partial\Omega)} \le C\varepsilon||\nabla\theta||_{L^2(\Omega)}||\Delta u||_{L^2(\partial\Omega)} \\ &\le \frac{1}{2}\int_\Omega |\nabla\theta|^2 dx + C\varepsilon^2\int_{\partial\Omega}|\Delta u|^2 d\sigma.\end{aligned}$$

Combining (13)–(15) it follows that there exist $C_1 > 0$ and $\gamma > 0$ such that

$$(16)\qquad \frac{dH}{dt}\,(t) \le -C_1\varepsilon H(t) + C\varepsilon^2\int_{\partial\Omega}|\Delta u|^2 d\sigma, \qquad \forall t \ge 0$$

for any $\varepsilon > 0$ such that

$$(17)\qquad \varepsilon(1 + H(0)) \le \gamma.$$

From (16) it follows that

$$\begin{aligned}(18)\qquad H(t) &\le e^{-C_1\varepsilon t}H(0) + C\varepsilon^2 e^{-C_1\varepsilon t}\int_0^t e^{C_1\varepsilon s}\int_{\partial\Omega}|\Delta u(s)|^2 d\sigma\,ds \\ &\le e^{-C_1\varepsilon t}H(0) + C\varepsilon^2\int_0^t\int_{\partial\Omega}|\Delta u(s)|^2 d\sigma\,ds.\end{aligned}$$

Combining (18) and (7) we obtain

$$(19)\qquad H(t) \le e^{-C_1\varepsilon t}H(0) + C\varepsilon^2\left(H(0) + t(H(0) + H^{\frac{3}{2}}(0))\right).$$

In terms of the original energy, E, (19) yields

$$E(t) \leq \left\{ 4e^{-C_1 \varepsilon t} + C\varepsilon^2 \left(1 + t(1 + E^{\frac{1}{2}}(0))\right) \right\} E(0). \tag{20}$$

We claim that $\varepsilon > 0$ and $t > 0$ in (20) can be chosen such that

$$4e^{-C_1 \varepsilon t} + C\varepsilon^2 \left(1 + t(1 + E^{\frac{1}{2}}(0))\right) \leq \rho < 1. \tag{21}$$

Indeed (21) holds provided

$$C\varepsilon^2 \leq \frac{\rho}{3}; \quad 4e^{-C_1 \varepsilon t} \leq \frac{\rho}{3}, \quad C\varepsilon^2 t(1 + E^{\frac{1}{2}}(0)) \leq \frac{\rho}{3}. \tag{22}$$

The first condition in (22) is fulfilled if $\varepsilon > 0$ is small enough independently of the initial data. The second one requires t to be sufficiently large of the order of $\frac{1}{\varepsilon}$ as $\varepsilon \to 0$. The last one requires $\varepsilon(1 + E^{\frac{1}{2}}(0))$ to be small which is a weaker restriction than (17).

Thus, given a solution of (1) with initial energy $E(0) = R$ we choose $\varepsilon \sim \frac{\gamma}{1+R^2}$ such that (17) holds and then for a suitable τ independent of R we get

$$E\left(\frac{1 + R^2}{\tau}\right) \leq \rho E(0) \tag{23}$$

with $0 < \rho < 1$ independent of R.

The exponential decay rate (6) follows from (23) and the semigroup property.

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