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COMPLEXITY AND REGULARITY OF MAXIMAL ENERGY DOMAINS FOR THE WAVE EQUATION WITH FIXED INITIAL DATA

Yannick Privat

CNRS, Sorbonne Universités, UPMC Univ Paris 06, UMR 7598
Laboratoire Jacques-Louis Lions, F-75005, Paris, France

Emmanuel Trélat

Sorbonne Universités, UPMC Univ Paris 06, CNRS UMR 7598
Laboratoire Jacques-Louis Lions and Institut Universitaire de France, F-75005, Paris, France

Enrique Zuazua

BCAM - Basque Center for Applied Mathematics, Mazarredo, 14 E-48009 Bilbao-Basque Country-Spain
Ikerbasque, Basque Foundation for Science, Alameda Urquijo 36-5, Plaza Bizkaia, 48011, Bilbao-Basque Country-Spain

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Abstract. We consider the homogeneous wave equation on a bounded open connected subset Ω of $\mathbb{R}^n$. Some initial data being specified, we consider the problem of determining a measurable subset ω of Ω maximizing the L^2-norm of the restriction of the corresponding solution to ω over a time interval $[0,T]$, over all possible subsets of Ω having a certain prescribed measure. We prove that this problem always has at least one solution and that, if the initial data satisfy some analyticity assumptions, then the optimal set is unique and moreover has a finite number of connected components. In contrast, we construct smooth but not analytic initial conditions for which the optimal set is of Cantor type and in particular has an infinite number of connected components.

1. Introduction.

Let $n\geq 1$ be an integer. Let T be a positive real number and Ω be an open bounded connected subset of $\mathbb{R}^n$. We consider the homogeneous wave equation with Dirichlet boundary conditions

$$\begin{array}{ll} \partial_{tt}y-\triangle y=0 & \text{in } (0,+\infty)\times\Omega, \\ y=0 & \text{on } (0,+\infty)\times\partial\Omega. \end{array} \tag{1}$$

For all initial data $(y^0,y^1)\in H^1_0(\Omega)\times L^2(\Omega)$, there exists a unique solution y of (1) in the space $\mathcal{C}^0([0,T];H^1_0(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$, such that $y(0,x)=y^0(x)$ and $\partial_t y(0,x)=y^1(x)$ for almost every $x\in\Omega$.

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Note that the energy of the solution y over the whole domain Ω, defined by

$$E(t) = \frac{1}{2}\int_\Omega \left(\|\nabla_x y(t,x)\|^2 + |y_t(t,x)|^2\right)\, dx,$$

is a constant function of t. This conservation property is no longer true when considering the integral over a proper subset of Ω. But, from a control theoretical point of view and, in particular, motivated by the problem of optimal observation or optimal placement of observers, it is interesting to consider such energies over a certain horizon of time. This motivates the consideration of the functional

$$G_{\gamma,T}(\chi_\omega) = \int_0^T\int_\omega \left(\gamma_1\|\nabla_x y(t,x)\|^2 + \gamma_2|y(t,x)|^2 + \gamma_3|\partial_t y(t,x)|^2\right)\, dx\, dt, \qquad (2)$$

where $\gamma = (\gamma_1,\gamma_2,\gamma_3) \in [0,+\infty)^3 \setminus \{(0,0,0)\}$ is fixed and ω is any arbitrary measurable subset of Ω of positive measure. Here and throughout the article, the notation χ_ω stands for the characteristic function of ω.

For every subset ω, the quantity $G_{\gamma,T}(\chi_\omega)$ provides an account for the amount of energy of the solution y restricted to ω over the horizon of time $[0,T]$.

In this paper we address the problem of determining the optimal shape and location of the subdomain ω of a given measure, maximizing $G_{\gamma,T}(\chi_\omega)$.

> **Optimal design problem** $(\mathcal{P}_{\gamma,T})$. *Let $L \in (0,1)$, let $\gamma \in [0,+\infty)^3 \setminus \{(0,0,0)\}$, and let $T > 0$ be fixed. Given $(y^0,y^1) \in H_0^1(\Omega)\times L^2(\Omega)$ arbitrary, we investigate the problem of maximizing the functional $G_{\gamma,T}$ defined by (2) over all possible measurable subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$, where $y \in \mathcal{C}^0([0,T];H_0^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$ is the unique solution of (1) such that $y(0,\cdot) = y^0(\cdot)$ and $\partial_t y(0,\cdot) = y^1(\cdot)$.*

We stress that the search of an optimal set is done over all possible measurable subsets of Ω having a certain prescribed Lebesgue measure. This class of domains is very large and it results in a non-trivial infinite-dimensional shape optimization problem, in which we optimize not only the placement of ω but also its shape. If we were to restrict our search to domains having a prescribed shape, for instance, a certain number of balls with a fixed radius where the unknowns are the centers of the balls, then the problem would turn into a more classical finite-dimensional one. Here, we do not make any restriction on the shape of the unknown domain ω.

Note that, in the above problem, the initial conditions (y^0,y^1) are arbitrary, but fixed. Therefore the optimal set ω, whenever it exists, depends on the initial data under consideration. This problem is a mathematical benchmark, in view of addressing other more intricate optimal design problems where one could as well search for the optimal set ω for a certain class of initial data. The problem we address here, where the initial data are fixed and therefore we are considering a single solution, is simpler but, as we shall see, it reveals interesting properties.

In this paper we provide a complete mathematical analysis of the optimal design problem $(\mathcal{P}_{\gamma,T})$. The article is structured as follows. In Section 2 (see in particular Theorem 2.2) we give a sufficient condition ensuring the existence and uniqueness of an optimal set. More precisely, we prove that, if the initial data under consideration belong to a suitable class of analytic functions, then there always exists a unique optimal domain ω, which has a finite number of connected components for any value of $T > 0$ but some isolated values. In Section 3, we investigate the sharpness of the assumptions made in Theorem 2.2. More precisely, in Theorem 3.1 we build initial data (y^0,y^1) of class $\mathcal{C}^\infty$ such that the problem $(\mathcal{P}_{\gamma,T})$ has a unique solution

ω, which is a fractal set and thus has an infinite number of connected components. In Section 4, we present some possible generalizations of the results in this article with further potential applications.

2. **Existence and uniqueness results.**

2.1. **Existence.**

Throughout the section, we fix arbitrary initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$. For almost every $x \in \Omega$, we define

$$\varphi_{\gamma,T}(x) = \int_0^T \left(\gamma_1 \|\nabla_x y(t,x)\|^2 + \gamma_2 |y(t,x)|^2 + \gamma_3 |\partial_t y(t,x)|^2\right) \, dt \tag{3}$$

where $y \in \mathcal{C}^0([0,T]; H_0^1(\Omega)) \cap \mathcal{C}^1([0,T]; L^2(\Omega))$ is the unique solution of (1) such that $y(0,\cdot) = y^0(\cdot)$ and $\partial_t y(0,\cdot) = y^1(\cdot)$. Note that the function $\varphi_{\gamma,T}$ is integrable on Ω, and that

$$G_{\gamma,T}(\chi_\omega) = \int_\omega \varphi_{\gamma,T}(x) \, dx, \tag{4}$$

for every measurable subset ω of Ω.

Theorem 2.1. *For any fixed initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ the optimal design problem $(\mathcal{P}_{\gamma,T})$ has at least one solution. Moreover, there exists a real number λ such that ω^* is a solution of $(\mathcal{P}_{\gamma,T})$ if and only if*

$$\chi_{\{\varphi_{\gamma,T}(x) > \lambda\}} \leq \chi_{\omega^*}(x) \leq \chi_{\{\varphi_{\gamma,T} \geq \lambda\}}(x), \tag{5}$$

for almost every $x \in \Omega$.

In other words, any optimal set, solution of $(\mathcal{P}_{\gamma,T})$, is characterized in terms of a level set of the function $\varphi_{\gamma,T}$. Note that (5) means that $\{\varphi_{\gamma,T} > \lambda\} \subset \omega^* \subset \{\varphi_{\gamma,T} \geq \lambda\}$, this inclusion being understood within the class of all measurable subsets of Ω quotiented by the set of all measurable subsets of Ω of zero Lebesgue measure.

Note also that the real number λ is independent on the solution of $(\mathcal{P}_{\gamma,T})$.

Proof. The proof uses a standard argument of decreasing rearrangement (see, e.g., [20, Chapter 1] or [7, 9]). We consider the decreasing rearrangement of the measurable nonnegative function $\varphi_{\gamma,T}$, which is the nonincreasing function $\varphi^*_{\gamma,T} : (0, |\Omega|) \to \mathbb{R}$ defined by

$$\varphi^*_{\gamma,T}(s) = \inf \left\{ t \in \mathbb{R} \mid |\{\varphi_{\gamma,T} > t\}| \leq s \right\}.$$

Recall that, according to the (first) Hardy-Littlewood inequality, we have

$$G_{\gamma,T}(\chi_\omega) = \int_\omega \varphi_{\gamma,T}(x) \, dx \leq \int_0^{L|\Omega|} \varphi^*_{\gamma,T}(s) \, ds,$$

for every measurable subset ω of Ω such that $|\omega| = L|\Omega|$. We are going to prove that the real number $\int_0^{L|\Omega|} \varphi^*_{\gamma,T}(s) \, ds$ is actually the maximal value of $G_{\gamma,T}$ over the set of all measurable subsets ω of Ω such that $|\omega| = L|\Omega|$, and thus is the maximal value of $(\mathcal{P}_{\gamma,T})$. We distinguish between two cases.

If $|\{\varphi_{\gamma,T} = \varphi^*_{\gamma,T}(L|\Omega|)\}| = 0$, then the (measurable) set $\omega^* = \{\varphi_{\gamma,T} > \varphi^*_{\gamma,T}(L|\Omega|)\}$ satisfies $|\omega^*| = L|\Omega|$ and

$$\int_{\omega^*} \varphi_{\gamma,T}(x) \, dx = \int_0^{L|\Omega|} \varphi^*_{\gamma,T}(s) \, ds.$$

If $|\{\varphi_{\gamma,T} = \varphi^*_{\gamma,T}(L|\Omega|)\}| > 0$, then clearly there exists a measurable subset ω^* of Ω such that $|\omega^*| = L|\Omega|$ and $\{\varphi_{\gamma,T} > \varphi^*_{\gamma,T}(L|\Omega|)\} \subset \omega^* \subset \{\varphi_{\gamma,T} \geq \varphi^*_{\gamma,T}(L|\Omega|)\}$ in the sense recalled above. Moreover, we have

$$\begin{aligned}\int_{\omega^*} \varphi_{\gamma,T}(x)\,dx &= \int_{\omega^*\backslash\{\varphi_{\gamma,T}>\varphi^*_{\gamma,T}(L|\Omega|)\}} \varphi_{\gamma,T}(x)\,dx + \int_{\{\varphi_{\gamma,T}>\varphi^*_{\gamma,T}(L|\Omega|)\}} \varphi_{\gamma,T}(x)\,dx \\ &= \varphi^*_{\gamma,T}(L|\Omega|)\,|\omega^*\backslash\{\varphi_{\gamma,T} > \varphi^*_{\gamma,T}(L|\Omega|)\}| + \int_{\{\varphi_{\gamma,T}>\varphi^*_{\gamma,T}(L|\Omega|)\}} \varphi^*_{\gamma,T}(s)\,ds \\ &= \int_0^{L|\Omega|} \varphi^*_{\gamma,T}(s)\,ds.\end{aligned}$$

At this step, we have proved that $(\mathcal{P}_{\gamma,T})$ has at least one solution, which satisfies (5) with $\lambda = \varphi^*_{\gamma,T}(L|\Omega|)$.

Moreover, the computation above also shows that every set ω^* satisfying (5) with $\lambda = \varphi^*_{\gamma,T}(L|\Omega|)$ is a solution of $(\mathcal{P}_{\gamma,T})$.

It remains to prove that, if ω is a solution of $(\mathcal{P}_{\gamma,T})$, then it must satisfy (5), with $\lambda = \varphi^*_{\gamma,T}(L|\Omega|)$ (not depending on ω). Assume by contradiction that ω is a solution of $(\mathcal{P}_{\gamma,T})$ such that (5) is not satisfied. Let $f = {\varphi_{\gamma,T}}_{|\omega}$ be the restriction of the function $\varphi_{\gamma,T}$ to the set ω. We clearly have $|\{f > t\}| \leq |\{\varphi_{\gamma,T} > t\}|$ for almost every $t \in \mathbb{R}$, and since (5) is not satisfied, there exist $\varepsilon > 0$ and $I \subset (0, |\Omega|)$ of positive Lebesgue measure such that $|\{f > t\}| + \varepsilon \leq |\{\varphi_{\gamma,T} > t\}|$ for almost every t in I. We infer that

$$\int_\omega \varphi_{\gamma,T}(x)\,dx = \int_0^{L|\Omega|} f^*(x)\,dx < \int_0^{L|\Omega|} \varphi^*_{\gamma,T}(x)\,dx,$$

where $f^* : (0, |\Omega|) \to \mathbb{R}$ is the decreasing rearrangement of f, defined by

$$f^*(s) = \inf\{t \in \mathbb{R} \mid |\{f > t\}| \leq s\},$$

and hence ω is not a solution of the problem $(\mathcal{P}_{\gamma,T})$. The theorem is proved. $\square$

Figure 1 below illustrates the construction of the optimal set ω^*, from the knowledge of $\varphi_{\gamma,T}$ and its decreasing rearrangement $\varphi^*_{\gamma,T}$.

Remark 1 (Relaxation)**.** In Calculus of Variations, it is usual to consider a relaxed formulation of the problem $(\mathcal{P}_{\gamma,T})$. Defining the set of unknowns

$$\mathcal{U}_L = \{\chi_\omega \in L^\infty(\Omega, \{0, 1\}) \mid |\omega| = L|\Omega|\},$$

the relaxation procedure consists in considering the convex closure of $\mathcal{U}_L$ for the L^∞ weak star topology, that is

$$\overline{\mathcal{U}}_L = \{a \in L^\infty(\Omega, [0, 1]) \mid \int_\Omega a(x)\,dx = L|\Omega|\},$$

and then in extending the functional $G_{\gamma,T}$ to $\overline{\mathcal{U}}_L$ by setting

$$G_{\gamma,T}(a) = \int_\Omega a(x)\varphi_{\gamma,T}(x)\,dx,$$

for every $a \in \overline{\mathcal{U}}_L$. The relaxed version of $(\mathcal{P}_{\gamma,T})$ is then defined as the problem of maximizing the relaxed functional $G_{\gamma,T}$ over the set $\overline{\mathcal{U}}_L$.

Since $a \mapsto G_{\gamma,T}(a)$ is clearly continuous for the weak star topology of L^∞, we claim that

$$\max_{\chi_\omega\in\mathcal{U}_L} G_{\gamma,T}(\chi_\omega) = \max_{a\in\overline{\mathcal{U}}_L} G_{\gamma,T}(a).$$

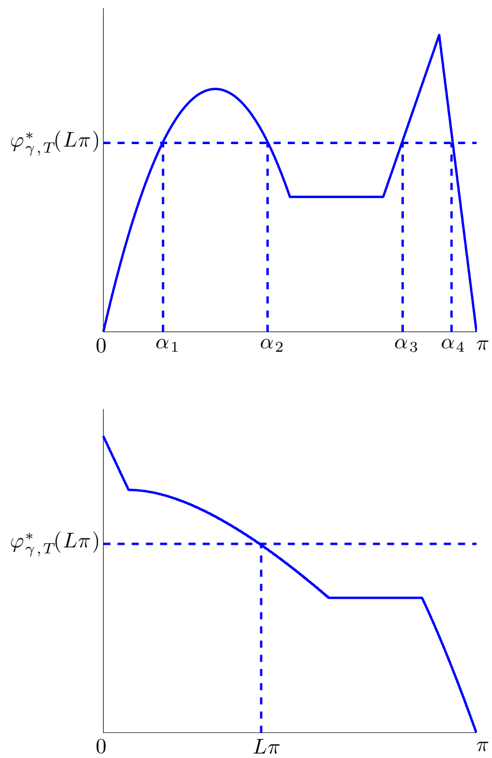

FIGURE 1. $\Omega = (0,\pi)$; Graph of a function $\varphi_{\gamma,T}$ (left) and graph of its decreasing rearrangement $\varphi^*_{\gamma,T}$ (right). With the notations of the figure, $\omega^* = (\alpha_1,\alpha_2)\cup(\alpha_3,\alpha_4)$.

It is easy to characterize all solutions of the relaxed problem. Indeed, adapting the proof of Theorem 2.1, we get that a is solution of the relaxed problem if and only if $a = 0$ on the set $\{\varphi_{\gamma,T} < \lambda\}$, $a = 1$ on the set $\{\varphi_{\gamma,T} > \lambda\}$, and $a(x) \in [0,1]$ for almost every $x \in \{\varphi_{\gamma,T} = \lambda\}$ and $\int_\Omega a(x)\,dx = L|\Omega|$.

2.2. **Uniqueness.** Let us now discuss the issue of the uniqueness of the optimal set.

Note that the characterization of the solutions of $(\mathcal{P}_{\gamma,T})$ in Theorem 2.1 enables situations where infinitely many different subsets maximize the functional $G_{\gamma,T}$ over the class of Lebesgue measurable subsets of Ω of measure $L|\Omega|$. This occurs if and only if the set $\{\varphi_{\gamma,T} = \lambda\}$, where λ is the positive real number introduced in Theorem 2.1, has a positive Lebesgue measure. In the next theorem, we provide sufficient regularity conditions on the initial data (y^0, y^1) of the wave equation (1) to guarantee the uniqueness of the solution of $(\mathcal{P}_{\gamma,T})$.

We denote by $A = -\triangle$ the Dirichlet-Laplacian, unbounded linear operator in $L^2(\Omega)$ with domain $D(A) = \{u \in H^1_0(\Omega) \mid \triangle u \in L^2(\Omega)\}$. Note that $D(A) = H^2(\Omega)\cap H^1_0(\Omega)$ whenever the boundary of Ω is C^2. Note however that our results hereafter, we do not require any regularity assumption on the boundary of Ω.

Theorem 2.2. *Let $\gamma \in [0,+\infty)^3 \setminus \{(0,0,0)\}$. We assume that:*

$(\mathbf{A_1})$ *the function* ψ *defined by*

$$\psi(x) = \gamma_1 \|\nabla_x y^0(x)\|^2 + \gamma_2 |y^0(x)|^2 + \gamma_3 |y^1(x)|^2,$$

for every $x \in \overline{\Omega}$, *is nonconstant;*

$(\mathbf{A_2})$ *there exists* $R > 0$ *such that*

$$\sum_{j=1}^{+\infty} \frac{R^j}{j!} \left(\|A^{j/2} y^0\|^2_{L^2(\Omega)} + \|A^{(j-1)/2} y^1\|^2_{L^2(\Omega)} \right)^{1/2} < +\infty.$$

Then, for every value of $T > 0$ *but some isolated values, the problem* $(\mathcal{P}_{\gamma,T})$ *has a unique*[1] *solution* ω *satisfying moreover the following properties:*

- ω *is a semi-analytic*[2] *subset of the open set* Ω *and, thus, in particular, for every compact subset* $K \subset \Omega$, *the set* $\omega \cap K$ *has a finite number of connected components;*
- *If* Ω *is symmetric with respect to an hyperplane,* σ *being the symmetry operator, and if* $y^0 \circ \sigma = y^0$ *and* $y^1 \circ \sigma = y^1$, *then* ω *enjoys the same symmetry property;*
- *If* $\gamma_1 = 0$ *and* $\gamma_2 + \gamma_3 > 0$, *then there exists* $\eta > 0$ *such that* $d(\omega, \partial\Omega) > \eta$, *where* d *denotes the Euclidean distance in* $\mathbb{R}^n$. *In that case, in particular,* ω *has a finite number of connected components.*

Proof of Theorem 2.2. Let us first prove that, under the assumption $(\mathbf{A_2})$, the corresponding solution y of the wave equation is analytic in the open set $(0, +\infty) \times \Omega$. We first note that the quantity

$$\|A^{j/2} y(t, \cdot)\|^2_{L^2(\Omega)} + \|A^{(j-1)/2} \partial_t y(t, \cdot)\|^2_{L^2(\Omega)}$$

is constant with respect to t (it is a higher-order energy over the whole domain Ω). Therefore, using $(\mathbf{A_2})$, there exists $C_1 > 0$ such that

$$\sum_{j=1}^{+\infty} \frac{R^j}{j!} \left(\|A^{j/2} y(t, \cdot)\|^2_{L^2(\Omega)} + \|A^{(j-1)/2} \partial_t y(t, \cdot)\|^2_{L^2(\Omega)} \right)^{1/2} = C_1 < +\infty, \quad (6)$$

for every time t.

Let U be any open subset of Ω. Using (6) and the well-known Sobolev imbedding theorem (see, e.g., [1, Theorem 4.12]), it is clear that $y(t, \cdot)$ and $\partial_t y(t, \cdot)$ are of class C^∞ on U, for every time t. Moreover, since $\|A^{j/2} u\|_{L^2(\Omega)} = \|\triangle^{j/2} u\|_{L^2(\Omega)}$ if j is even, and $\|A^{j/2} u\|_{L^2(\Omega)} = \|\nabla \triangle^{(j-1)/2} u\|_{L^2(\Omega)}$ if j is odd, we get in particular that

$$\sum_{j=1}^{+\infty} \frac{R^{2j}}{(2j)!} \|\triangle^j y(t, \cdot)\|_{L^2(U)} \leq C_1 \quad \text{and} \quad \sum_{j=0}^{+\infty} \frac{R^{2j+1}}{(2j+1)!} \|\triangle^j \partial_t y(t, \cdot)\|_{L^2(U)} \leq C_1. \quad (7)$$

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[1] Similarly to the definition of elements of L^p-spaces, the subset ω is unique within the class of all measurable subsets of Ω quotiented by the set of all measurable subsets of Ω of zero Lebesgue measure.

[2] A subset ω of an open subset $\Omega \subset \mathbb{R}^n$ is said to be semi-analytic if it can be written in terms of equalities and inequalities of analytic functions, that is, for every $x \in \omega$, there exists a neighborhood U of x in Ω and $2pq$ analytic functions g_{ij}, h_{ij} (with $1 \leq i \leq p$ and $1 \leq j \leq q$) such that

$$\omega \cap U = \bigcup_{i=1}^{p} \{ y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0, \ j = 1, \ldots, q \}.$$

We recall that such semi-analytic (and more generally, subanalytic) subsets enjoy nice properties, for instance they are stratifiable in the sense of Whitney (see [4, 8]).

Now, it follows from [15, Theorem 7] that $y(t,\cdot)$ and $\partial_t y(t,\cdot)$ are analytic functions in U, for every time t. Moreover, following the proof of [15, Theorem 7], we get that there exist $C_2>0$ and $\delta>0$ such that

$$\max_{|\alpha|\leq k}\|\partial^\alpha y(t,\cdot)\|_{C^0(U)}+\max_{|\alpha|\leq k}\|\partial^\alpha\partial_t y(t,\cdot)\|_{C^0(U)}\leq\ C_2\frac{k!}{\delta^k},$$

for every $t\geq 0$ and every integer k, and we stress on the fact that C_2 does not depend on t. To derive the analytic regularity both in time and space, we write that $\partial_t^{2k}y=\triangle^k y$ and $\partial_t^{2k+1}y=\triangle^k\partial_t y$ for every $k\in\mathbb{N}$, and therefore[3]

$$\begin{aligned}
&\max_{|\alpha|\leq k_2}\|\partial_t^{2k_1}\partial^\alpha y(t,\cdot)\|_{C^0((0,+\infty)\times U)}+\max_{|\alpha|\leq k_2}\|\partial_t^{2k_1+1}\partial^\alpha y(t,\cdot)\|_{C^0((0,+\infty)\times U)}\\
\leq\ &\max_{|\alpha|\leq k_2}\|\triangle^{k_1}\partial^\alpha y(t,\cdot)\|_{C^0((0,+\infty)\times U)}+\max_{|\alpha|\leq k_2}\|\triangle^{k_1}\partial^\alpha\partial_t y(t,\cdot)\|_{C^0((0,+\infty)\times U)}\\
\leq\ & n^{k_1}\left(\max_{|\alpha|\leq k_2+2k_1}\|\partial^\alpha y(t,\cdot)\|_{C^0(U)}+\max_{|\alpha|\leq k_2+2k_1}\|\partial^\alpha\partial_t y(t,\cdot)\|_{C^0(U)}\right)\\
\leq\ & C_2\frac{n^{k_1}}{\delta^{2k_1+k_2}}(2k_1+k_2)!
\end{aligned}$$

The analyticity of y now follows from standard theorems.

Let us now prove that the function $\varphi_{\gamma,T}$ defined by (3) is analytic in Ω. We have $\varphi_{\gamma,T}(x)=\int_0^T f(t,x)\,dt$, with

$$f(t,x)=\gamma_1\|\nabla_x y(t,x)\|^2+\gamma_2|y(t,x)|^2+\gamma_3|\partial_t y(t,x)|^2.$$

It follows from the above estimates that, for every $t\in[0,T]$, the function $f(t,\cdot)$ is analytic in Ω, and moreover is bounded by some constant on the bounded set Ω. Therefore $\varphi_{\gamma,T}$ is analytic in Ω.

As a consequence, given a value of $T>0$, either $\varphi_{\gamma,T}$ is constant on the whole Ω, or $\varphi_{\gamma,T}$ cannot be constant on any subset of Ω of positive Lebesgue measure. In the latter case this implies that the set $\{\varphi_{\gamma,T}=\lambda\}$, where λ is defined as in (5), has a zero Lebesgue measure for such values of T, and therefore the optimal set is unique (according to the characterization of the optimal set following from Theorem 2.1).

Let us then prove that the function $\varphi_{\gamma,T}$ is nonconstant for every value T but some isolated values. According to Assumption $(\mathbf{A_1})$, there exists $(x_1,x_2)\in\Omega^2$ such that $\psi(x_1)\neq\psi(x_2)$. Since f is in particular continuous with respect to its both variables, we have

$$\lim_{T\searrow 0}\frac{\varphi_{\gamma,T}(x_2)-\varphi_{\gamma,T}(x_1)}{T}=\psi(x_2)-\psi(x_1)\neq 0.$$

The function $T\mapsto\varphi_{\gamma,T}(x_2)-\varphi_{\gamma,T}(x_1)$ is clearly analytic (thanks to the estimates above) and thus can vanish only at some isolated values of T. Hence $\varphi_{\gamma,T}$ cannot be constant on a subset of positive measure for such values of T, whence the claim above. This ensures the uniqueness of the optimal set ω.

The first additional property follows from the analyticity properties. The symmetry property follows from the fact that $\varphi_{\gamma,T}\circ\sigma(x)=\varphi_{\gamma,T}(x)$ for every $x\in\Omega$. If ω were not symmetric with respect to this hyperplane, the uniqueness of the solution of the first problem would fail, which is a contradiction.

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[3]Here, $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ is a multi-index, $|\alpha|=\alpha_1+\dots+\alpha_n$, and $\partial^\alpha u=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}u$ with the coordinates $x=(x_1,\dots,x_n)$.

Finally, if $\gamma_1 = 0$ and $\gamma_2 + \gamma_3 > 0$ then $\varphi_{\gamma,T}(x) = 0$ for every $x \in \partial\Omega$, and hence $\varphi_{\gamma,T}$ reaches its global minimum on the boundary of Ω. The conclusion follows according to the characterization of ω made in Theorem 2.1. □

Let us now comment on the assumptions done in the theorem. Several remarks are in order.

Remark 2. In the proof above, the boundary of Ω is not assumed to be regular to ensure the analyticity of the solution under the assumption $(\mathbf{A_2})$. This might seem surprising since one could expect the lack of regularity (analyticity) of the boundary to possibly interact microlocally with the interior analyticity, but this is not the case. A complete analysis of the interaction of microlocal boundary and interior regularity is not done, as far as we know, but it is not required to address the problem under consideration.

We recall however that, if the boundary of Ω is C^∞, then the Dirichlet spaces $D(A^{j/2})$, $j \in \mathbb{N}$, are the Sobolev spaces with Navier boundary conditions defined by

$$D(A^{j/2}) = \{u \in H^j(\Omega) \mid u_{|\partial\Omega} = \triangle u_{|\partial\Omega} = \cdots = \triangle^{\left[\frac{j-1}{2}\right]} u_{|\partial\Omega} = 0\}.$$

Analyticity of solutions up to the boundary would require the boundary of Ω be analytic. Under this condition ω would be a semi-analytic subset of $\bar{\Omega}$ and, thus, in particular, constituted by a finite number of connected components.

Remark 3. In the case where $\gamma_1 = 0$ and $\gamma_2 + \gamma_3 > 0$, the conclusion of Theorem 2.2 actually holds true for every $T > 0$. Indeed, it suffices to note that, if the initial data satisfy $(\mathbf{A_2})$, then the function $\varphi_{\gamma,T}$ is analytic and vanishes on $\partial\Omega$. Hence it cannot be constant, whence the result.

Remark 4. The optimal shape depends on the initial data under consideration. But there are infinitely many initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ leading to the same function $\varphi_{\gamma,T}$ and thus to the same solution(s) of $(\mathcal{P}_{\gamma,T})$. Indeed, consider for the sake of simplicity, the one-dimensional case $\Omega = (0, \pi)$ with $T = 2k\pi$, $k \in \mathbb{N}^*$. Expanding the initial data y^0 and y^1 as

$$y^0(x) = \sum_{j=1}^{+\infty} a_j \sin(jx) \quad \text{and} \quad y^1(x) = \sum_{j=1}^{+\infty} j b_j \sin(jx),$$

it follows that

$$G_{\gamma,T}(\chi_\omega) = k\pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \int_0^\pi \chi_\omega(x) \left(\gamma_1 j^2 \cos(jx)^2 + (\gamma_2 + j^2\gamma_3) \sin^2(jx)\right) dx, \tag{8}$$

and

$$\varphi_{\gamma,T}(x) = k\pi \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) \left(\gamma_1 j^2 \cos(jx)^2 + (\gamma_2 + j^2\gamma_3) \sin^2(jx)\right). \tag{9}$$

This justifies the claim above. Similar considerations have been discussed in [17, Section 1.2].

Remark 5 (Genericity with respect to T)**.** As settled in Theorem 2.2, the uniqueness of the optimal domain holds true for every value of $T > 0$ but some isolated values (independently on the initial conditions). For exceptional values of T the uniqueness property may fail. For example, assume that $\Omega = (0,\pi)$, that $T = 2k\pi$ with $k \in \mathbb{N}^*$, and that $\gamma_2 = 0$. Then, using the notations of Remark 4, we have

$$\varphi_{\gamma,T}(x) = \pi j^2 (\gamma_1 + \gamma_3) \sum_{j=1}^{+\infty} (a_j^2 + b_j^2) = 2\pi (\gamma_1 + \gamma_3) \|(y^0, y^1)\|^2_{H_0^1 \times L^2},$$

for every $x \in \Omega$. Then, obviously, the characteristic function of any measurable subset ω of Lebesgue measure $L|\Omega|$ is a solution of $(\mathcal{P}_{\gamma,T})$.

Remark 6 (Sharpness of Assumption $(\mathbf{A_1})$)**.** If Assumption $(\mathbf{A_1})$ does not hold then the uniqueness of the optimal set may fail. Indeed, assume that $\Omega = (0,\pi)$, that $\gamma = (1,1,0)$, and let $y^0(x) = \sin x$ and $y^1(x) = 0$ for every $x \in \Omega$. The solution y of (1) is $y(t,x) = \cos t \sin x$ for all $(t,x) \in (0,+\infty) \times (0,\pi)$, and then,

$$\psi(x) = 1 \quad \text{and} \quad \varphi_{\gamma,T}(x) = \frac{T}{2} + \frac{\sin(2T)}{4},$$

for every $x \in (0,\pi)$. Since the functions ψ and $\varphi_{\gamma,T}$ are constant in Ω, any subset ω of Lebesgue measure $L|\Omega|$ is optimal.

Remark 7. Assumption $(\mathbf{A_2})$ may look intricate but it is actually standard in the PDE setting, within the framework of the Dirichlet spaces (Sobolev with Navier boundary conditions). This assumption guarantees both the analyticity of the initial data and the boundary compatibility conditions that are required to ensure the analyticity of the solution.

Note that the analyticity of (y^0, y^1) by itself is not sufficient to ensure the analyticity of the corresponding solution y of the wave equation since boundary singularities associated to the lack of boundary compatibility conditions propagate inside the domain according to the D'Alembert formula.

Indeed, to simplify the explanation assume that we are in dimension one and that $\Omega = (0,\pi)$. The solution y is analytic inside the characteristic cone, in accordance with the Cauchy-Kowalevska theorem. Assumption $(\mathbf{A_2})$ ensures that the initial data are in the usual Dirichlet spaces and in particular requires that all their derivatives of even order vanish on the boundary, but this condition is actually necessary to ensure analyticity. Consider functions y^0 and y^1 that are analytic on $(0,\pi)$ and whose derivatives of even order vanish at 0 and π. Then we first consider the odd extension of these functions on $(0,2\pi)$, and then we extend them by periodicity to the whole real line. It is clear that the corresponding solution y of the wave equation is analytic. If one of the derivatives of odd order of y^0 or y^1 were not equal to zero at 0 or π, then this would cause a singularity propagating as soon as $t > 0$, and the corresponding solution would not be analytic. Therefore Assumption $(\mathbf{A_2})$ is in some sense a sharp condition to ensure the analyticity of the solution.

Remark 8 (Further comments on Assumption $(\mathbf{A_2})$)**.** We have seen that the optimal solution may not be unique whenever the function $\varphi_{\gamma,T}$ is constant on some subset of Ω of positive measure. More precisely, assume that $\varphi_{\gamma,T}$ is constant, equal to c, on some subset I of Ω of positive measure $|I|$. If $|\{\varphi_{\gamma,T} > c\}| < L|\Omega| < |\{\varphi_{\gamma,T} \geq c\}|$ then there exists an infinite number of measurable subsets ω of Ω maximizing (4), all of them containing the subset $\{\varphi_{\gamma,T} > c\}$. The set $\omega \cap \{\varphi_{\gamma,T} = c\}$ can indeed be chosen arbitrarily.

We do not know any simple characterization of all initial data for which this non-uniqueness phenomenon occurs, however to get convinced that this may indeed happen it is convenient to consider the one-dimensional case $\Omega = (0,\pi)$ with $T = 2\pi$

and $\gamma=(0,0,1)$. Indeed in that case the functional $G_{\gamma,T}$ reduces to (8) and the corresponding function $\varphi_{\gamma,T}$ reduces to (9). Using the notations of Remark 4, and noting that

$$\frac{\pi}{2}\sum_{j=1}^{+\infty} j^2(a_j^2+b_j^2)=\|(y^0,y^1)\|^2_{H_0^1\times L^2},$$

one gets

$$\varphi_{\gamma,T}(x)=\frac{\pi}{2}\sum_{j=1}^{+\infty} j^2(a_j^2+b_j^2)-\frac{\pi}{2}\sum_{j=1}^{+\infty} j^2(a_j^2+b_j^2)\cos(2jx),$$

for almost every $x\in(0,\pi)$. Then $\varphi_{\gamma,T}$ can be written as a Fourier series whose sine Fourier coefficients vanish and cosine coefficients are nonpositive and summable. Hence, to provide an explicit example where the non-uniqueness phenomenon occurs, consider any nontrivial even function ψ of class C^∞ on $\mathbb{R}$ whose support is contained in $[-\alpha,\alpha]$ for some $\alpha\in(0,\pi/4)$. The C^1 regularity ensures that its Fourier coefficients are summable. To ensure the nonpositivity of its Fourier coefficients, it suffices to consider the π-periodic function $\varphi_{\gamma,T}$ defined on $(0,\pi)$ by the convolution

$$\varphi_{\gamma,T}(x)=\int_{\mathbb{R}}\psi(y)\psi(\pi-y)\,dy-\int_{\mathbb{R}}\psi(y)\psi\,(x-y)\,dy.$$

Indeed, the function $\varphi_{\gamma,T}$ defined in such a way vanishes at $x=0$ and $x=\pi$, is of class C^∞ on $(0,\pi)$, with support contained in $[0,2\alpha]\cup[\pi-2\alpha,\pi]$, and all its Fourier coefficients are nonpositive. More precisely $\varphi_{\gamma,T}$ has a Fourier series expansion of the form

$$\varphi_{\gamma,T}(x)=\left(\int_{\mathbb{R}}\psi(y)\,dy\right)^2-\sum_{j=1}^{+\infty}\beta_j\cos(2jx),$$

with $\beta_j\geq 0$ for every $j\geq 1$. To construct an example where the solution of $(\mathcal{P}_{\gamma,T})$ is not unique, it suffices to define the initial data y^0 and y^1 by their Fourier expansion, and with the notations of Remark 4 in such a way that $\frac{\pi}{2}j^2(a_j^2+b_j^2)=\beta_j$, for every $j\geq 1$. Since the function $\varphi_{\gamma,T}$ vanishes (at least) on $[2\alpha,\pi-2\alpha]$, it suffices to choose $L>4\alpha/\pi$ and it follows that a is a solution of the relaxed problem $\min_{a\in\overline{\mathcal{U}}_L}G_{\gamma,T}(a)$ introduced in Remark 1 if and only if the three following conditions hold:

(i) $a(x)=1$ on $\operatorname{supp}\varphi_{\gamma,T}$,
(ii) $a(x)\in[0,1]$ for almost every $x\in(0,\pi)\setminus\operatorname{supp}\varphi_{\gamma,T}$,
(iii) $\int_0^\pi a(x)\,dx=L\pi$.

The non uniqueness of solutions is thus obvious.

2.3. **Numerical simulations.** We provide hereafter a numerical illustration of the results presented in this section. According to Theorem 2.2, the optimal domain is characterized as a level set of the function $\varphi_{\gamma,T}$. Some numerical simulations are provided on Figure 2, with $\Omega=(0,\pi)^2$, $\gamma=(0,0,1)$, $L=0.6$, $T=3$, $y^1=0$ and

$$y^0(x)=\sum_{n,k=1}^{N_0}a_{n,k}\sin(nx_1)\sin(kx_2),$$

where $N_0\in\mathbb{N}^*$ and $(a_{n,k})_{n,k\in\mathbb{N}^*}$ are real numbers. The level set is numerically computed using a simple dichotomy procedure.

3. **On the complexity of the optimal set.**

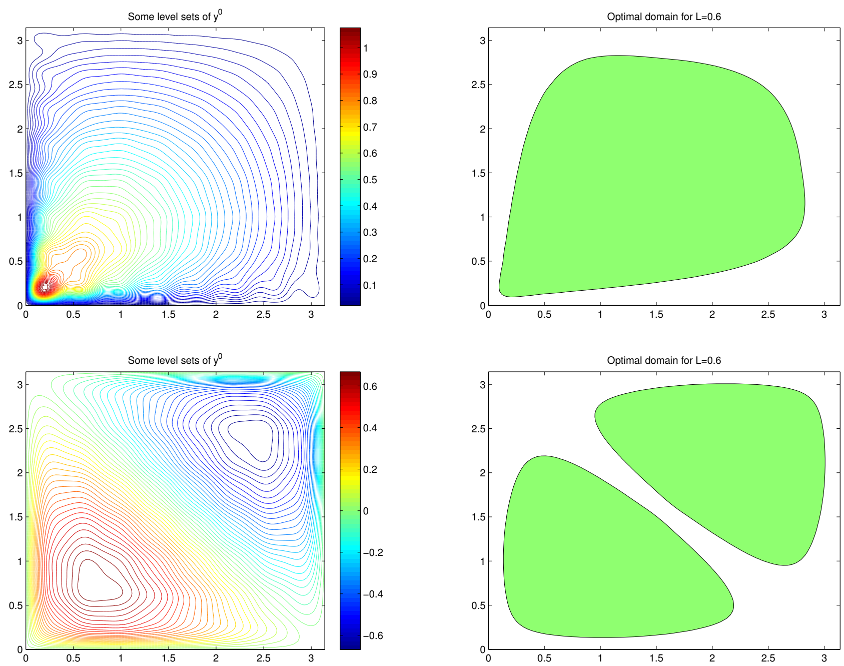

FIGURE 2. $\Omega = (0,\pi)^2$ with Dirichlet boundary conditions, $L = 0.6$, $T = 3$ and $y^1 = 0$. At the top: $N_0 = 15$ and $a_{n,k} = 1/[n^2+k^2]$. At the bottom: $N_0 = 15$ and $a_{n,k} = [1-(-1)^{n+k}]/[n^2k^2]$. On the left: some level sets of y^0. On the right: optimal domain (in green) for the corresponding choice of y^0.

3.1. **Main result.** It is interesting to raise the question of the complexity of the optimal sets solutions of the problem $(\mathcal{P}_{\gamma,T})$, according to its dependence on the initial data. In Theorem 2.2 we proved that, if the initial data belong to some analyticity spaces, then the (unique) optimal set ω is the union of a finite number of connected components. Hence, analyticity implies finiteness and it is interesting to wonder whether this property still holds true for less regular initial data.

In what follows we show that, in the one-dimensional case and for particular values of T, there exist C^∞ initial data for which the optimal set ω has a fractal structure and, more precisely, is of Cantor type.

The proof of the following theorem is quite technical and relies on a careful harmonic analysis construction. In order to facilitate the use of Fourier series, it is more convenient to assume hereafter that $\Omega = (0, 2\pi)$.

Theorem 3.1. *Let $\Omega = (0, 2\pi)$ and let $T = 4k\pi$ for some $k \in \mathbb{N}^*$. Assume either that $\gamma_1 > \gamma_3 \geq 0$ and $\gamma_1 - \gamma_3 - 4\gamma_2 > 0$, or that $0 \leq \gamma_1 < \gamma_3$ and $\gamma_1 - \gamma_3 - 4\gamma_2 < 0$. There exist C^∞ initial data (y^0, y^1) defined on Ω for which the problem $(\mathcal{P}_{\gamma,T})$ has a unique solution ω. The set ω has a fractal structure and, in particular, it has an infinite number of connected components.*

Remark 9. The generalization to the hypercube $\Omega = (0,2\pi)^n$ of the following construction of a fractal optimal set is not immediate since the solutions of the multi-dimensional wave equation fail to be time-periodic.

Remark 10. Theorem 3.1 states that there exist smooth initial data for which the optimal set has a fractal structure, but for which the relaxation phenomenon (see Remark 1) does not occur. At the opposite, the optimal design observability problem for the wave equation settled in [18] admits only relaxed solutions and, as highlighted in [5, 17], numerous difficulties such as the so-called *spillover phenomenon* arise when trying to compute numerically the optimal solutions.

Note that this particular feature cannot be guessed from numerical simulations. It illustrates the variety of behaviors of any maximizing sequence of the problem $(\mathcal{P}_{\gamma,T})$, and the difficulty to capture such phenomena numerically. Similar discussions are led in [19] where the problem of determining the shape and position of the control domain minimizing the norm of the HUM control for given initial data have been investigated.

Proof of Theorem 3.1. Without loss of generality, we set $T = 4\pi$. The eigenfunctions of the Dirichlet-Laplacian on Ω are given by

$$e_n(x) = \sin(nx/2),$$

for $n \in \mathbb{N}^*$, associated with the eigenvalues $n^2/4$. We compute

$$\begin{aligned}\varphi_{\gamma,T}(x) &= 2\pi\sum_{j=1}^{+\infty}(a_j^2+b_j^2)\left(\frac{\gamma_1 j^2}{4}\cos\left(\frac{jx}{2}\right)^2 + \left(\gamma_2+\frac{\gamma_3 j^2}{4}\right)\sin\left(\frac{jx}{2}\right)^2\right)\\ &= \pi\sum_{j=1}^{+\infty}(a_j^2+b_j^2)\left(\frac{j^2(\gamma_1+\gamma_3)}{4}+\gamma_2\right)+\pi\sum_{j=1}^{+\infty}(a_j^2+b_j^2)\left(\frac{j^2(\gamma_1-\gamma_3)}{4}-\gamma_2\right)\cos(jx).\end{aligned}$$

Note that the coefficients $j^2(a_j^2+b_j^2)$ are nonnegative and of converging sum. The construction of the set ω having an infinite number of connected components is based on the following result.

Proposition 1. *There exist a measurable open subset C of $[-\pi,\pi]$, of Lebesgue measure $|C| \in (0,2\pi)$, and a smooth nonnegative function f on $[-\pi,\pi]$, satisfying the following properties:*

- *C is of fractal type and, in particular, it has an infinite number of connected components;*
- *$f(x) > 0$ for every $x \in C$, and $f(x) = 0$ for every $x \in [-\pi,\pi] \setminus C$;*
- *f is even;*
- *for every integer n,*

$$\alpha_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx > 0;$$

- *The series $\sum \alpha_n$ is convergent.*

Indeed, consider the function f introduced in this proposition and choose the initial data y^0 and y^1 such that their Fourier coefficient a_j and b_j satisfy

$$\pi(a_j^2+b_j^2)\left(\frac{j^2(\gamma_1-\gamma_3)}{4}-\gamma_2\right) = \alpha_j$$

for every $j \in \mathbb{N}^*$. Let us extend the characteristic function χ_C of C as a 2π-periodic function on $\mathbb{R}$, and let $\tilde{C} = [0, 2\pi] \cap \{\chi_C = 1\}$. We have

$$\varphi_{\gamma,T}(x) = \sum_{j=1}^{+\infty} \alpha_j \frac{\frac{j^2(\gamma_1+\gamma_3)}{4} + \gamma_2}{\frac{j^2(\gamma_1-\gamma_3)}{4} - \gamma_2} + \sum_{j=1}^{+\infty} \alpha_j \cos(jx).$$

Note that the function $\varphi_{\gamma,T}$ constructed in such a way is nonnegative[4], and there exists $\lambda > 0$ such that the set $\omega = \{\varphi_{\gamma,T} \geq \lambda\}$ has an infinite number of connected components. In other words it suffices to choose $L = |\{\varphi_{\gamma,T} > \lambda\}|$ and the optimal set ω can be chosen as the complement of the fractal set $\tilde{C}$ in $[0, 2\pi]$. The uniqueness of ω is immediate, by construction of the function $\varphi_{\gamma,T}$.

Theorem 3.1 follows from that result. Proposition 1 is proved in the next subsection. □

3.2. **Proof of Proposition 1.** There are many possible variants of such a construction. We provide hereafter one possible way of proving this result.

Let $\alpha \in (0, 1/3)$. We assume that α is a rational number, that is, $\alpha = \frac{p}{q}$ where p and q are relatively prime integers, and moreover we assume that $p + q$ is even. Let us first construct the fractal set $C \subset [-\pi, \pi]$. Since C will be symmetric with respect to 0, we describe below the construction of $C \cap (0, \pi)$. Set $s_0 = 0$ and

$$s_k = \pi - \frac{\pi}{2^k}(\alpha + 1)^k,$$

for every $k \in \mathbb{N}^*$. Around every such point s_k, $k \in \mathbb{N}^*$, we define the interval

$$I_k = \left[s_k - \frac{\pi}{2^k}\alpha(1-\alpha)^k, s_k + \frac{\pi}{2^k}\alpha(1-\alpha)^k \right]$$

of length $|I_k| = \frac{\pi}{2^{k-1}}\alpha(1-\alpha)^k$.

Lemma 3.2. *We have the following properties:*

- $\inf I_1 > \alpha\pi$*;*
- $\sup I_k < \inf I_{k+1} < \pi$ *for every* $k \in \mathbb{N}^*$*.*

Proof. Since $\alpha < 1/3$ it follows that $\inf I_1 = \pi - \frac{\pi}{2}(\alpha + 1) > \alpha\pi$. For the second property, note that the inequality $\sup I_k < \inf I_{k+1}$ is equivalent to

$$\alpha(1-\alpha)^{k-1}(3-\alpha) < (\alpha+1)^k,$$

which holds true for every $k \in \mathbb{N}^*$ since $\alpha(3-\alpha) < \alpha + 1$. □

It follows in particular from that lemma that the intervals I_k are two by two disjoint. Now, we define the set C by

$$C \cap (0, \pi) = [0, \alpha\pi] \cup \bigcup_{k=1}^{+\infty} I_k.$$

The resulting set C (symmetric with respect to 0) is then of fractal type and has an infinite number of connected components (see Figure 3).

We now define the function f such that f is continuous, piecewise affine, equal to 0 outside C, and such that $f(s_k) = b_k$ for every $k \in \mathbb{N}$, b_k being positive real numbers to be chosen (see Figure 3).

Let us compute the Fourier series of f. Since f is even, its sine coefficients vanish. In order to compute its cosine coefficients, we will use the following result.

[4] Note also that, as expected, the function $\varphi_{\gamma,T}$ constructed in such a way satisfies $\varphi_{\gamma,T}(0) = \varphi_{\gamma,T}(\pi) = 0$ whenever $\gamma_1 = 0$.

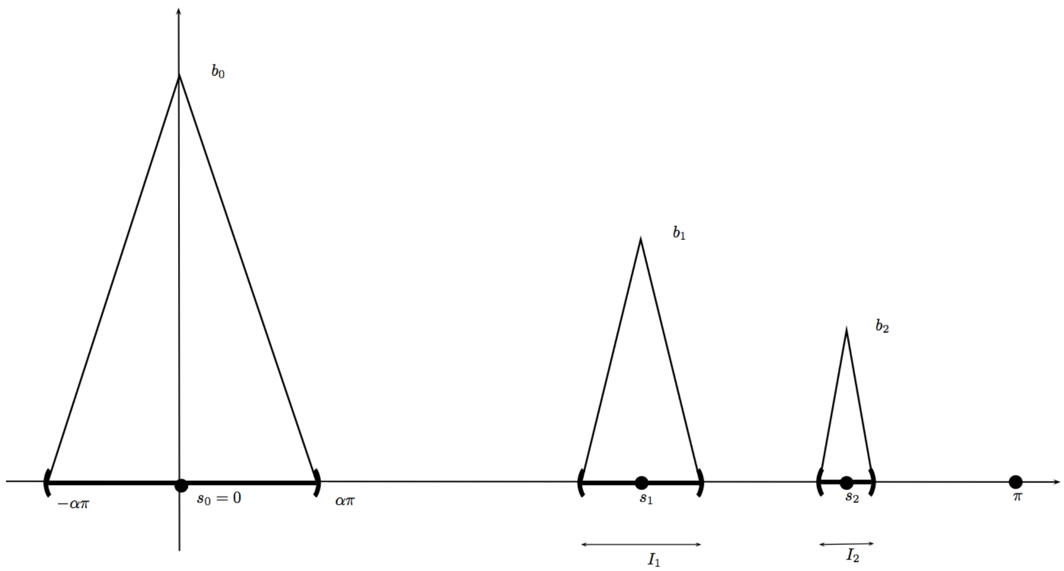

FIGURE 3. Drawing of the function f and of the set C

Lemma 3.3. *Let $a \in \mathbb{R}$, $\ell > 0$ and $b > 0$. Let g be the function defined on $\mathbb{R}$ by*

$$g(x) = \begin{cases} \frac{2b}{\ell}(x - a + \frac{\ell}{2}) & \text{if } a - \frac{\ell}{2} \leq x \leq a, \\ \frac{2b}{\ell}(a + \frac{\ell}{2} - x) & \text{if } a \leq x \leq a + \frac{\ell}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, g is a hat function, i. e. its graph is a positive triangle of height b above the interval $[a - \frac{\ell}{2}, a + \frac{\ell}{2}]$. Then

$$\int_{\mathbb{R}} g(x)\cos(nx)\,dx = \frac{4b}{\ell n^2}\cos(na)\left(1 - \cos\frac{n\ell}{2}\right),$$

for every $n \in \mathbb{N}^*$.

It follows from this lemma that

$$\int_0^{\alpha\pi} f(x)\cos(nx)\,dx = \frac{b_0}{\alpha\pi n^2}(1 - \cos(n\alpha\pi)), \tag{10}$$

and

$$\int_{I_k} f(x)\cos(nx)\,dx = \frac{2^{k+1}b_k}{\alpha(1-\alpha)^k\pi n^2}\cos\left(n\pi - \frac{n\pi}{2^k}(\alpha+1)^k\right)\left(1 - \cos\left(\frac{n\pi}{2^k}\alpha(1-\alpha)^k\right)\right), \tag{11}$$

for every $k \in \mathbb{N}^*$. Note that

$$\left|\int_{I_k} f(x)\cos(nx)\,dx\right| \leq \frac{4b_k}{\alpha\pi n^2}\left(\frac{2}{1-\alpha}\right)^k, \tag{12}$$

for every $k \in \mathbb{N}^*$. Formally, the n^{th} cosine Fourier coefficient of f is given by

$$\alpha_n = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 2\int_0^{\alpha\pi} f(x)\cos(nx)\,dx + 2\sum_{k=1}^{+\infty}\int_{I_k} f(x)\cos(nx)\,dx.$$

Our next task consists of choosing adequately the positive real numbers b_k, $k \in \mathbb{N}$, so that the series in the above formal expression α_n converges, α_n being nonnegative.

Let us first consider the integral (10) (first peak). It is clearly nonnegative for every $n \in \mathbb{N}^*$, and is positive except whenever n is a multiple of $2q$. Taking advantage of the rationality of α, we can moreover derive an estimate from below, as follows. Set

$$\sigma_0 = \min \left\{ 1 - \cos(n\frac{p}{q}\pi) \mid n = 1, \dots, 2q-1 \right\}.$$

One has $\sigma_0 > 0$, and there holds

$$\int_0^{\alpha\pi} f(x)\cos(nx)\,dx \geq \frac{b_0\sigma_0}{\alpha\pi n^2}, \tag{13}$$

for every $n \in \mathbb{N}^* \setminus (2q\mathbb{N}^*)$. At this step, assume that

$$b_k \leq \left(\frac{1-\alpha}{2}\right)^k \frac{1}{2^k}\frac{\sigma_0 b_0}{8}, \tag{14}$$

for every $k \in \mathbb{N}^*$ ($b_0 > 0$ is arbitrary). Under this assumption, using (12) it follows that the formal expression of α_n above is well defined, and that

$$\left| \sum_{k=1}^{+\infty} \int_{I_k} f(x)\cos(nx)\,dx \right| \leq \frac{1}{2}\frac{b_0\sigma_0}{\alpha\pi n^2} \leq \frac{1}{2}\int_0^{\alpha\pi} f(x)\cos(nx)\,dx,$$

for every $n \in \mathbb{N}^* \setminus (2q\mathbb{N}^*)$, ensuring therefore $\alpha_n > 0$ for such integers n.

If $n = 2rq$, with $r \in \mathbb{N}^*$, then the integral (10) vanishes. We then focus on the second peak, that is, on the integral (11) with $k = 1$. Since $n = 2rq$, its value is

$$\int_{I_1} f(x)\cos(nx)\,dx = \frac{4b_1}{\alpha(1-\alpha)\pi n^2}\cos\left(2rq\pi - rq\pi(\frac{p}{q}+1)\right)\left(1 - \cos\left(rq\pi\frac{p}{q}(1-\frac{p}{q})\right)\right).$$

Since $p+q$ is even, it follows that $\cos\left(2rq\pi - rq\pi(\frac{p}{q}+1)\right) = 1$. Hence, we have

$$\int_{I_1} f(x)\cos(nx)\,dx = \frac{4b_1}{\alpha(1-\alpha)\pi n^2}\left(1 - \cos\left(r\pi\frac{p}{q}(q-p)\right)\right) \geq 0.$$

Moreover, since the integers p and q are relatively prime integers and $q-p$ is even, in this last expression one has $\cos(r\pi\frac{p}{q}(q-p)) = 1$ if and only if r is multiple of q, that is, if and only if n is multiple of $2q^2$. As before we derive an estimate from below, setting

$$\sigma_1 = \min\left\{1 - \cos\left(r\pi\frac{p}{q}(q-p)\right) \mid r = 1, \dots, 2q-1\right\}.$$

One has $\sigma_1 > 0$, and there holds

$$\int_{I_1} f(x)\cos(nx)\,dx \geq \frac{4b_1\sigma_1}{\alpha(1-\alpha)\pi n^2}, \tag{15}$$

for every $n \in (2q\mathbb{N}^*) \setminus (2q^2\mathbb{N}^*)$. At this step, additionally to (14), assume that

$$b_k \leq \left(\frac{1-\alpha}{2}\right)^{k-1} \frac{1}{2^{k+1}} b_1\sigma_1, \tag{16}$$

for every $k \geq 2$. Under this assumption, using (12) it follows that

$$\left| \sum_{k=2}^{+\infty} \int_{I_k} f(x)\cos(nx)\, dx \right| \leq \frac{1}{2} \frac{4b_1\sigma_1}{\alpha(1-\alpha)\pi n^2} \leq \frac{1}{2} \int_{I_1} f(x)\cos(nx)\, dx,$$

for every $n \in (2q\mathbb{N}^*) \setminus (2q^2\mathbb{N}^*)$, ensuring therefore $\alpha_n > 0$ for such integers n.

The construction can be easily iterated. At iteration m, assume that $n = 2rq^m$, with $r \in \mathbb{N}^*$. Then the integrals over the m first peaks vanish, that is,

$$\int_0^{\alpha\pi} f(x)\cos(nx)\, dx = \int_{I_k} f(x)\cos(nx)\, dx = 0$$

for every $k = 1, \ldots, m-1$. We then focus on the $(m+1)^{\text{th}}$ peak, that is, on the integral (11) with $k = m$. Since $n = 2rq^m$, its value is

$$\int_{I_m} f(x)\cos(nx)\, dx = \frac{2^{m+1}b_m}{\alpha(1-\alpha)^m\pi n^2} \cos\left(2rq^m\pi - \frac{rq^m\pi}{2^{m-1}}\left(\frac{p}{q}+1\right)^m\right) \times \left(1 - \cos\left(\frac{rq^m\pi}{2^{m-1}}\frac{p}{q}\left(1-\frac{p}{q}\right)^m\right)\right).$$

Since $p+q$ is even, it follows that

$$\cos\left(2rq^m\pi - \frac{rq^m\pi}{2^{m-1}}\left(\frac{p}{q}+1\right)^m\right) = 1,$$

and hence,

$$\int_{I_m} f(x)\cos(nx)\, dx = \frac{2^{m+1}b_m}{\alpha(1-\alpha)^m\pi n^2}\left(1 - \cos\left(\frac{r\pi}{2^{m-1}}\frac{p}{q}(q-p)^m\right)\right) \geq 0.$$

Moreover, since the integers p and q are relatively prime integers and $q-p$ is even, it follows easily that q and $(\frac{q-p}{2})^m$ are relatively prime integers, and therefore this last expression vanishes if and only if r is multiple of q, that is, if and only if n is multiple of $2q^{m+1}$. Setting

$$\sigma_m = \min\left\{1 - \cos\left(\frac{r\pi}{2^{m-1}}\frac{p}{q}(q-p)^m\right) \;\middle|\; r = 1, \ldots, 2q-1\right\},$$

one has $\sigma_m > 0$ and

$$\int_{I_m} f(x)\cos(nx)\, dx \geq \frac{2^{m+1}b_m\sigma_m}{\alpha(1-\alpha)^m\pi n^2},$$

for every $n \in (2q^m\mathbb{N}^*) \setminus (2q^{m+1}\mathbb{N}^*)$. Additionally to (14), (16) and the following iterative assumptions, we assume that

$$b_k \leq \left(\frac{1-\alpha}{2}\right)^{k-m} \frac{1}{2^{k-m+2}} b_m\sigma_m, \tag{17}$$

for every $k \geq m+1$. Under this assumption, using (12) it follows that

$$\left| \sum_{k=m+1}^{+\infty} \int_{I_k} f(x)\cos(nx)\, dx \right| \leq \frac{1}{2} \frac{2^{m+1}b_m\sigma_m}{\alpha(1-\alpha)^m\pi n^2} \leq \frac{1}{2} \int_{I_m} f(x)\cos(nx)\, dx,$$

for every $n \in (2q^m\mathbb{N}^*) \setminus (2q^{m+1}\mathbb{N}^*)$, ensuring therefore $\alpha_n > 0$ for such integers n.

The construction of the function f goes in such a way by iteration. By construction, its Fourier cosine coefficients α_n are positive, and moreover, the series $\sum_{n=0}^{+\infty} \alpha_n$ is convergent. We have thus constructed a function f satisfying all requirements of the statement except the fact that f is smooth.

Let us finally show that, using appropriate convolutions, we can modify f in order to obtain a smooth function keeping all required properties. Set $f_0 = f_{[-\alpha\pi,\alpha\pi]}$ and $f_k = f_{I_k}$ for every $k \in \mathbb{N}^*$. For every $\varepsilon > 0$, let ρ_ε be a real nonnegative function which is even, whose support is $[-\varepsilon, \varepsilon]$, whose integral over $\mathbb{R}$ is equal to 1, and whose Fourier (cosine) coefficients are all positive. Such a function clearly exists. Indeed, only the last property is not usual, but to ensure this Fourier property it suffices to consider the convolution of any usual bump function with itself. Then, for every $k \in \mathbb{N}$, consider the (nonnegative) function $\tilde{f}_k$ defined by the convolution $\tilde{f}_k = \rho_{\varepsilon(k)} \star f_k$, where each $\varepsilon(k)$ is chosen small enough so that the supports of all functions $\tilde{f}_k$ are still disjoint two by two and contained in $[-\pi, \pi]$ as in Lemma 3.2. Then, we define the function $\tilde{f}$ as the sum of all functions $\tilde{f}_k$, and we symmetrize it with respect to 0. Clearly, every Fourier (cosine) coefficient of $\tilde{f}$ is the sum of the Fourier (cosine) coefficients of $\tilde{f}_k$, and thus is positive, and their sum is still convergent. The function $\tilde{f}$ is smooth and satisfies all requirements of the statement of the proposition. This ends the proof.

4. **Conclusions and further comments.**

4.1. **Other partial differential equations.**

The study developed in this article can be extended in several directions and in particular to other partial differential equations.

Indeed, to ensure the existence of an optimal set for the problem

$$\sup_{|\omega|=L|\Omega|} \int_\omega \varphi_{\gamma,T}(x)\, dx,$$

it is just required that $\varphi_{\gamma,T} \in L^1(\Omega)$. Then the conclusion of Theorem 2.1 clearly holds true for other evolution PDE's, either parabolic or hyperbolic.

Theorem 2.2 holds true as soon as one is able to ensure that the function $\varphi_{\gamma,T}$ is analytic on Ω. Depending on the model under consideration it may however be more or less difficult to ensure this property by prescribing suitable regularity properties on the initial conditions. Let us next provide some details for two examples: the Schrödinger equation and the heat equation. In these cases, and since these equations are of first order in time, it is definitely more relevant to replace the cost functional $G_{\gamma,T}$ with

$$\widetilde{G}_T(\chi_\omega) = \int_0^T\int_\omega |y(t,x)|^2\, dx\, dt.$$

Then, the function $\varphi_{\gamma,T}$ defined in Section 2.1 becomes

$$\varphi_{\gamma,T}(x) = \int_0^T |y(t,x)|^2\, dt.$$

We then have the following results.

Schrödinger equation. Consider the Schrödinger equation on Ω

$$i\partial_t y = \triangle y, \tag{18}$$

with Dirichlet boundary conditions, and $y(0,\cdot) = y^0(\cdot) \in L^2(\Omega, \mathbb{C})$. The conclusion of Theorem 2.2 (including the uniqueness of the optimal set) holds true for every $T > 0$ when replacing the sufficient condition $(\mathbf{A_2})$ with

$$\sum_{j=0}^{+\infty} \frac{R^j}{j!} \|A^{j/2} y^0\|_{L^2(\Omega,\mathbb{C})} < +\infty.$$

Indeed, the function φ is then analytic, and cannot be constant since it vanishes on $\partial\Omega$, whence the conclusion (see Remark 3).

As concerns the complexity of the optimal set, it is clear that the construction of C^∞ initial data for which the optimal set ω has a fractal structure, made in Theorem 3.1 can be applied to this case as well.

The generalization of the fractal optimal set in several space dimensions is simpler in this case. Indeed, by considering Cartesian products of this one-dimensional fractal set constructed in the proof of Theorem 3.1, it is immediate to generalize the construction to $\Omega = (0,\pi)^n$ since any solution of (18) remains periodic in this case, which ensures that $\widetilde{G}_T$ does not involve any additional crossed terms. As mentioned above, the problem is more complex for the multi-dimensional wave equation due to the lack of periodicity and, accordingly, due to the nondiagonal terms in the expansion of the functional.

Heat equation. Consider the homogeneous heat equation on Ω

$$\partial_t y = \triangle y,$$

with Dirichlet boundary conditions, and $y(0,\cdot) = y^0(\cdot) \in L^2(\Omega)$. Let $(\phi_j)_{j\in\mathbb{N}^*}$ be a Hilbert basis of $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet-Laplacian operator on Ω, associated with the negative eigenvalues $(-\lambda_j^2)_{j\in\mathbb{N}^*}$. We expand y^0 as

$$y^0 = \sum_{j=1}^{+\infty} a_j \phi_j(\cdot)$$

with $a_j = \langle y^0, \phi_j\rangle_{L^2}$. We claim that the function $\varphi_{\gamma,T}$ is analytic provided that the initial datum y^0 satisfies the condition

$$\sum_{j=1}^{\infty} \lambda_j^{\frac{n}{2}-1} |a_j| < +\infty. \tag{19}$$

Indeed, using the well-known estimates $\|\phi_j\|_{L^\infty(\Omega)} \leq C\lambda_j^{\frac{n-1}{2}}$, it follows that, for almost every $x \in \Omega$,

$$\begin{aligned}
\varphi_{\gamma,T}(x) &\leq \int_0^T \sum_{j,k=1}^{+\infty} |a_j a_k| e^{-(\lambda_j+\lambda_k)t} |\phi_j(x)\phi_k(x)|\, dt \\
&\leq \sum_{j,k=1}^{+\infty} |a_j a_k| \frac{(\lambda_j\lambda_k)^{\frac{n-1}{2}}}{\lambda_j+\lambda_k} \\
&\leq \frac{1}{2}\sum_{j,k=1}^{+\infty} |a_j a_k| (\lambda_j\lambda_k)^{\frac{n}{2}-1} = \frac{1}{2}\left(\sum_{j=1}^{+\infty} |a_j| \lambda_j^{\frac{n}{2}-1}\right)^2 .
\end{aligned}$$

Here above, we used the inequality $(\lambda_j\lambda_k)^{\frac{1}{2}} \leq \frac{1}{2}(\lambda_j+\lambda_k)$. The claim then follows from standard analyticity results by noting that for a given $t>0$, the function $|y(t,\cdot)|^2$ is analytic in Ω. As a consequence, if the condition (19) holds, then the conclusion of Theorem 2.2 holds true for every $T>0$ when replacing the cost functional $G_{T,\eta}$ with $\widetilde{G}_T$, the function $\varphi_{\gamma,T}$ with φ, with arguments similar to the ones used for the Schrödinger equation.

4.2. **Other boundary conditions.** Throughout the paper, for the clarity of the exposition we restricted our study to the wave equation with Dirichlet boundary conditions. With minor changes, our results can be extended to the case of other boundary conditions. For example, in the Neumann case, it is necessary to consider initial data (y^0, y^1) in $H^1(\Omega) \times L^2(\Omega)$. In this frame, it is still relevant to consider the cost functional $G_{\gamma,T}$ defined by (2).

Denoting now by A the Neumann-Laplacian, the theorems 2.1 and 2.2 still hold true in this case (except the last claim, that is, the fact that there is a positive distance between the optimal set and the boundary of Ω, which is specific to the Dirichlet case).

Theorem 2.2 is still valid for more general choices of boundary conditions such as Neumann, mixed Dirichlet-Neumann, or Robin boundary conditions for the wave equation, and the corresponding appropriate functional spaces are discussed in [18].

4.3. **Perspectives.** In this paper, given fixed initial data, we have solved the problem of determining the best shape and location of a subdomain of a given measure maximizing the energy of the corresponding solution of the wave equation with Dirichlet boundary conditions, restricted to the subdomain and over a certain horizon of time. We have discussed the question of the uniqueness of an optimal solution. We have also investigated the complexity of the optimal set, showing that it depends on the regularity of the initial data. In particular, we have constructed an example where the optimal set is fractal.

We have considered the problem of optimally observing solutions. The dual problem of optimally controlling the solutions to zero can also be considered. Similar results have been established in [14, 16] and [19] in the context of optimal design of the control support.

We have considered the problem of choosing the optimal observation set for fixed initial data. But, in engineering applications this problem can be viewed as a first step towards modeling the problem of optimizing the shape and location of sensors or actuators location problems. Recall that the equation (1) is said to be *observable* on ω in time T if there exists a positive constant C such that the inequality

$$C\|(y^0, y^1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \int_0^T \int_\omega y(t,x)^2 \, dx \, dt, \tag{20}$$

holds for every $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. This is the so-called *observability inequality* (see [12]). It is well known that within the class of $\mathcal{C}^\infty$ domains Ω, this observability property holds if the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [2, 3]), according to which every ray of Geometric Optics that propagates in Ω and is reflected on its boundary $\partial\Omega$ intersects ω within time T. We denote by $C_T(\chi_\omega)$ the largest constant in the observability inequality above, that is

$$C_T(\chi_\omega) = \inf \left\{ \frac{G_{\gamma,T}(\chi_\omega)}{\|(y^0, y^1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}} \mid (y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega) \backslash \{(0,0)\} \right\}. \tag{21}$$

An a priori natural way of modeling the problem of optimal shape and placement of sensors or controllers for the wave equation consists of maximizing the functional $C_T(\chi_\omega)$ over the set of all measurable subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$. Moreover, it can be argued that the observability constant $C_T(\chi_\omega)$, appearing in the observability inequality (20), gives an account for the quality of some inverse problem consisting of reconstructing the initial data (y^0, y^1) from the observed variable $\chi_\omega y$ over $[0,T]$. This optimal design problem, settled as such, is a difficult one. We refer to [17, 18] for a study of a simplified version in which the criterium to be optimized is reduced to a purely spectral one by some randomization procedure or by some time-asymptotic procedure.

It can be also of interest in practice to address a variant of the above criterion by considering suitable class of initial data:

$$J(\chi_\omega) = \inf_{(y^0,y^1)\in\mathcal{V}} \frac{G_{\gamma,T}(\chi_\omega)}{\|(y^0,y^1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}},$$

where $\mathcal{V}$ is a set of initial data. In the criterion above (observability constant), one has $\mathcal{V} = L^2(\Omega)\times H^{-1}(\Omega)$, that is the set of *all* possible initial data, but it may be interesting to consider subsets of $L^2(\Omega)\times H^{-1}(\Omega)$, such as:

- The subset of initial data having a certain number of nonzero Fourier components. This is the case in practice when the measurement devices can only measure, say, the first N frequencies of the solutions of the wave equation. Note however that there is then an intrinsic instability feature called spillover (see [17, 18]).
- A set of data in a neighborhood of a given fixed datum in $L^2(\Omega)\times H^{-1}(\Omega)$. It may happen in practice that only such initial data be relevant for physical reasons.
- A set of initial data which is defined as a parametrized (finite- or infinite-dimensional) submanifold of $L^2(\Omega)\times H^{-1}(\Omega)$, appearing in the model of some physical experiment. For instance the works in [10, 11, 13, 22] consider a finite number of possibilities for the unknown domains and run the optimization over this finite-dimensional manifold.

A systematic and complete analysis of all these issues is still to be done.

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E-mail address: yannick.privat@upmc.fr
E-mail address: emmanuel.trelat@upmc.fr
E-mail address: zuazua@bcamath.org