On the optimality of the observability inequalities for Kirchhoff plate systems with potentials in unbounded domains*

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Summary. In this paper, we derive a sharp observability inequality for Kirchhoff plate equations with lower order terms in an unbounded domain Ω of $\mathbb{R}^n$. More precisely, when the observation is assumed to be located in a subdomain ω such that $\Omega \setminus \overline{\omega}$ is bounded and the observation time $T > 0$ is sufficiently large, we establish an observability estimate with an explicit observability constant for Kirchhoff plate systems with an arbitrary finite number of components and in any space dimension with lower order bounded potentials. Also, when $\Omega = \mathbb{R}^n$, by means of the Meshkov construction for the bi-Laplacian equation, we prove the optimality of this estimate for systems of more than two components and in even space dimensions $n \geq 2$.

2000 Mathematics Subject Classification. Primary 93B07; Secondary 93B05, 35B37.

Key Words. Kirchhoff plate system, observability constant, Carleman inequalities, potential, Meshkov's construction.

1 Introduction

Let $n \geq 1$ and $N \geq 1$ be two integers. Let D be a bounded convex domain in $\mathbb{R}^n$ with C^4 boundary Γ, $\Omega = \mathbb{R}^n \setminus \overline{D}$, and ω be an open subset in Ω so that $\Omega \setminus \overline{\omega}$ is bounded. Obviously, when $D = \emptyset$, one has $\Omega = \mathbb{R}^n$. Let $T > 0$

* The work is supported by the Grant MTM2005-00714 of the Spanish MEC, the DOMINO Project CIT-370200-2005-10 in the PROFIT program of the MEC (Spain), the SIMUMAT projet of the CAM (Spain), the EU TMR Project "Smart Systems", and the NSF of China under grants 10371084 and 10525105.

be given and sufficiently large. Put $Q \stackrel{\triangle}{=} (0,T) \times \Omega$ and $\Sigma \stackrel{\triangle}{=} (0,T) \times \Gamma$. For simplicity, we will use the notation $y_i = \frac{\partial y}{\partial x_i}$, where x_i is the i-th coordinate of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. Throughout this paper, we will use $C = C(T, \Omega, \omega)$ to denote generic positive constants depending on their arguments which may vary from line to line.

Denote by $\mathcal{D}'(\Omega)$ the usual space of distributions in Ω. Set

$$\mathcal{H} \stackrel{\triangle}{=} \Big\{ (\varphi, \psi) \in \mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega) \Big| \ \varphi, \Delta\varphi, \psi \in H_0^1(\Omega),\ \Delta\psi \in L^2(\Omega) \Big\}^N .$$

Clearly, $\mathcal{H}$ is a Hilbert space with the following norm:

$$||(\varphi, \psi)||_{\mathcal{H}} \stackrel{\triangle}{=} \sqrt{||\varphi||^2_{(H^1(\Omega))^N} + ||\Delta\varphi||^2_{(H^1(\Omega))^N} + ||\psi||^2_{(H^1(\Omega))^N} + ||\Delta\psi||^2_{(L^2(\Omega))^N}} .$$

We consider the following $\mathbb{R}^N$-valued plate system with a potential $a \in L^\infty(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n/3, \infty]$:

$$\begin{cases} y_{tt} + \Delta^2 y - \Delta y_{tt} + ay = 0 & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad y_t(0) = y^1 & \text{in } \Omega, \end{cases} \tag{1}$$

where $y = (y_1, \cdots, y_N)^\top$, and the initial datum (y^0, y^1) is supposed to belong to $\mathcal{H}$, the state space of system (1). It is easy to show that system (1) admits one and only one weak solution $y \in C([0,T]; \mathcal{H})$.

In what follows, we shall denote by $|\cdot|$ and $||\cdot||_p$ the (canonical) norms on $\mathbb{R}^N$ and $L^\infty(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$, respectively.

We shall study the observability constant $K(a)$ of system (1), defined as the smallest constant such that the following observability estimate for system (1) holds:

$$\begin{aligned} &||(y^0, y^1)||^2_{\mathcal{H}} \\ &\le K(a) \int_0^T \int_\omega \left(|y|^2 + |\nabla y|^2 + |\Delta y|^2 + |\nabla \Delta y|^2 \right) dxdt, \quad \forall\, (y^0, y^1) \in \mathcal{H}. \end{aligned} \tag{2}$$

This inequality, the so-called *observability inequality*, allows estimating the total energy of solutions in terms of the energy localized in the observation subdomain ω. It is relevant for control problems. In particular, in this linear setting, this (observability) inequality is equivalent to the so-called exact controllability property, i.e., that of driving solutions to rest by means of control forces localized in ω (see [4, 10]). This type of inequality, with explicit estimates on the observability constant, is also relevant for the control of semilinear problems ([9]). Similar inequalities are also useful for solving a variety of Inverse Problems ([7]).

Obviously the observability constant $K(a)$ in (2) not only depends on the potential a, but also on the domains Ω and ω and on the time T. The main purpose of this paper is to analyze only its explicit and sharp dependence on the potential a.

The main tools to derive the explicit observability estimates are the so-called *Carleman inequalities*. Here we have chosen to work in the space $\mathcal{H}$ in which Carleman inequalities can be applied more naturally. But some other choices of the state space are possible. For example, one may consider similar problems in state spaces of the form $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ or $\left\{\varphi \in \mathcal{D}'(\Omega) \middle| \ \varphi \in H_0^1(\Omega),\ \Delta\varphi \in L^2(\Omega)\right\}^N \times (H_0^1(\Omega))^N$ where the Kirchhoff plate system is also well posed. But the corresponding analysis on the observability constants, in turn, is technically more involved.

The same problem was considered in [8] in the case of bounded domains. In that paper, under some assumptions on the observation domain and time, an observability inequality of the form (2) was established with the following estimate on the observability constant:

$$K(a) \leq C \exp\left(C||a||_p^{\frac{1}{3-5n/2p}} \right). \tag{3}$$

In this paper we show that the same estimate holds in unbounded domains. Similar (boundary and/or internal) observability estimates (in suitable spaces) have been established for the heat and wave equations in [2], and for the Euler-Bernoulli plate equations in [3]. Furthermore, in [2, 3], the optimality of these estimates has been proved based on the Meshkov construction (see [6, 2]) of highly concentrated solutions of elliptic equations with potentials. Unfortunately, in [8], we failed to show the optimality of (3) for bounded domains, because of the difficulty one encounters to correct the rescaled Meshkov-type solutions of the bi-harmonic equation so that they fulfill the homogeneous boundary conditions. In our present case, we shall see that this difficulty disappears when we consider the problem in the whole space, i.e., $\Omega = \mathbb{R}^n$. However, for general unbounded domains optimality of the observability constants is an open problem.

The rest of this paper is organized as follows. In Section 2 we give some preliminary energy estimates for Kirchhoff plate systems, and recall some fundamental weighted pointwise estimates for the wave and elliptic operators. In Section 3 we present the sharp observability estimate for the Kirchhoff plate system. Finally, Section 4 is devoted to the analysis of the optimality of the observability estimate.

2 Preliminaries

In this section, we collect some preliminary energy estimates for Kirchhoff plate systems, and weighted pointwise estimates for the wave and elliptic operators.

2.1 Energy estimates for Kirchhoff plate systems

Denote the energy of system (1) by

$$E(t) = \frac{1}{2}\Big[||(y,y_t)||^2_{\mathcal{H}} + ||\nabla y_t||^2_{(L^2(\Omega))^{nN}}\Big]$$
$$\equiv \frac{1}{2}\int_\Omega \big(|y|^2 + |\nabla y|^2 + |\Delta y|^2 + |\nabla \Delta y|^2 + |y_t|^2 + 2|\nabla y_t|^2 + |\Delta y_t|^2\big)\,dx.$$

By system (1), it is easy to verify the following energy law:

$$\frac{dE(t)}{dt} = \int_\Omega (y - ay)\cdot(y_t - \Delta y_t)dx.$$

For $s_0 = \frac{n}{3p}$, consider also the modified energy function:

$$\mathcal{E}(t) = E(t) + \frac{1}{2}||a||_p^{\frac{2}{2-s_0}}|y(t,\cdot)|^2_{(L^2(\Omega))^N}.$$

It is clear that both energies are equivalent. Indeed,

$$E(t) \le \mathcal{E}(t) \le C\left(1 + ||a||_p^{\frac{2}{2-s_0}}\right)E(t).$$

Similar to [8], the following estimate holds for the modified energy:

Lemma 1. *Let $a \in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$ for some $p \in [n/3,\infty]$. Then there is a constant $C_0 = C_0(\Omega,p,n) > 0$, independent of T, such that*

$$\mathcal{E}(t) \le C_0 e^{C_0||a||_p^{\frac{1}{2-s_0}}|t-s|}\mathcal{E}(s), \qquad \forall\, t,s \in [0,T]. \tag{4}$$

2.2 Pointwise weighted estimates for the wave and elliptic operators

In this subsection, we present some pointwise weighted estimates for the wave and elliptic equations that will play a key role when deriving the sharp observability estimates for the Kirchhoff plate system.

As in [8], one of the key points to derive inequality (2) for system (1) is the possibility of decomposing the Kirchhoff plate operator $\partial_t^2 + \Delta^2 - \partial_t^2\Delta$ as follows:

$$\partial_t^2 + \Delta^2 - \partial_t^2\Delta = (\partial_{tt} - \Delta)(I - \Delta) + \Delta, \tag{5}$$

where I is the identity operator. Actually, we set

$$z = y - \Delta y, \tag{6}$$

where y is the solution of (1). By the first equation of (1) and noting (5), it follows that

$$-ay = y_{tt} + \Delta^2 y - \Delta y_{tt} = (\partial_{tt} - \Delta)(y - \Delta y) + \Delta y = z_{tt} - \Delta z + y - z.$$

Therefore the Kirchhoff plate system (1) can be written equivalently as the following coupled elliptic-wave system

$$\begin{cases} \Delta y + z - y = 0 & \text{in } Q, \\ z_{tt} - \Delta z + y - z + ay = 0 & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \Delta y^0, \quad z_t(0) = y^1 - \Delta y^1 & \text{in } \Omega. \end{cases} \tag{7}$$

Consequently, in order to derive the desired observability inequality (2) for system (1), it is natural to proceed in cascade by applying the global Carleman estimates to the second order operators in the two equations in system (7).

In what follows, to simplify the argument to derive the desired observability inequality (2), we assume $D \neq \emptyset$ (When $D = \emptyset$, we can modify an argument in [5, Case 2 in the proof of Theorem 5.1] to derive the same result).

For any (large) $\lambda > 0$, any $x_0 \in D$ and $c \in \mathbb{R}$, set

$$\ell \equiv \ell(t,x) = \lambda\Big[|x - x_0|^2 - c\Big(t - \frac{T}{2}\Big)^2\Big], \qquad \theta = e^{\ell}. \tag{8}$$

Without loss of generality, we may assume $\overline{D} \cap \overline{\omega} = \emptyset$ (Otherwise one uses a smaller observation domain, say $\omega \setminus \Big\{x \in \mathbb{R}^n \;\Big|\; \text{dist}\,(x, \overline{D}) \leq 1\Big\}$, to replace the original ω). Put

$$\Omega_0 = \Omega \setminus \overline{\omega}, \qquad \Omega_1 = \Big\{x \in \mathbb{R}^n \;\Big|\; \text{dist}\,(x, \overline{\Omega_0}) \leq 1\Big\} \setminus \overline{D}. \tag{9}$$

Clearly, $\Omega_0 \subset \Omega_1$ and both of them are bounded domains in $\mathbb{R}^n$.

Since $x_0 \in D \subset \mathbb{R}^n \setminus \overline{\Omega_1}$, it holds

$$0 < R_0 \overset{\triangle}{=} \min_{x \in \Omega_1} |x - x_0| < R_1 \overset{\triangle}{=} \max_{x \in \Omega_1} |x - x_0|. \tag{10}$$

Also, for any $\beta > 0$, we set

$$\Theta = \Theta(t) \overset{\triangle}{=} \exp\Big\{-\frac{\beta R_1}{t} - \frac{\beta R_1}{T - t}\Big\}, \qquad 0 < t < T. \tag{11}$$

It is easy to see that $\Theta(t)$ decays rapidly to 0 as $t \to 0$ or $t \to T$. We recall the following two known pointwise Carleman-type estimates (with singular weight Θ) for the wave and elliptic operators, respectively:

Lemma 2. ([8]) *Let $u \in C^2([0,T] \times \overline{\Omega_1})$ and $v = \theta u$. Then there exist four constants $T_0 > 0$, $\lambda_0 > 0$, $\beta_0 > 0$ and $c_0 > 0$, independent of u, such that for all $T \geq T_0$, $\beta \in (0, \beta_0)$ and $\lambda \geq \lambda_0$ it holds*

$$\begin{aligned}
&\theta^2\Theta|u_{tt}-\Delta u|^2+2\Big\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t\\
&\qquad\qquad\qquad +(A+\Psi)\ell_t v^2\Big]\Big\}_t\\
&\quad+2\Theta\sum_{i=1}^n\Big\{2v_i(\nabla\ell)\cdot(\nabla v)-\ell_i|\nabla v|^2+\Psi vv_i-2\ell_t v_t v_i+\ell_i v_t^2\\
&\qquad\qquad -(A+\Psi)\ell_i v^2\Big\}_i\\
&\geq c_0\lambda\theta^2\Theta(u_t^2+|\nabla u|^2+\lambda^2u^2),
\end{aligned} \tag{12}$$

where

$$\begin{cases}\Psi \stackrel{\triangle}{=} \lambda(2n-2c-1+k),\\ A=4\lambda^2[c^2(t-T/2)^2-|x-x_0|^2]+\lambda(4c+1-k).\end{cases} \tag{13}$$

Lemma 3. ([8]) *Let $p=p(t,x)\in C^2([0,T]\times\overline{\Omega_1})$, and set $q=\theta p$. Then there exist two constants $\lambda_0>0$ and $c_0>0$, independent of p, such that for all $T>0$, $\beta>0$ and $\lambda\geq\lambda_0$ it holds*

$$\begin{aligned}
&\theta^2\Theta|\Delta p|^2+2\Theta\sum_{i=1}^n\Big\{2q_i(\nabla\ell)\cdot(\nabla q)-\ell_i|\nabla q|^2+\widetilde{\Psi}qq_i-(\widetilde{A}+\widetilde{\Psi})\ell_i q^2\Big\}_i\\
&\geq c_0\lambda\theta^2\Theta(|\nabla p|^2+\lambda^2p^2),
\end{aligned} \tag{14}$$

where

$$\begin{cases}\widetilde{\Psi}\stackrel{\triangle}{=}\lambda(2n-1),\quad \widetilde{A}=-4\lambda^2|x-x_0|^2+\lambda,\\ \widetilde{B}=40\lambda^3|x-x_0|^2+O(\lambda^2),\quad \textit{uniformly w.r.t. } t\in[0,T].\end{cases} \tag{15}$$

3 Sharp observability estimate

In this section we establish a sharp observability estimate for system (1).

One of the main results in this paper is the following observability inequality with explicit dependence of the observability constant on the potential a for system (1):

Theorem 1. *Let D be a bounded convex domain in $\mathbb{R}^n$, $\Omega=\mathbb{R}^n\setminus\overline{D}$ and ω an open subset in Ω so that $\Omega\setminus\overline{\omega}$ is bounded. Let $p\in[5n/2,\infty]$. Then there is a constant $C>0$ such that for any $T>T_0$, with T_0 as in Lemma 2, and any $a\in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$, the weak solution y of system (1) satisfies estimate (2) with the observability constant $K(a)>0$ verifying (3).*

Remark 1. If D is not assumed to be convex, the result in Theorem 1 still holds provided that ω is further assumed to contain a set of the form

$$\Big\{x\in\Gamma\;\Big|\;(x-x_0)\cdot\nu(x)>0\Big\}$$

for some $x_0 \in \mathbb{R}^n$, where $\nu(x)$ is the unit outward normal vector of Ω at $x \in \Gamma$.

We now sketch the main points in the proof of Theorem 1. As in [8], we decompose the Kirchhoff plate equation into a coupled system of wave and elliptic equations as in (7) and apply the pointwise estimates of the previous section in cascade. However, in the present setting in which the system holds in an unbounded domain, we need to use a cut-off argument. Indeed, roughly speaking, the observed quantity, i.e., the right hand side terms of inequality (2), provides full information on the solution in the observed subdomain ω. One then needs to recover the missing information in $\Omega \setminus \omega$. It is therefore natural to split the solution into two parts, i.e., to distinguish the restriction of the solution to ω, which is known, and the unknown restriction to $\Omega \setminus \omega$. This argument, needed for deriving the observability on unbounded domains, was used for instance in [1] for parabolic equations in unbounded domains.

To be more precise, we fix any smooth cut-off function ϕ so that

$$\begin{cases} \phi \equiv 1 \quad \text{in } \Omega_0, \\ \operatorname{supp} \phi \subset \Omega_1 \cup \overline{D}. \end{cases} \tag{16}$$

Put

$$Y = \phi y, \qquad Z = \phi z. \tag{17}$$

First, we apply Lemma 2 to Z. Integrating (12) in $Q_1 = (0,T) \times \Omega_1$, noting that $\Theta(t)$ decays rapidly to 0 as $t \to 0+$ or $t \to T-$, by (16)–(17) and recalling that $Z|_\Sigma = 0$ (and hence $\nabla Z = \frac{\partial Z}{\partial \nu}\nu$ and $Z_i = \frac{\partial Z}{\partial \nu}\nu_i$ on Σ), using the fact that $(x - x_0) \cdot \nu(x) \leq 0$ on Σ (because we assume D is convex), that

$$Z_{tt} - \Delta Z = Z - Y - aY - 2\nabla\phi \cdot \nabla z - z\Delta\phi \qquad \text{in } Q_1,$$

and that

$$\int_0^T \int_{\partial\Omega_1 \setminus \Gamma} \theta^2 \Theta \left| \frac{\partial Z}{\partial \nu} \right|^2 (x - x_0) \cdot \nu(x) dx dt = 0$$

(because, by the second condition in (16), one has $Z \equiv 0$ on $(0,T) \times (\partial\Omega_1 \setminus \Gamma)$), one may deduce that

$$
\begin{aligned}
&\lambda\int_{Q_1}\theta^2\Theta(|Z_t|^2+|\nabla Z|^2)dxdt+\lambda^3\int_{Q_1}\theta^2\Theta|Z|^2dxdt\\
&\le C\Big\{\int_{Q_1}\theta^2\Theta|Z_{tt}-\Delta Z|^2dxdt+4\lambda\Big[\int_{\Sigma}\theta^2\Theta\Big|\frac{\partial Z}{\partial\nu}\Big|^2(x-x_0)\cdot\nu(x)dxdt\\
&\qquad\qquad+\int_0^T\int_{\partial\Omega_1\setminus\Gamma}\theta^2\Theta\left|\frac{\partial Z}{\partial\nu}\right|^2(x-x_0)\cdot\nu(x)dxdt\Big]\Big\}\\
&\le C\int_{Q_1}\theta^2\Theta|Z_{tt}-\Delta Z|^2dxdt \qquad (18)\\
&\le C\int_{Q_1}\theta^2\Theta[|aY|^2+|Y|^2+|Z|^2+|\nabla\phi\cdot\nabla z|^2+|z\Delta\phi|^2]dxdt\\
&\le C\Big\{\int_{Q_1}\theta^2\Theta[|aY|^2+|Y|^2+|Z|^2]dxdt\\
&\qquad\qquad+e^{C\lambda}\int_0^T\int_\omega(|z|^2+|\nabla z|^2)dxdt\Big\}.
\end{aligned}
$$

Similarly, applying Lemma 3 respectively to Y and Y_t, we deduce that

$$
\begin{aligned}
&\lambda\int_{Q_1}\theta^2\Theta|\nabla Y|^2dxdt+\lambda^3\int_{Q_1}\theta^2\Theta|Y|^2dxdt\\
&\le C\Big\{\int_{Q_1}\theta^2\Theta(|Y|^2+|Z|^2)dxdt+e^{C\lambda}\int_0^T\int_\omega(|y|^2+|\nabla y|^2)dxdt\Big\},
\end{aligned}
\qquad (19)
$$

and

$$
\begin{aligned}
&\lambda\int_{Q_1}\theta^2\Theta|\nabla Y_t|^2dxdt+\lambda^3\int_{Q_1}\theta^2\Theta|Y_t|^2dxdt\\
&\le C\Big\{\int_{Q_1}\theta^2\Theta[|Y_t|^2+|Z_t|^2+|\nabla\phi\cdot\nabla y_t|^2+|y_t\Delta\phi|^2]dxdt\Big\}\\
&\le C\Big\{\int_{Q_1}\theta^2\Theta[|ay|^2+|Y_t|^2+|Z_t|^2]dxdt\\
&\qquad+e^{C\lambda}\int_0^T\int_\omega\left(|y|^2+|\nabla y|^2+|\Delta y|^2+|\nabla\Delta y|^2\right)dxdt\Big\}.
\end{aligned}
\qquad (20)
$$

Similar to [8], from (18)-(20), it follows

$$
\begin{aligned}
&\lambda \int_{Q_1} \theta^2 \Theta[|\Delta Y_t|^2 + |\nabla \Delta Y|^2] dxdt + \lambda^2 \int_{Q_1} \theta^2 \Theta |\nabla Y_t|^2 dxdt \\
&\quad + \lambda^3 \int_{Q_1} \theta^2 \Theta[|\Delta Y|^2 + |Y_t|^2] dxdt + \lambda^4 \int_{Q_1} \theta^2 \Theta |\nabla Y|^2 dxdt \\
&\quad + \lambda^6 \int_{Q_1} \theta^2 \Theta |Y|^2 dxdt \qquad (21) \\
&\leq C \Big\{ ||\theta \sqrt{\Theta} a y||^2_{L^2(Q)} \\
&\qquad + e^{C\lambda} \int_0^T \int_\omega \left(|y|^2 + |\nabla y|^2 + |\Delta y|^2 + |\nabla \Delta y|^2 \right) dxdt \Big\}.
\end{aligned}
$$

Adding both sides of (21) by

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n \setminus \overline{\Omega_0 \cup D}} \theta^2 \Theta \Big\{ \lambda[|\Delta Y_t|^2 + |\nabla \Delta Y|^2] + \lambda^2 |\nabla Y_t|^2 \\
&\qquad\qquad + \lambda^3[|\Delta Y|^2 + |Y_t|^2] + \lambda^4 |\nabla Y|^2 + \lambda^6 |Y|^2 \Big\} dxdt,
\end{aligned}
$$

one may deduce that

$$
\begin{aligned}
&\lambda \int_{Q} \theta^2 \Theta[|\Delta y_t|^2 + |\nabla \Delta y|^2] dxdt + \lambda^2 \int_{Q} \theta^2 \Theta |\nabla y_t|^2 dxdt \\
&\quad + \lambda^3 \int_{Q} \theta^2 \Theta[|\Delta y|^2 + |y_t|^2] dxdt + \lambda^4 \int_{Q} \theta^2 \Theta |\nabla y|^2 dxdt \\
&\quad + \lambda^6 \int_{Q} \theta^2 \Theta |y|^2 dxdt \qquad (22) \\
&\leq C \Big\{ ||\theta \sqrt{\Theta} a y||^2_{L^2(Q)} \\
&\qquad + e^{C\lambda} \int_0^T \int_\omega \left(|y|^2 + |\nabla y|^2 + |\Delta y|^2 + |\nabla \Delta y|^2 \right) dxdt \Big\}.
\end{aligned}
$$

Finally, using the same argument as in [8], the desired estimates (2) and (3) follow from (22) and Lemma 1.

4 Optimality of the observability constant for Kirchhoff plate systems in the whole space

This section is devoted to analyze the optimality of the observability inequality for system (1).

4.1 Optimality in the whole space

First, we shall show that when $p = \infty$, the term $||a||_p^{\frac{1}{3-5n/2p}}$ (i.e., $||a||_\infty^{1/3}$) in the estimate (3) is sharp in what concerns the exponential dependence on the potential a for systems with at least two equations in $\mathbb{R}^n$ for even $n \geq 2$. More precisely, the following holds:

Theorem 2. *Assume that $\Omega = \mathbb{R}^n$ (i.e., $D = \emptyset$), $n \geq 2$ is even and that $N \geq 2$. Let ω be an open non-empty subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \overline{\omega} \neq \emptyset$. Then, there exist a constant $c > 0$, a family of time-independent potentials $\{a_R\}_{R>0} \subset L^\infty(\Omega; \mathbb{R}^{N\times N})$ satisfying*

$$||a_R||_\infty \to \infty, \qquad \text{as } R \to \infty$$

and a family of initial data $\{(y_R^0, y_R^1)\}_{R>0} \in \mathcal{H}$ such that for any $T > 0$, the corresponding weak solutions $\{y_R\}_{R>0}$ of (1) satisfy

$$\lim_{R\to\infty} \left\{ \frac{||(y_R^0, y_R^1)||_{\mathcal{H}}^2}{\exp\left(c||a_R||_\infty^{1/3}\right) \int_0^T \int_\omega \left(|y|^2 + |\nabla y|^2 + |\Delta y|^2 + |\nabla \Delta y|^2\right) dtdx} \right\} = \infty. \tag{23}$$

The main idea to prove this optimality result is the same as that in [2], which is based on a suitable construction of u and q satisfying the following bi-Laplacian equation:

$$\Delta^2 u = qu, \qquad \text{in } \mathbb{R}^n, \tag{24}$$

which decays at infinity sufficiently fast. More precisely, following Meshkov's construction [6, 2, 3], we have the following result on u and q for (24):

Lemma 4. *Let $n \geq 2$ be even. Then there exist two nontrivial complex-valued functions:*

$$u \in C^\infty(\mathbb{R}^n; \mathbb{C}), \quad q \in C^\infty(\mathbb{R}^n; \mathbb{C}) \bigcap L^\infty(\mathbb{R}^n; \mathbb{C})$$

such that (24) is satisfied, and for some constant C:

$$|u(x)| + |\nabla u(x)| + |\Delta u(x)| + |\nabla \Delta u(x)| \leq Ce^{-|x|^{4/3}}, \qquad \forall\, x \in \mathbb{R}^n. \tag{25}$$

Since we assume that $\Omega = \mathbb{R}^n$, the proof of Theorem 2 via Lemma 4 is quite easy. Indeed, consider the solution u and potential q on $\mathbb{R}^n$ given by Lemma 4. Recalling that both u and q are complex-valued, by setting

$$u_R(x) = \begin{pmatrix} \operatorname{Re} u(Rx) \\ \operatorname{Im} u(Rx) \end{pmatrix}, \qquad a_R(x) = -R^4 \begin{pmatrix} \operatorname{Re} u(Rx) & -\operatorname{Im} q(Rx) \\ \operatorname{Im} q(Rx) & \operatorname{Re} q(Rx) \end{pmatrix}, \tag{26}$$

we obtain a family of potentials $\{a_R\}_{R>0}$ and solutions $\{u_R\}_{R>0}$ satisfying

$$\Delta^2 u_R + a_R(x) u_R = 0 \qquad \text{in } \mathbb{R}^n, \tag{27}$$

and

$$\begin{aligned} &|u_R(x)| + |\nabla u_R(x)| + |\Delta u_R(x)| + |\nabla \Delta u_R(x)| \\ &\le C \exp\Big(- R^{4/3} |x|^{4/3} \Big) \qquad \text{in } \mathbb{R}^n. \end{aligned} \tag{28}$$

Furthermore, for some constant $C > 0$, the potential a_R is such that

$$C^{-1} R^4 \le ||a_R||_\infty \le C R^4, \quad \text{if } n \text{ is even.} \tag{29}$$

Set

$$\psi_R(t, x) = u_R(x), \qquad (t, x) \in [0, T] \times \mathbb{R}^n. \tag{30}$$

The functions $\{\psi_R\}_{R>0}$ may also be viewed as a family of stationary solutions of the Cauchy problem

$$\begin{cases} \psi_{R,\,tt} + \Delta^2 \psi_R - \Delta \psi_{R,\,tt} + a_R \psi_R = 0 & \text{in } (0,T) \times \mathbb{R}^n, \\ \psi_R(0) = u_R, \quad \psi_{R,\,t}(0) = 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{31}$$

with potentials $a_R = a_R(x)$ as in (26).

By (28) and (30), we have

$$\begin{aligned} &|\psi_R(x,t)| + |\nabla \psi_R(x,t)| + |\Delta \psi_R(x,t)| + |\nabla \Delta \psi_R(x,t)| \\ &\le C \exp\Big(- R^{4/3} |x|^{4/3} \Big) \quad \text{in } \mathbb{R}^n. \end{aligned} \tag{32}$$

Without loss of generality, assume that $\omega \subset \mathbb{R}^n \setminus B$, where B is the unit ball in $\mathbb{R}^n$. Then

$$\begin{aligned} &\int_0^T \int_\omega \Big(|\psi_R|^2 + |\nabla \psi_R|^2 + |\nabla \Delta \psi_R|^2 + |\Delta \psi_R|^2 \Big) dt dx \\ &\le C \int_{\mathbb{R}^n \setminus B} \exp(-2R^{4/3} |x|^{4/3}) dx \\ &\le C \exp(-R^{4/3}) \int_{\mathbb{R}^n \setminus B} \exp(-R^{4/3} |x|^{4/3}) dx \le C \exp(-R^{4/3}). \end{aligned} \tag{33}$$

Now, combining (26), (29) and (33), one establishes (23) immediately.

4.2 An open problem: optimality for general domains

One could expect to be able to use Lemma 4 to establish similar optimality results in general domains (i.e., $\Omega \neq \mathbb{R}^n$). However, this is an open problem. Indeed, as in the above subsection, based on the construction of u and q in Lemma 4, one can find a family of rescaled potentials $a_R(x) = R^4 q(Rx)$ with an L^∞-norm of the order of R^4 and a family of solutions $u_R(x) = u(Rx)$ of the corresponding bi-harmonic problem in Ω. These solutions can be regarded also as solutions of the Kirchhoff plate system for suitable initial data. However, they do not fulfill homogeneous boundary conditions. Therefore, one needs to compensate them by subtracting the solution taking their boundary data and zero initial ones. In turn, one has to show that these solutions are as small as $\exp\Big(-||a_R||_\infty^{1/3}\Big)$ in the energy space $\mathcal{H}$. However, by inequality (4) in Lemma 1, the energy estimate yields an exponential growth $\exp\Big(T||a_R||_\infty^{1/2}\Big)$ for the energy evolution, and it has to be used in the whole time duration $[0, T]$. Since one needs to take the time T to be large enough, this breaks down the concentration effect that Meshkov's construction guarantees.

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