Boundary observability for the space semi-discretizations of the $1 - d$ wave equation

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Abstract

We consider space semi-discretizations of the $1 - d$ wave equations in a bounded interval with homogeneous Dirichlet boundary conditions. We analyze the problem of boundary observability, i.e., the problem of whether the total energy of solutions can be estimated uniformly in terms of the energy concentrated on the boundary as the net-spacing $h \to 0$. We prove that, due to the spurions modes that the numerical scheme introduces at high frequencies, there is no such a uniform bound. We prove however a uniform bound in a subspace of solutions generated by the low frequencies of the discrete system. When $h \to 0$ this finite-dimensional spaces increase and eventually cover the whole space. We thus recover the well-known observability property of the continuous systems as the limit of discrete observability estimates as the mesh size tends to zero. We consider both finite-difference and finite-element semi-discretizations.

1   Introduction: Finite-difference semi-discretizations

Consider the $1 - d$ wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < L, \quad 0 < t < T \\ u(0,t) = u(L,t) = 0, & 0 < t < T \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & 0 < x < L. \end{cases} \tag{1.1}$$

System (1.1) is well-posed in the energy space $H_0^1(0,L) \times L^2(0,L)$. More precisely, for any $(u_0, u_1) \in H_0^1(0,L) \times L^2(0,L)$ there exists a unique solution $u \in C\left([0,T]; H_0^1(0,L)\right) \cap C^1\left([0,T]; L^2(0,L)\right)$.

The energy of solutions is given by

$$E(t) = \frac{1}{2} \int_0^L \left[\mid u_t(x,t) \mid^2 + \mid u_x(x,t) \mid^2 \right] dx \tag{1.2}$$

and it is conserved along time, i.e.

$$E(t) = E(0), \quad \forall 0 < t < T. \tag{1.3}$$

It is by now well known that when $T > 2L$, the total energy of solutions can be estimated uniformly by means of the energy concentrated on one extreme of the boundary, say, $x = L$. More precisely, for any $T > 2L$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T | u_x(L, t) |^2 \, dt \tag{1.4}$$

for every finite energy solution of (1.1).

When the energy concentrated on the boundary is measured in both extremes $x = 0$ and L, the inequality holds for all $T > L$.

In this paper we focus on inequality (1.4).

Inequalities of the form (1.4) are related to the boundary controllability of the wave equations. We refer to [K] and [L] for a systematic analysis of these issues, both in the context of wave equations and plate models.

In this paper we analyze the analogue of (1.4) for several space semi-discretizations of the wave equation.

Let us consider first the finite-difference semi-discretization to illustrate the kind of problems we habe in mind.

Given $n \in I\!N$ we set $h = L/(N+1)$ and introduce the net

$$x_0 = 0 < x_1 = h < \cdots < x_N = Nh < x_{N+1} = L \tag{1.5}$$

with $x_j = jh$, $j = 0, \cdots, N+1$.

We then introduce the following finite-difference semi-discretization of (1.1)

$$\begin{cases} u_j''(t) = \frac{[u_{j+1}(t) + u_{j-1}(t) - 2u_j(t)]}{h^2}, 0 < t < T, j = 1 \cdots N \\ u_0(t) = u_{N+1}(t) = 0, \quad 0 < t < T \\ u_j(0) = u_j^0, \, u_j'(0) = u_j^1, \quad j = 0, \cdots, N+1. \end{cases} \tag{1.6}$$

In (1.6) $'$ denotes derivation with respect to time.

System (1.6) is a system of N linear differential equations with N unknowns $u_1, \cdots, u_N$, since, in view of the boundary conditions, $u_0 \equiv u_{N+1} \equiv 0$.

Obviously, $u_j(t)$ is an approximation of $u(x_j, t)$, u being the solution of (1.1), provided the initial data $\left(u_j^0, u_j^1\right), j = 0, \cdots, N+1$ are an approximation of the initial data in (1.1).

The energy of system (1.6) is given by

$$E_h(t) = \frac{h}{2} \sum_{j=0}^N \left[| u_j'(t) |^2 + \left| \frac{u_{j+1}(t) - u_j(t)}{h} \right|^2 \right] \tag{1.7}$$

which is a discretization of the continuous energy E.

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It is easy to see that the energy E_h is conserved along time for the solutions of (1.6), i.e.

$$E_h(t) = E_h(0), \quad \forall 0 < t < T. \tag{1.8}$$

The main goal of this paper is to analyze the following discrete version of (1.4):

$$E_h(0) \le C(T, h) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt. \tag{1.9}$$

Note that in (1.9), $u_N(t)/h$ represents an approximation of the normal derivative $u_x(L, t)$ of the solution of the continuous system (1.1) since $u_{N+1}(t) \equiv 0$.

In view of (1.4) one may except that, when $T > 2L$, there exists $C = C(T) > 0$ independent of h such that (1.9) holds for every solution of (1.6) and for every $0 < h < 1$.

The first result of this paper asserts that this is false:

Theorem 1.1 *For any $T > 0$, we have*

$$\sup_{u \text{solution of } (1.6)} \left[\frac{E_h(0)}{\int_0^T | u_N(t)/h |^2 \, dt} \right] \to \infty \; as \; h \to 0. \tag{1.10}$$

As we shall see, this is due to the spurious modes that the numerical scheme introduces at high frequencies. This was already observed by R. Glowinski et al. in [G], [GLL] and [GL], in connection with the exact boundary controllability of the wave equation in several space dimensions and the numerical implementation of the so-called HUM method (see J.L. Lions [L]). In these works two methods were proposed to cure this high frequency pathology: (a) A Tychonoff regularization procedure for the quadratic functional to be minimized when computing the controls; (b) A filtering technique to eliminate the short wave length components of the solutions of the discrete system. The efficiency of both methods was exhibited in these works by various numerical experiments.

To prove Theorem 1.1 we analyze the spectrum of (1.6) and we use discrete multiplier techniques to derive sharp observability inequalities of the form (1.9) which are uniform as $h \to 0$. As we mentioned above, in order for these inequalities to be uniform, one has to rule out the high frequency spurious modes introduced by the numerical scheme. This will be done by considering suitable classes of solutions of (1.6) generated by the low frequency eigenvectors of (1.6), or, in other words, by a suitable truncation of the Fourier development of solutions of (1.6). Thus, our approach is very close to the filtering technique mentioned above (we refer to R. Glowinski [G] for a complete discussion of this issue).

To make our statements precise, let us consider the eigenvalue problem associated with (1.6):

$$\begin{cases} -\left[\frac{\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j}{h^2} \right] = \lambda \varphi_j, \quad j = 0, \cdots, N + 1 \\ \varphi_0 = \varphi_{N+1} = 0. \end{cases} \tag{1.11}$$

Let us denote by $\lambda_1(h), \cdots, \lambda_N(h)$ the N eigenvalues of (1.11):

$$0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_N(h). \tag{1.12}$$

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These eigenvalues can be computed explicetely. We have (see [IK], p. 456):

$$\lambda_k(h) = \frac{4}{h^2} \sin^2 \left(\frac{\pi_k h}{2L} \right), j = 1, \cdots, N. \tag{1.13}$$

The eigenfunction $\varphi^k = \left(\varphi_1^k, \cdots, \varphi_n^k \right)$ associated to the eigenvalue $\lambda_k(h)$ can also be computed explicitely:

$$\varphi_j^k = \sin \left(j \frac{\pi h k}{2L} \right), j = 1, \cdots, N. \tag{1.14}$$

Solutions of (1.6) admit a Fourier development on the basis of eigenvectors of system (1.11). More precisely, every solution $u = (u_1, \cdots, u_N)$ of (1.6) can be written as

$$u(t) = \sum_{k=1}^{N} \left[a_k \sin \left(\sqrt{\lambda_k(h)} t \right) + b_k \cos \left(\sqrt{\lambda_k(h)} t \right) \right] \varphi^k \tag{1.15}$$

for suitable coefficients $a_k, b_k \in I\!R$, $k = 1, \cdots, N$, that can be computed explicitly in terms of the initial data in (1.6).

Before getting into the discussion of the observability of solutions of (1.6) it is interesting to analyze the boundary observability of the eigenvectors. The following Lemma provides the answer:

Lemma 1.1 *For any eigenvector $\varphi = (\varphi_1, \cdots, \varphi_N)$ of system (1.11) the following identity holds:*

$$h \sum_{j=0}^{N} \left| \frac{\varphi_{j+1} - \varphi_j}{h} \right|^2 = \frac{2L}{4 - \lambda h^2} \left| \frac{\varphi_N}{h} \right|^2. \tag{1.16}$$

This identity provides an explicit relation between the total energy of the eigenvectors (the left hand side of (1.16)) and the energy concentrated on the extreme $x = L$ which is represented by the quantity $\mid \varphi_N / h \mid^2$.

On the other hand, it is easy to check that

$$\lambda h^2 < 4 \tag{1.17}$$

for all $h > 0$ and all the eigenvalues of (1.11). But, obviously, (1.17) does not exclude the blow up of the constant in the right hand side of (1.16). In fact, on can check that

$$\lambda_N(h) h^2 \to 4 \text{ as } h \to 0. \tag{1.18}$$

Therefore blow-up occurs. This immediately yields the negative result of Theorem 1.1.

In order to obtain a positive counterpart to Theorem 1.1 we have to introduce suitable subclasses of solutions of (1.6). Given any $\gamma < 4$ we introduce the class $\mathcal{C}_h(\gamma)$ of solutions of (1.6) generated by eigenvectors of (1.11) associated with eigenvalues such that

$$\lambda h^2 \leq \gamma. \tag{1.19}$$

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More precisely,

$$\mathcal{C}_h(\gamma) := \left\{ u = \sum_{\lambda_k(h) \leq \gamma h^{-2}} \left[a_k \sin\left(\sqrt{\lambda_k(h)}t\right) + b_k \cos\left(\sqrt{\lambda_k(h)}t\right) \right] \varphi^k \text{ with } a_k, b_k \in I\!R \right\}. \tag{1.20}$$

According to Lemma 1.1, the energy of every eigenvector entering in $\mathcal{C}_h(\gamma)$ can be estimated uniformly in terms of the energy concentrated on the boundary.

The following result guarantees that this is in fact the case for all solutions of (1.6) in the class $\mathcal{C}_h(\gamma)$ provided the length T of the time interval is large enough:

Theorem 1.2 *Assume that $\gamma < 4$. Then, there exists $T(\gamma) \geq 2L$ such that for all $T > T(\gamma)$ there exists $C = C(T, \gamma)$ such that (1.9) holds for every solution of (1.6) in the class $\mathcal{C}_h(\gamma)$, uniformly as $h \to 0$.*

Moreover,

(a) *$T(\gamma) \nearrow \infty$ as $\gamma \nearrow 4$ and $T(\gamma) \searrow 2L$ as $\gamma \searrow 0$.*

(b) *$C(T, \gamma) \nearrow$ as $\gamma \nearrow 4$ and $C(T, \gamma) \searrow \frac{L}{2(T-2L)}$ as $\gamma \searrow 0$.*

Remark 1.1 Theorem 1.2 asserts that the uniform observability inequality (1.9) holds in the class $\mathcal{C}_h(\gamma)$ provided T is large enough. In fact, $T(\gamma) \to \infty$ as $\gamma \to 4$. This is due to the fact that the gap between the roots of consecutive eigenvalues vanishes as they approach the critical value. However, as $\gamma \to 0$ the observability time $T(\gamma)$ converges to $2L$, which is the observability time for system (1.1). Note that, according to this result, the uniform observability inequality (1.9) holds for $T > 2L$ for solutions of (1.6) of the form

$$u = \sum_{\lambda_k(h) \leq \mu(h)} \left[a_k \sin\left(\sqrt{\lambda_k(h)}t\right) + b_k \cos\left(\sqrt{\lambda_k(h)}t\right) \right] \varphi k \tag{1.21}$$

with $\mu(h)$ such that

$$\mu(h)h^2 \to 0 \text{ as } h \to 0. \tag{1.22}$$

This allows to recover the observability of the original system (1.1) as the limit as $h \to 0$ of the observability of solutions of the form (1.21)-(1.22) of the semi-discrete system (1.6).

We also observe that the constant $C(T, \gamma)$ of the observability inequality (1.9) converges to $L/2(T-2L)$, which is the constant that one obtains by multiplier techniques for the observability of the continuous system (1.1).

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Roughly speaking, Theorem 1.2 guarantees that the semi-discrete systems are uniformly observable as $h \to 0$ provided the high frequencies are filtered.

We shall give two proof of Theorem 1.2. The first one is an adaptation of the classical multiplier techniques that are used to prove the observability of wave and plate equations (see

[K] and [L]). The second one is based on the classical inequality by Inghan [I] for non-harmonic Fourier series.

One may think that this results are due to the particular finite-difference discretization we have considered. But this is not the case. We also consider the finite-element space semi-discretization and obtain similar results, both, in what concerns the negative result of Theorem 1.1 but also the positive one of Theorem 1.2.

It is also worth mentioning that the phenomena we have described here for the discretizations of the wave equation have been found earlier in the context of the observability of the $1 - d$ wave equation with rapidly oscillating periodic coefficients.

Roughly, in both cases, the interaction of waves with the microstructure or the discrete mesh produces spurious high frequency vibrations that are not observed in the limit continuous model.

The rest of the paper is organized as follows. Section 2 is devoted to the analysis of the finite-difference approximation. Inparticular, we develop and prove the results stated in this introduction. Section 3 is enterely dedicated to the analysis of the finite-element discretization.

2 Finite-difference semi-discretization

In this section we analyze in detail the problem of the observability of the finite-difference space semi-discretization (1.6) of the wave equation (1.1) that we have discussed in the introduction.

First of all we perform a careful analysis of the spectrum. In particular we prove Lemma 1.1 and, as an immediate consequence of it, Theorem 1.1. We then prove Theorem 1.2 in detail using multiplier techniques. We also indicate how the same results can be recovered using well-known results on non-harmonic Fourier series.

2.1 Spectral analysis

Let us recall the system that eigenfunctions $\varphi = (\varphi_1, \cdots, \varphi_N)$ and eigenvalues λ of system (1.6) satisfy:

$$\begin{cases} -\left[\frac{\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j}{h^2}\right] = \lambda\varphi_j, \ j = 1, \cdots, N \\ \varphi_0 = \varphi_{N+1}. \end{cases} \tag{2.1}$$

The eigenvalues and eigenvectors of system (2.1) can be computed explicitely. We have (see [IK], p. 456):

$$\lambda_k(h) = \frac{4}{h^2} \sin^2\left(\frac{\pi_k h}{2L}\right), \ j = 1, \cdots, N \tag{2.2}$$

and

$$\varphi^k = \left(\varphi_j^k\right) = \sin\left(\frac{j\pi h_k}{2L}\right), \ k = 1, \cdots, N, \ j = 1, \cdots, N. \tag{2.3}$$

Observe in particular that eigenvectors of the discrete system coincide with the eigenfunctions $\sin\left(\frac{\pi k x}{L}\right)$ of the continuous one. On the other hand, for k fixed,

$$\lambda_k(h) \to \frac{\pi^2 k^2}{L^2}, \ \text{ as } h \to 0 \tag{2.4}$$

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which is the $k-th$ eigenvalue of the continuous system.

The gap between consecutive eigenvalues plays an important role on the analysis of the boundary observability problem, since very eigenvalues produces time harmonics at $x = L$ that are almost indistinguisible. For the continuous model we have

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = \frac{(k+1)\pi}{L} - \frac{k\pi}{L} = \frac{\pi}{L}. \tag{2.5}$$

Thus, the gap, $\frac{\pi}{L}$, is independent of the frequency.

However, as we shall see, the gap between consecutive eigenvalues in the discrete problem decreases at high frequencies and it is of the order of h as $h \to 0$.

We have

$$
\begin{aligned}
\sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} &= \frac{2}{h}\left[\sin\left(\frac{\pi h(k+1)}{2L}\right) - \sin\left(\frac{\pi hk}{2L}\right)\right] \\
&= \frac{2}{h}\left[\sin\frac{\pi hk}{2L}\left(\cos\left(\frac{\pi h}{2L}\right) - 1\right) + \sin\left(\frac{\pi h}{2L}\right)\cos\left(\frac{\pi hk}{2L}\right)\right] \\
&\leq \frac{2}{h}\left|1 - \cos\left(\frac{\pi h}{2L}\right)\right| + \frac{\pi}{L}\cos\left(\frac{\pi hk}{2L}\right)
\end{aligned}
$$

and, taking into account that $(N+1)h = L$,

$$
\begin{aligned}
\cos\left(\frac{\pi hk}{2L}\right) &= \cos\left(\frac{\pi}{2} + \frac{(k-(N+1))h\pi}{2L}\right) \\
&= \sin\left(\frac{((N+1)-k)h\pi}{2L}\right)
\end{aligned}
$$

and that

$$\frac{2}{h}\left|1 - \cos\left(\frac{\pi h}{2L}\right)\right| \leq \frac{\pi^2 h}{2L^2}$$

we deduce

$$\sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} \leq \frac{\pi^2 h}{2L^2} + \frac{\pi}{L}\sin\left(\frac{((N+1)-k)h\pi}{2L}\right). \tag{2.6}$$

Therefore, as soon as

$$N + 1 - k \leq j \Leftrightarrow k \geq N + 1 - j \tag{2.7}$$

for some $j \in I\!N$ we have

$$\sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} \leq \frac{\pi^2 h}{2L^2} + \frac{\pi^2 jh}{2L^2} = \frac{\pi^2}{2L^2}h(j+1) \tag{2.8}$$

This shows that the gap between the roots of the largest j eigenvalues is of the order of h, with a multiplicative factor that increases as j increases.

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Reciprocally, it can be shown that the gap remains bounded below for the low eigenvalues. Indeed,

$$\sqrt{\lambda_{k+1}(h)} - \sqrt{\lambda_k(h)} \geq \frac{2}{h} \sin\left(\frac{\pi h}{2L}\right) \cos\left(\frac{\pi h k}{2L}\right) - \frac{\pi^2 h}{2L^2} \tag{2.9}$$
$$= \left[\frac{\pi}{L} - O(h)\right] \cos\left(\frac{\pi h k}{2L}\right) - \frac{\pi^2 h}{2L^2}.$$

The right hand side of this inequality converges to $\frac{\pi}{L}$ as $h \to 0$ when k remains bounded, or even if k is unbounded but $hk \to 0$. Recall that π/L is the gap between the roots of the eigenvalues in the continuous model.

2.2 Boundary observability of eigenvectors

The goal of this section is to prove the identity (1.16) of Lemma 1.1.

In view of the explicit values of the eigenvalues and eigenvectors in (2.2) and (2.3) this identity can be checked easily. However, we shall prove it using multiplier techniques.

We multiply in (2.1) by $j(\varphi_{j+1} - \varphi_{j-1})/2$ and add for $j = 1, \cdots, N$. Note that this is a discrete version of the multiplier $x\varphi_x$. We obtain

$$\frac{1}{h^2} \sum_{j=1}^N [\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j] \, j \frac{(\varphi_{j+1} - \varphi_{j-1})}{2}$$
$$= \frac{1}{h^2} \sum_{j=1}^N \frac{j}{2} \left[|\varphi_{j+1}|^2 - |\varphi_{j-1}|^2\right] - \frac{1}{h^2} \sum_{j=1}^N j \left(\varphi_{j+1} - \varphi_{j-1}\right) \varphi_j$$
$$= -\frac{1}{h^2} \sum_{j=1}^N |\varphi_j|^2 + \frac{N+1}{2h^2} \mid \varphi_N \mid^2 + \frac{1}{h^2} \sum_{j=1}^N \varphi_j \varphi_{j+1}$$

On the other hand,

$$\sum_{j=1}^N j\varphi_j \left(\frac{\varphi_{j+1} - \varphi_{j-1}}{2}\right) = -\frac{1}{2} \sum_{j=1}^N \varphi_j \varphi_{j+1}.$$

Therefore

$$\frac{1}{h^2} \left[\sum_{j=1}^N \mid \varphi_j \mid^2 - \frac{N+1}{2} \mid \varphi_N \mid^2 - \sum_{j=1}^N \varphi_j \varphi_{j+1}\right] = -\frac{\lambda}{2} \sum_{j=1}^N \varphi_j \varphi_{j+1}. \tag{2.10}$$

On the other hand, multiplying by φ_j we get

$$-\frac{1}{h^2} \sum_{j=1}^N [\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j] \varphi_j = \lambda \sum_{j=1}^N \varphi_j^2. \tag{2.11}$$

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We also have

$$\sum_{j=1}^{N} \left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right]\varphi_j = 2\sum_{j=1}^{N}\left(\varphi_{j+1}\varphi_j - \varphi_j^2\right). \qquad (2.12)$$

Combining (2.11) and (2.12) we deduce that

$$\frac{2}{h^2}\sum_{j=1}^{N}\left[\varphi_j^2 - \varphi_j\varphi_{j+1}\right] = \lambda\sum_{j=1}^{N}\varphi_j^2. \qquad (2.13)$$

We normalize the eigenvector so that

$$h\sum_{j=1}^{N}\varphi_j^2 = 1. \qquad (2.14)$$

In view of (2.13) we deduce that

$$\frac{h}{2}\left(\frac{2}{h^2} - \lambda\right) = \sum_{j=1}^{N}\varphi_j\varphi_{j+1} = \sum \qquad (2.15)$$

Going back to (2.10) we deduce that

$$\frac{1}{h^2}\left[\frac{1}{h} - \frac{N+1}{2}\mid\varphi_N\mid^2 -\frac{h}{2}\left(\frac{2}{h^2} - \lambda\right)\right] = -\frac{\lambda h}{4}\left(\frac{2}{h^2} - \lambda\right).$$

In other words,

$$\frac{(N+1)h}{2}\left|\frac{\varphi_N}{h}\right|^2 = \frac{L}{2}\left|\frac{\varphi_N}{h}\right|^2 = \frac{1}{h^2}\left[1 - \frac{h^2}{2}\left(\frac{2}{h^2} - \lambda\right)\right] + \frac{\lambda h^2}{4}\left(\frac{2}{h^2} - \lambda\right) \qquad (2.16)$$

$$= \lambda - \frac{\lambda^2 h^2}{4} = \lambda\left(1 - \frac{\lambda h^2}{4}\right).$$

On the other hand, taking into account that

$$2\sum_{j=1}^{N}\left[\varphi_j^2 - \varphi_j\varphi_{j+1}\right] = \sum_{j=0}^{N}\mid\varphi_j - \varphi_{j+1}\mid^2$$

and (2.13) we have

$$h\sum_{j=0}^{N}\left|\frac{\varphi_j - \varphi_{j+1}}{h}\right|^2 = \lambda. \qquad (2.17)$$

Combining (2.16) and (2.17) we deduce that

$$h\sum_{j=0}^{N}\left|\frac{\varphi_j - \varphi_{j+1}}{h}\right|^2 = \frac{2L}{(4 - \lambda h^2)}\left|\frac{\varphi_N}{h}\right|^2. \qquad (2.18)$$

This completes the proof of Lemma 1.1.

∎

2.3 Proof of the non-uniform observability

This section is devoted to prove Theorem 1.1. As indicated in the Introduction, it is an inmediate consequence of Lemma 1.1. Indeed, let u be the solution of (1.6) associated to the $N-th$ eigenvectors, i.e.,

$$u = e^{i\sqrt{\lambda_n(h)t}}\varphi_N. \tag{2.19}$$

According to Lemma 1.1 we have

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = T \left|\frac{\varphi_{N,N}}{h}\right|^2 = \frac{Th}{2L}\left(4 - \lambda_N(h)h^2\right)\sum_{j=0}^N \left|\frac{\varphi_{N,j} - \varphi_{N,j+1}}{h}\right|^2. \tag{2.20}$$

On the other hand

$$
\begin{aligned}
E_h(0) &= \frac{h}{2}\sum_{j=0}^N \left[\frac{|\varphi_{N,j} - \varphi_{N,j+1}|^2}{h^2} + \lambda_N(h)\,|\varphi_{N,j}|^2\right] \\
&= h\sum_{j=0}^N \left[\frac{|\varphi_{N,j} - \varphi_{N,j+1}|^2}{h^2}\right].
\end{aligned}
\tag{2.21}
$$

In view of (2.20) and (2.21) we deduce that

$$\frac{E_h(0)}{\displaystyle\int_0^T |u_N(t)/h|^2\,dt} = \frac{2L}{T\left(4 - \lambda_N(h)h^2\right)}. \tag{2.22}$$

Moreover, in view of (2.2):

$$
\begin{aligned}
\lambda_N(h)h^2 &= 4\sin^2\left(\frac{\pi N h}{2L}\right) = 4\sin^2\left(\frac{\pi}{2} - \frac{h\pi}{2L}\right) \\
&= 4\cos\left(\frac{h\pi}{2L}\right) \to 4 \text{ as } h \to 0.
\end{aligned}
\tag{2.23}
$$

Combining (2.22) and (2.23), Theorem 1.1 follows inmediately.

∎

2.4 Boundary observability of the discrete wave equation: the multiplier method

This section is devoted to prove Theorem 1.2 using the multiplier method. First of all we stablish some basic identities.

Lemma 2.1 (Conservation of energy) *For any $h > 0$ and u solution of (1.6) we have*

$$E_h(t) = E_h(0), \quad \forall t \in [0, T]. \tag{2.24}$$

Proof. We multiply in (1.6) by $u_j'(t)$ and add for $j = 1, \cdots, N$.

We have

$$\sum_{j=1}^{N} u_j'' u_j' = \frac{1}{h^2} \sum_{j=1}^{N} (u_{j+1} + u_{j-1} - 2u_j) \, u_j'. \tag{2.25}$$

On the other hand,

$$\sum_{j=1}^{N} u_j'' u_j' = \frac{1}{2} \frac{d}{dt} \sum_{j=1}^{N} |u_j'|^2, \tag{2.26}$$

and

$$-\sum_{j=1}^{N} (u_{j+1} + u_{j-1} - 2u_j) \, u_j' = \frac{1}{2} \frac{d}{dt} \sum_{j=0}^{N} |u_j - u_{j+1}|^2 \tag{2.27}$$

Combining (2.25)-(2.27) we deduce that

$$\frac{d}{dt} \sum_{j=0}^{N} \left[|u_j'|^2 + |u_j - u_{j+1}|^2 \right] = 0$$

which is equivalent to (2.24).

$\blacksquare$

Lemma 2.2 *For any $h > 0$ and u solution of (1.6) we have*

$$\frac{h}{2} \sum_{j=0}^{N} \int_0^T \left[u_j' u_{j+1}' + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right] dt + X_h(t) \Big|_0^T = \frac{L}{2} \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{2.28}$$

with

$$X_h(t) = h \sum_{j=1}^{N} j \left(\frac{u_{j+1} - u_{j-1}}{2} \right) u_j'. \tag{2.29}$$

Remark. Identity (2.28) is the discrete analogue of the well-known identity for the $1 - d$ wave equation (1.1) obtained by multipliers that reads as follows (see [L]):

$$\frac{1}{2} \int_0^T \int_0^L \left[|u_t|^2 + |u_x|^2 \right] dx dt + X(t) \Big|_0^T = \frac{L}{2} \int_0^T |u_x(1, t)|^2 \, dt \tag{2.30}$$

with

$$X(t) = \int_0^L x \, u_x \, u_t \, dx. \tag{2.31}$$

Note that main difference between (2.28)-(2.29) and (2.30)-(2.31) is that, in (2.28), we get

$$h \sum_{j=0}^{N} \int_0^T u_j' u_{j+1}' dt$$

as a discretization of $\int_0^T \int_0^L u_t^2 dx dt$, which is not a positive definite quantity.

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■

Proof of Lemma 2.2.

We multiply in (1.6) by $j\left(u_{j+1}-u_{j-1}\right)/2$ which is a discrete version of the classical multiplier xu_x for the wave equation. We obtain

$$\sum_{j=1}^{N}\int_{0}^{T}u_j''\frac{(u_{j+1}-u_{j-1})}{2}dt=\frac{1}{h^2}\sum_{j=1}^{N}\int_{0}^{T}(u_{j+1}+u_{j-1}-2u_j)\,j\frac{(u_{j+1}-u_{j-1})}{2}dt. \qquad (2.32)$$

We now develop the two terms in this identity. For the first one we have:

$$\begin{aligned}
\sum_{j=1}^{N}\int_{0}^{T}u_j''j\frac{(u_{j+1}-u_{j-1})}{2}dt &= -\sum_{j=1}^{N}\int_{0}^{T}u_j'j\frac{\left(u_{j+1}'-u_{j-1}'\right)}{2}dt \qquad (2.33)\\
&\quad +\sum_{j=1}^{N}u_j'j\left.\frac{(u_{j+1}-u_{j-1})}{2}\right|_0^T\\
&= \frac{1}{2}\sum_{j=1}^{N}\int_{0}^{T}u_j'u_{j+1}'dt+\sum_{j=1}^{N}u_j'j\left.\frac{(u_{j+1}-u_{j-1})}{2}\right|_0^T.
\end{aligned}$$

On the other hand

$$\sum_{j=1}^{N}\int_{0}^{T}(u_{j+1}+u_{j-1}-2u_j)\,j\frac{(u_{j+1}-u_{j-1})}{2}dt=\frac{1}{2}\sum_{j=1}^{N}\int_{0}^{T}j\left[|\,u_{j+1}\,|^2-|\,u_{j-1}\,|^2\right]dt \quad (2.34)$$

$$-\sum_{j=1}^{N}\int_{0}^{T}ju_j(u_{j+1}-u_{j-1})dt=-\sum_{j=1}^{N}\int_{0}^{T}|\,u_j\,|^2+\frac{(N+1)}{2}\int_{0}^{T}|\,u_N\,|^2\,dt+\sum_{j=1}^{N}\int_{0}^{T}u_ju_{j+1}dt$$

$$=-\frac{1}{2}\sum_{j=0}^{N}\int_{0}^{T}|\,u_j-u_{j+1}\,|^2\,dt+\frac{(N+1)}{2}\int_{0}^{T}|\,u_N\,|^2\,dt.$$

Combining (2.32)-(2.34) we deduce that

$$\frac{h}{2}\sum_{j=0}^{N}\int_{0}^{T}\left[u_j'u_{j+1}'+\frac{|\,u_j-u_{j+1}\,|^2}{h^2}\right]dt+X_h(t)\Big|_0^T$$

$$=\frac{(N+1)h}{2}\int_{0}^{T}\left|\frac{u_N(t)}{h}\right|^2dt=\frac{L}{2}\int_{0}^{T}\left|\frac{u_N(t)}{h}\right|^2dt.$$

This concludes the proof of the Lemma.

■

Lemma 2.3 (Equipartition of energy) *For any $h > 0$ and u solution of (1.6) the following identity holds:*

$$-h \sum_{j=1}^{N} \int_{0}^{T} |u_j'|^2 \, dt + h \sum_{j=0}^{N} \left| \frac{u_j - u_{j+1}}{h} \right|^2 + Y_h(t) \Big|_{0}^{T} = 0 \tag{2.35}$$

with

$$Y_n(t) = h \sum_{j=1}^{N} u_j' u_j \Big|_{0}^{T} . \tag{2.36}$$

Proof. We multiply in (1.6) by u_j. Note that this is the discrete version of the classical multiplier u for the wave equation. We obtain

$$\sum_{j=1}^{N} \int_{0}^{T} u_j'' u_j dt - \sum_{j=1}^{N} \frac{(u_{j+1} + u_{j-1} - 2u_j)}{h^2} u_j = 0. \tag{2.37}$$

On the other hand,

$$\sum_{j=1}^{N} \int_{0}^{T} u_j'' u_j dt = -\sum_{j=1}^{N} \int_{0}^{T} |u_j'|^2 \, dt + \sum_{j=1}^{N} u_j' u_j \Big|_{0}^{T} \tag{2.38}$$

and

$$\sum_{j=1}^{N} (u_{j+1} - u_{j-1} - 2u_j) u_j = -\sum_{j=0}^{N} |u_j - u_{j+1}|^2 . \tag{2.39}$$

Combining (2.37)-(2.39) we deduce that (2.35) holds.

∎

We may now proceed to the proof of Theorem 1.2.

In view of the conservation of energy, identity (2.28) may be rewritten as

$$TE_h(0) \quad + \quad \frac{h}{2} \sum_{j=0}^{N} \int_{0}^{T} \left[u_j' u_{j+1}' - |u_j'|^2 \right] dt + X_h(t) \Big|_{0}^{T} \tag{2.40}$$

$$= \quad \frac{L}{2} \int_{0}^{T} \left| \frac{u_N(t)}{h} \right|^2 dt$$

On the other hand

$$\sum_{j=0}^{N} \int_{0}^{T} \left[u_j' u_{j+1}' - |u_j'|^2 \right] dt = -\frac{1}{2} \sum_{j=0}^{N} \int_{0}^{T} |u_j' - u_{j+1}'|^2 \, dt \tag{2.41}$$

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The right hand side of (2.41) can be estimated as follows. Let Λ be the largest eigenvalue in the Fourier development of u. Then

$$u = \sum_{|\mu_k| \leq \sqrt{\Lambda}} a_k e^{i\mu_k t} \varphi_k \tag{2.42}$$

with $\mu_k = \sqrt{\lambda_k}$ for $k > 0$ and $\mu_{-k} = -\mu_k$. Therefore

$$u' = i \sum_{|\mu_k| \leq \sqrt{\Lambda}} a_k \mu_k e^{i\mu_k t} \varphi_k. \tag{2.43}$$

Thus

$$\sum_{j=0}^{N} \left| u'_j - u'_{j+1} \right|^2 = \sum_{j=0}^{N} \left| \sum_{|\mu_k| \leq \sqrt{\Lambda}} a_k \mu_k e^{i\mu_k t} \left(\varphi_{k,j} - \varphi_{k,j+1} \right) \right|^2 \tag{2.44}$$

$$\sum_{j=0}^{N} \sum_{|\mu_k| \leq \sqrt{\Lambda}} \mu_k^2 \mid a_k \mid^2 \left| \varphi_{k,j} - \varphi_{k,j+1} \right|^2$$

$$+ \sum_{j=0}^{N} \sum_{\substack{|\mu_k| \leq \sqrt{\Lambda} \\ |\mu_\ell| \leq \sqrt{\Lambda} \\ \mu_k \neq \mu_\ell}} \mu_k \mu_\ell e^{i(\mu_k - \mu_t)t} \left(\varphi_{k,j} - \varphi_{k,j+1} \right) \left(\varphi_{k,\ell} - \varphi_{k,\ell+1} \right).$$

The following identities holds:

Lemma 2.4 *For any eigenvectors φ with eigenvalues λ of (2.1) the following identity holds:*

$$\sum_{j=0}^{N} \left| \frac{\varphi_j - \varphi_{j+1}}{h} \right|^2 = \lambda \sum_{j=1}^{N} \mid \varphi_j \mid^2. \tag{2.45}$$

If φ_k and φ_ℓ are eigenvectors associated to eigenvalues $\lambda_k \neq \lambda_\ell$ it follows that

$$\sum_{j=0}^{N} \left(\varphi_{k,j} - \varphi_{k,j+1} \right) \left(\varphi_{\ell,j} - \varphi_{\ell,j+1} \right) = 0. \tag{2.46}$$

Proof. Multiplying in (2.1) by φ_j and adding for $j = 1, \cdots, N$ the identity (2.45) follows inmediately.

In order to get (2.46) we multiply by $\varphi_{\ell,j}$ the equation satisfied by $\varphi_{k,j}$. It follows that

$$\lambda_k \sum_{j=1}^{N} \varphi_{k,j} \varphi_{\ell,j} = -\frac{1}{h^2} \sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} - 2\varphi_{k,j} \right) \varphi_{\ell,j}. \tag{2.47}$$

Now multiplying by $\varphi_{k,j}$ the equation satisfied by $\varphi_{\ell,j}$ we deduce that

$$\lambda_\ell \sum_{j=1}^{N} \varphi_{k,j} = -\frac{1}{h^2} \sum_{j=1}^{N} \left(\varphi_{\ell,j+1} + \varphi_{\ell,j-1} - 2\varphi_{\ell,j} \right) \varphi_{k,j}. \tag{2.48}$$

On the other hand,

$$\sum_{j=1}^{N} \left(\varphi_{\ell,j+1} + \varphi_{\ell,j-1} - 2\varphi_{\ell,j} \right) \varphi_{k,j} = \sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} - 2\varphi_{k,j} \right) \varphi_{\ell,j}. \tag{2.49}$$

Therefore

$$(\lambda_\ell - \lambda_k) \sum_{j=1}^{N} \varphi_{k,j}\varphi_{\ell,j} = 0.$$

Thus

$$\sum_{j=1}^{N} \varphi_{k,j}\varphi_{\ell,j} = 0 \tag{2.50}$$

and consequently

$$\sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} - 2\varphi_{k,j} \right) \varphi_{\ell,j} = \sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} \right) \varphi_{\ell,j} = 0.$$

Therefore

$$\sum_{j=1}^{N} \varphi_{k,j+1}\varphi_{\ell,j} = -\sum_{j=1}^{N} \varphi_{k,j-1}\varphi_{\ell,j} = -\sum_{j=1}^{N} \varphi_{k,j}\varphi_{\ell,j+1}.$$

In other words

$$\sum_{j=1}^{N} \left[\varphi_{k,j+1}\varphi_{\ell,j} + \varphi_{k,j}\varphi_{\ell,j+1} \right] = 0, \tag{2.51}$$

which, in view of (2.50) is equivalent to (2.46).

$\blacksquare$

In view of the identities (2.45)-(2.46) the term in (2.44) can be rewritten as

$$\begin{aligned}
\sum_{j=0}^{N} \mid u_j' - u_{j+1}' \mid^2 &= \sum_{|\mu_k| \leq \sqrt{\Lambda}} \mid a_k \mid^2 \lambda_k^2 h^2 \sum_{j=1}^{N} |\varphi_{k,j}|^2 \\
&\leq \Lambda \sum_{|\mu_k| \leq \sqrt{\Lambda}} \mid a_k \mid^2 \lambda_k h^2 \sum_{j=1}^{N} |\varphi_{k,j}|^2 \\
&= \Lambda h^2 \sum_{j=1}^{N} \left| u_j' \right|^2.
\end{aligned}$$

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Therefore

$$\sum_{j=0}^{N} \int_0^T \left[u_j' u_{j+1}' - \left| u_j' \right|^2 \right] dt \geq -\frac{\Lambda h^2}{2} \sum_j \int_0^T \left| u_j' \right|^2 dt. \tag{2.52}$$

Combining (2.40) and (2.52) we deduce that

$$TE_h(0) - \frac{\Lambda h^2}{4} h \sum_{j=0}^{N} \int_0^T \left| u_j' \right|^2 dt + X_h(t) \Big|_0^T \leq \frac{L}{2} \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt. \tag{2.53}$$

In view of the equipartition of energy identity (2.35) it follows that

$$h \sum_{j=1}^{N} \left| u_j' \right|^2 dt = \int_0^T E_h(t) + \frac{1}{2} Y_h(t) \Big|_0^T TE_h(0) + \frac{1}{2} Y_h(t) \Big|_0^T. \tag{2.54}$$

Combining (2.53) and (2.54) we deduce that

$$T \left(1 - \frac{\Lambda h^2}{4} \right) E_h(0) + Z_h(t) \Big|_0^T \leq \frac{L}{2} \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{2.55}$$

with

$$\begin{aligned} Z_h(t) &= X_h(t) - \frac{\Lambda h^2}{8} Y_h(t) \\ &= h \sum_{j=1}^{N} u_j' \left[\frac{(u_{j+1} - u_{j-1})}{2} - \frac{\Lambda h^2}{8} u_j \right], \end{aligned} \tag{2.56}$$

for every solution of (1.6) in which Λ is the largest eigenvalue entering in its Fourier expansion.

The following provides an estimate on the term Z_h:

Lemma 2.5 *For any $h > 0, t \in [0, T]$ and u solution of (1.6) in which Λ is the upper bound on the eigenvalues entering in its Fourier development, it follows that*

$$\mid Z_h(t) \mid \leq \sqrt{L^2 - \frac{\Lambda h^4}{16} + \frac{3\Lambda h^2}{16\lambda_1}} E_h(0). \tag{2.57}$$

Proof. We do not make explicit the time dependence to simplify the notation. We have

$$\mid Z_h \mid \leq \left[\sum_{j=1}^{N} \left| u_j' \right|^2 \right]^{1/2} \left[\sum_{j=1}^{N} \left| j \frac{(u_{j+1} - u_{j-1})}{2} + \eta u_j \right|^2 \right]^{1/2} \tag{2.58}$$

with $\eta = -\Lambda h^2/8$.

On the other hand

$$h\sum_{j=1}^{N}\left|\frac{(u_{j+1}-u_{j-1})}{2}+\eta u_j\right|^2 = h\sum_{j=1}^{N}\left[\frac{j^2}{4}\left|u_{j+1}-u_{j-1}\right|^2+\eta^2 u_j^2+\eta j\left(u_{j+1}-u_{j-1}\right)u_j\right]$$

$$\le h\sum_{j=0}^{N}\left[\frac{j^2}{2}\left|u_{j+1}-u_j\right|^2+\frac{j^2}{2}\left|u_j-u_{j-1}\right|^2+\eta^2 u_j^2-\eta u_j u_{j+1}\right]$$

$$\le L^2 h\sum_{j=0}^{N}\frac{\mid u_j-u_{j+1}\mid^2}{h^2}-\mid\eta\mid h\sum_{j=1}^{N}\left(u_j^2-u_j u_{j+1}\right)$$

$$+\left[\eta^2+\mid\eta\mid\right]h\sum_{j=1}^{N}u_j^2=\left(L^2-\frac{\mid\eta\mid h^2}{2}\right)h\sum_{j=0}^{N}\left|\frac{u_j-u_{j+1}}{h}\right|^2$$

$$+\left[\eta^2+\mid\eta\mid\right]h\sum_{j=1}^{N}u_j^2=\left[L^2-\frac{\mid\eta\mid h^2}{2}+\frac{(\eta^2+\mid\eta\mid)}{\lambda_1}\right]h\sum_{j=0}^{N}\left|\frac{u_j-u_{j+1}}{h}\right|^2\le$$

$$\le\left[L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}\right]h\sum_{j=0}^{N}\left|\frac{u_j-u_{j+1}}{h}\right|^2. \tag{2.59}$$

In the last step we have used the fact that

$$h\sum_{j=1}^{N}u_j^2\le\frac{1}{\lambda_1}h\sum_{j=0}^{N}\left|\frac{u_j-u_{j+1}}{h}\right|^2$$

which is the discrete version of Poincaré's inequality and can be deduced easily from (2.45)-(2.46), and also that $\eta^2+\mid\eta\mid\le\frac{3}{2}\mid\eta\mid$ which is a consequence of the fact that $\mid\eta\mid=\Lambda h^2/8\le 1/2$ as it is inmediately seen from (2.2).

Combining (2.58)-(2.59) we deduce that

$$\mid Z_h\mid\le\sqrt{L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}}\left(h\sum_{j=1}^{N}\left|u_j'\right|^2\right)^{1/2}\left(h\sum_{j=0}^{N}\left|\frac{u_{j+1}-u_j}{h}\right|^2\right)^{1/2} \tag{2.60}$$

$$=\sqrt{L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}}E_h(0).$$

∎

Combining (2.55) and (2.57) we deduce that

$$\left[T/1-\Lambda h^2/4\right)-2\sqrt{L^2-\frac{\Lambda h^4}{16}+\frac{3\Lambda h^2}{16\lambda_1}}\right]E_h(0)\le\frac{L}{2}\int_0^T\left|\frac{u_N(t)}{h}\right|^2 dt. \tag{2.61}$$

As a consequence of (2.61) and taking into account that

$$\Lambda=\gamma/h^2$$

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in the class of solutions $\mathcal{C}_h(\gamma)$ of sistem (1.6) we deduce that

$$E_h(0) \leq \frac{L}{2\left(T(1-\gamma/4) - 2\sqrt{L^2 + \gamma(3/\lambda_1 - h^2)/16}\right)} \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \qquad (2.62)$$

provided

$$T > 2\sqrt{L^2 - \frac{\gamma h^2}{16} + \frac{3\gamma}{16\lambda_1}} / (1 - \gamma/4).$$

Taking into account that $\lambda_1 \geq \pi^2/2L^2$ for h sufficiently small, the statement of Theorem 1.2 holds with

$$T(\gamma) = \frac{2\sqrt{L^2(1 + 3\gamma/8\pi^2) - \gamma h^2/16}}{1 - \gamma/4}$$

and

$$C(T,\gamma) = \frac{L}{2\left(T(1-\gamma/4) - 2\sqrt{L^2(1 - 3\gamma/8\pi^2) - \gamma h^2/16}\right)}. \qquad (2.63)$$

2.5 Boundary observability of the discrete wave equation: non-harmonic Fourier series

In this section we prove Theorem 1.2 using well known results from the theory of nonharmonic Fourier series.

To do that we need an estimate between the roots of consecutive eigenvalues entering in the Fourier development of the solutions of (1.6) in the class $\mathcal{C}_h(\gamma)$. We have

Lemma 2.6 *Assume that*

$$\gamma = 4\sin^2\left(\frac{\pi\varepsilon}{2}\right) \qquad (2.64)$$

for some $0 \leq \varepsilon < 1$. Then

$$\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} \geq \frac{\pi}{L}\cos\left(\frac{\pi\varepsilon}{2}\right) \qquad (2.65)$$

for all eigenvalues in the range

$$\lambda h^2 \leq \gamma. \qquad (2.66)$$

Remark. Note that every $0 \leq \gamma < 4$ can be written in the form (2.65) for some $0 \leq \varepsilon < 1$. Note also that the gap given in (2.66) tends to π/L, the gap in the continuous wave equation, as $\varepsilon \to 1$, i.e., as $\gamma \to 4$. This is consistent with the estimates of section 2.2 in which we observed that gap for large eigenvalues is of the order of h.

∎

Proof. In view of the expression (2.2) the eigenvalues λ satisfy (2.66) with γ as in (2.64) if and only if

$$(j+1)h \le \varepsilon L. \tag{2.67}$$

Let us now compute the gap:

$$\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} = \frac{2}{h}\left[\sin\left(\frac{\pi(j+1)h}{2L}\right) - \sin\left(\frac{\pi_j h}{2L}\right)\right] = \frac{\pi}{L}\cos(\xi) \tag{2.68}$$

for some $\xi \in \left[\frac{\pi_j h}{2L}, \frac{\pi(j+1)h}{2L}\right]$. In view of (2.67), $0 \le \xi \le \frac{\pi_\varepsilon}{2}$ and therefore $\cos\xi \ge \cos(\pi\varepsilon/2)$. Therefore (2.65) holds.

∎

According to Ingham's inequality [I] and in view of Lemma 2.6 it follows that for any $0 < \varepsilon < 1$ and $T > \frac{2L}{\cos(\pi\varepsilon/2)}$ there exists positive constant $C_1(T,\varepsilon), C_2(T,\varepsilon) > 0$ such that

$$C_1(T,\varepsilon) \sum_{\lambda h^2 \le \gamma(\varepsilon)} |a_k|^2 \le \int_0^T \left|\sum_{\lambda h^2 \le \gamma(\varepsilon)} a_k e^{i\sqrt{\lambda_k}t}\right|^2 dt \tag{2.69}$$

$$\le C_2(T,\varepsilon) \sum_{\lambda h^2 \le \gamma(\varepsilon)} |a_k|^2,$$

with

$$\gamma(\varepsilon) = 4\sin^2(\pi\varepsilon/2). \tag{2.70}$$

On the other hand, in the range of eigenvalues

$$\lambda h^2 \le \gamma(\varepsilon), \tag{2.71}$$

according to the identity of Lemma 1.1 it follows that

$$h\sum_{j=0}^N \left|\frac{\varphi_{j+1} - \varphi_j}{h}\right|^2 \le \frac{2L}{4 - \gamma(\varepsilon)}\left|\frac{\varphi_N}{h}\right|^2 = \frac{L}{2\cos^2(\pi\varepsilon/2)}\left|\frac{\varphi_N}{h}\right|^2 \tag{2.72}$$

for any eigenvector $\varphi = (\varphi_1, \cdots, \varphi_N)$ associated to an eigenvalue λ in the range (2.71).

Let us now consider a solution u of (1.6) in the class $\mathcal{C}_h(\gamma(\varepsilon))$. It can be written as

$$u = \sum_{|\mu h \le \sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k(h)t}\varphi_k. \tag{2.73}$$

According to (2.69) and (2.72) we deduce that, for $T > 2L/cos(\pi\varepsilon/2)$,

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = \int_0^T \left|\sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} a_k e^{i\mu_k(h)t}\varphi_{k,N}\right|^2 \tag{2.74}$$

$$\ge C_1(T,\varepsilon) \sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} |a_k|^2 |\varphi_{k,N}|^2 \ge \frac{C_1(T,\varepsilon)L}{2\cos^2(\pi\varepsilon/2)} \sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} |a_k|^2 h\sum_{j=0}^N \left|\frac{\varphi_{k,j+1} - \varphi_{k,j}}{h}\right|^2.$$

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Moreover,

$$
\begin{aligned}
E_h(0) &= \frac{1}{2} \sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} \left[|a_k|^2 \lambda_k h \sum_{j=1}^{N} |\varphi_{k,j}|^2 \right] + \frac{1}{2} \sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} \left[|a_k|^2 h \sum_{j=0}^{N} \left| \frac{\varphi_{k,j+1} - \varphi_{k,j}}{h} \right|^2 \right] \\
&= \sum_{|\mu_k|h \le \sqrt{\gamma(\varepsilon)}} \left[|a_k|^2 h \sum_{j=0}^{N} \left| \frac{\varphi_{k,j+1} - \varphi_{k,j}}{h} \right|^2 \right].
\end{aligned}
$$

Therefore, as a consequence of (2.74) it follows that

$$
E_h(0) \le \frac{2 \cos^2(\pi \varepsilon/2)}{L C_1(T, \varepsilon)} \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{2.75}
$$

for any $T > 2L/\cos(\pi \varepsilon/2)$ and for any $u \in \mathcal{C}_h(\gamma(\varepsilon))$.

Therefore, Theorem 1.2 holds with

$$
T(\gamma) = \frac{2L}{\sqrt{1 - \gamma/4}} \tag{2.76}
$$

and

$$
C(T, \gamma) = \frac{2(1 - \gamma/4)}{L C_1(T, \varepsilon)} \tag{2.77}
$$

provided $\gamma = \gamma(\varepsilon)$.

Observe that the estimate (2.76) obtained by Ingham's inequality for the observability time is slightly better then (2.63) obtained by multipliers. However both expressions coincide in the limit when $h \to 0$.

Note also that, according to (2.69), the reverse inequality also holds, i.e.,

$$
\int_0^T \left| \frac{u_N(t)}{h} \right| dt \le C E_h(0). \tag{2.78}
$$

However, from the identity (2.28) it is easy to see that (2.78) holds for all $h > 0$, every solution u of (1.6) and all $T > 0$ with $C = C(T) > 0$.

3 Finite-element semi-discretization

3.1 Problem formulation

Let us consider the finite-element space semi-discretization of the wave equation (1.1):

$$
\begin{cases} \frac{2}{3} u_j'' + \frac{1}{6} u_{j+1}'' + \frac{1}{6} u_{j-1}'' = \frac{[u_{j+1} + u_{u-1} - 2u_j]}{h^2}, 0 < t < T, j = 1, \cdots, N \\ u_0 = u_{N+1} = 0. \end{cases} \tag{3.1}
$$

Let us recall that system (3.1) is obtained by a Galerkin approximation of the wave equation when the basis of $H_0^1(0, L)$ is built by means of finite elements

$$e_j(x) = \left[1 - \frac{|x - x_j|}{h} \right]^+, j = 1, \cdots, N \tag{3.2}$$

with $x_j = jh$.

The Galerkin approximation of (1.1) is given by

$$\begin{cases} u = \sum_{j=1}^{N} u_j(t) e_j(x) \\ \int_0^L u'', e_k dx = \int_0^L u_x e_{k,x} dx, \quad 0 < t < T, \quad \forall k = 1, \cdots, N. \end{cases} \tag{3.3}$$

System (3.1) can be easily derived taking into account that

$$\begin{cases} \int_0^1 e_j e_k dx = \frac{h}{6} \text{ if } j = j+1, j-1, \int_0^L e_j e_k dx = 0 \text{ if } |k - j| \geq 2 \\ \int_0^L |e_j|^2 dx = \frac{2h}{3}, \forall j = 1, \cdots, N, \end{cases} \tag{3.4}$$

$$\begin{cases} \int_0^L e_{j,x} e_{k,x} dx = -\frac{1}{h} \text{ if } k = j-1, j+1; \int_0^T e_{j,x} e_{k,x} dx = 0 \text{ if } |k - j| \geq 2, \\ \int_0^L |e_{j,x}|^2 dx \frac{2}{h}. \end{cases} \tag{3.5}$$

The conserved energy for system (3.1) is given by

$$E_h(t) = \frac{h}{6} \sum_{j=1}^{N} |u_j'|^2 + \frac{h}{12} \sum_{j=0}^{N} \left| u_j' + u_{j+1}' \right|^2 + \frac{h}{2} \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2, \tag{3.6}$$

i.e.,

$$E_h(t) = E_h(0), \quad \forall 0 < t < T \tag{3.7}$$

for all $h > 0$ and for every solution of (3.1).

As in Section 2 above, the goal is to obtain observability inequalities of the form

$$E_h(0) \leq C \int_0^t \left| \frac{u_N(t)}{h} \right|^2 dt. \tag{3.8}$$

Let us see first that the constant C in (3.8) may not be uniform when $h \to 0$. For that we analyze the eigenvectors of the system:

$$\begin{cases} -\left[\frac{\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j}{h^2} \right] = \lambda \left[\frac{2}{3} \varphi_j + \frac{1}{6} \varphi_{j+1} + \frac{1}{6} \varphi_{j-1} \right], j = 1, \cdots, N \\ \varphi_0 = \varphi_{N+1} = 0. \end{cases} \tag{3.9}$$

We denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N$ the eigenvalues of (3.9) and by $\{\varphi_k\}_{k=1}^{N}$ the associated eigenvectors.

The following identity holds:

Lemma 3.1 *For any $h > 0$ and any eigenvector of system (3.9) the following identity holds:*

$$h \sum_{j=0}^{N} \left| \frac{\varphi_{j+1} - \varphi_j}{h} \right|^2 = \frac{6 + \lambda h^2}{12 - \lambda h^2} L \left| \frac{\varphi_N}{n} \right|^2. \tag{3.10}$$

Moreover, as we shall see

$$\lambda_N(h) h^2 \to 12 \text{ as } h \to 0. \tag{3.11}$$

Therefore, as in the case of the finite-difference it follows that:

Theorem 3.1 *For any $T > 0$,*

$$\sup_{u \text{ solution of } (3.1)} \left[\frac{E_h(0)}{\int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt} \right] \to \infty \tag{3.12}$$

as $h \to 0$.

In order to prove uniform observability results, for any $0 < \gamma < 12$ we introduce the class of solutions of (3.1) in which only the terms of the Fourier development corresponding to $\lambda h^2 \leq \gamma$ do not vanish. More precisely,

$$\mathcal{C}_h(\gamma) = \left\{ u \text{ solution of } (3.1) : u = \sum_{\lambda h^2 \leq \gamma} a_k e^{i \sqrt{\lambda_k(h)} t} \varphi_k \right\}. \tag{3.13}$$

The following holds:

Theorem 3.2 *For any $0 < \gamma < 12$ there exists $T(\gamma) > 0$ such that for any $T > T(\gamma)$ there exists a positive constant $C(T, \gamma)$ such that*

$$E_h(0) \leq C(T, \gamma) \int_0^T \left[\left| \frac{u_N(t)}{h} \right|^2 + \left| \frac{u_N'(t)}{6} \right|^2 \right] dt \tag{3.14}$$

for any solution u of (3.1) in the class $\mathcal{C}_h(\gamma)$.
Moreover,

(a) *$T(\gamma) \nearrow \infty$ as $\gamma \nearrow 12$ and $T(\gamma) \searrow 2^L$ as $\gamma \searrow 0$,*

(b) *$C(T, \gamma) \to L/2(T - 2L)$ as $\gamma \searrow 0$.*

Note that the extra term $\int_0^T | u_N'(t) |^2 \, dt$ appears on the right hand side of the observability inequality (3.14). This was not the case in the context of the finite-difference. We shall see that, by using Ingham's inequality, thiss extra term can be absorbed by $\int_0^T | u_N(t)/h |^2 \, dt$ by increasing the observability constant.

As in section 2, the observability inequality (3.14) can be proved by Ingham's inequality. However we shall derive it using multipliers.

The rest of this section is organized as follows. First we analyze the spectrum of the system. Then we orive observability identity (3.10) for the eigenvectors. Then we prove Theorem 3.2 by multipliers. Finally, using Ingham's inequality we absorve the term $\int_0^T | u'_N(t) |^2\, dt$ from the right hand side of (3.14).

3.2 Spectral analysis

Let us first compute the eigenvalues and eigenvectors of the system. We assume that the eigenvectors is of the form $\varphi_k = \varphi(kh)$ with $\varphi(x) = \alpha^x$. Then, the eigenvalue problem (3.9) reduces to

$$\frac{\left[2 - \alpha^h - \alpha^{-h}\right]}{h^2} = \lambda \left[\frac{2}{3} + \frac{\alpha^h}{6} + \frac{\alpha^{-h}}{6}\right],$$

or, by selting, $\mu = \lambda h^2$ and $\omega = \alpha^h$, to

$$\omega\left(1 + \frac{\mu}{6}\right) + \omega^{-1}\left(1 + \frac{\mu}{6}\right) - \left(2 - \frac{2\mu}{3}\right) = 0. \tag{3.15}$$

In view of the structure of equation (3.15) we see that if $\phi(x) = \alpha^x$ solves the discrete equation, $\varphi(x) = \alpha^{-x}$ is also a solution. Then $\varphi(x) = a\alpha^x + b\alpha^{-x}$ is the general solution. Imposing the boundary conditions $\varphi(0) = \varphi(L) = 0$ we get that $a = -b$ and the equation

$$\omega^{N+1} - \omega^{-(N+1)} = 0 \Leftrightarrow \omega^{2(N+1)} = 1.$$

The roots of unity that provide linearly independent eigenvectors are given by

$$\omega_j = e^{ij\pi/(N+1)}, j = 1, \cdots, N.$$

The eigenvectors are then

$$\varphi_j(x) = \sin\left(\frac{\pi_j x}{L}\right), j = 1, \cdots, N$$

i.e.,

$$\varphi_{j,k} = \sin\left(\frac{\pi jkh}{L}\right), j, k = 1, \cdots, N. \tag{3.16}$$

From (3.15) we can obtain the values of $\mu_j = \lambda_j h^2$:

$$\mu_j = \frac{2 - \omega - \omega^{-1}}{\frac{\omega}{6} + \frac{\omega^{-1}}{6} + \frac{2}{3}}.$$

Therefore

$$\lambda_j = \frac{1}{h^2}\frac{2 - e^{ij\pi/(N+1)} - e^{-ij\pi/(N+1)}}{\frac{e^{ij\pi/(N+1)}}{6} + \frac{e^{-ij\pi/(N+1)}}{6} + \frac{2}{3}} = \frac{6}{h^2}\left[\frac{1 - \cos(j\pi h/L)}{2 + \cos(j\pi h/L)}\right] \tag{3.17}$$

As we mentioned above

$$\lambda_N(h)h^2 \to 12. \tag{3.18}$$

Indeed,

$$\lambda_N(h)h^2 = 6\left[\frac{1 - \cos(N\pi h/L)}{2 + \cos(N\pi h/L)}\right] = 6\left[\frac{1 - \cos(\pi - h\pi/L)}{2 + \cos(\pi - h\pi/L)}\right]$$

$$6\left[\frac{1 + \cos(h\pi/L)}{2 - \cos(h\pi/L)}\right] \to 12, \text{ as } h \to 0.$$

It is also easy to see that, for j fixed,

$$\lambda_j(h) \to \left(\frac{\pi_j}{L}\right)^2, \text{ as } h \to 0. \tag{3.19}$$

Indeed,

$$\lim_{h \to 0} \lambda_j(h) = 2\lim_{h \to 0} \frac{1 - \cos(j\pi h/L)}{h^2} = \frac{j\pi}{L}\lim_{h \to 0}\frac{\sin(j\pi h/L)}{h} = \left(\frac{j\pi}{L}\right)^2.$$

Let us now analyze the gap between the roots of the eigenvalues. We have

$$\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} = \frac{\sqrt{6}}{h}\left[\left(\frac{1 - \cos\left((j+1)\pi^h/L\right)}{2 + \cos\left((j+1)\pi h/L\right)}\right)^{1/2} - \left(\frac{1 - \cos(j\pi h/L)}{2 + \cos(j\pi h/L)}\right)^{1/2}\right]$$

$$\frac{\sqrt{6}\pi}{L}\left[\frac{1}{2}\left(\frac{2 + \cos(\xi)}{1 - \cos(\xi)}\right)^{1/2}\left(\frac{e\,\text{sen}\,\xi}{(2 + \cos\xi)^2}\right)\right] = 3\sqrt{\frac{3}{2}}\frac{\pi}{L}\frac{(1 + \cos\xi)^{1/2}}{(2 + \cos\xi)^{3/2}} \tag{3.20}$$

$$\frac{\pi}{\sqrt{2}L}(1 + \cos\xi)^{1/2}$$

for some $\xi \in [j\pi h/L, (j+1)\pi h/L]$.

Assume we consider eigenvalues corresponding to the indexes $(j+1)h \le \varepsilon L$ with $0 < \varepsilon < 1$. Then $\xi \le \varepsilon\pi$ and therefore $\cos\xi \ge \cos(\varepsilon\pi)$. Going back to (3.20) we deduce the following Lemma:

Lemma 3.2 *For any $0 < \varepsilon < 1$ the gap between the roots of consecutive eigenvalues associated to indexes such that*

$$(j+1)h \le \varepsilon L \tag{3.21}$$

satisfies

$$\sqrt{\lambda_{j+1}(h)} - \sqrt{\lambda_j(h)} \ge \frac{\pi}{L}\left(\frac{1 + \cos(\varepsilon\pi)}{2}\right)^{1/2}. \tag{3.22}$$

Observe that the lower bound on the gap vanishes as $\varepsilon \to 1$. However, it converges to π/L when $\varepsilon \to 0$.

3.3 Boundary observability of eigenvectors

This section is devoted to prove identity (3.10).

We multiply in both sides of (3.9) by $j\left(\varphi_{j+1} - \varphi_{j-1}\right)/2$. Proceeding is in the proof of Lemma 1.1 we obtain:

$$\frac{1}{h^2} \sum_{j=1}^{N} \left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right] j\left(\varphi_{j+1} - \varphi_{j-1}\right)/2 = \tag{3.23}$$

$$= -\frac{1}{h^2} \sum_{j=1}^{N} \mid \varphi_j \mid^2 + \frac{N+1}{2h^2} \mid \varphi_N \mid^2 + \frac{1}{h^2} \sum_{j=1}^{N} \varphi_j \varphi_{j+1}.$$

On the other hand,

$$\sum_{j=1}^{N} \left[\frac{2}{3}\varphi_j + \frac{1}{6}\varphi_{j+1} + \frac{1}{6}\varphi_{j-1}\right] j\frac{(\varphi_{j+1} - \varphi_{j-1})}{2} = \frac{1}{3}\sum_{j=1}^{N} j\varphi_j \left(\varphi_{j+1} - \varphi_{j-1}\right) \tag{3.24}$$

$$+ \frac{1}{12}\sum_{j=1}^{N} j\left(|\varphi_{j+1}|^2 - |\varphi_{j-1}|^2\right) = -\frac{1}{3}\sum_{j=1}^{N} \varphi_j\varphi_{j+1} - \frac{1}{6}\sum_{j=1}^{N} |\varphi_j|^2 + \frac{N+1}{12} \mid \varphi_N \mid^2$$

Combining (3.23) and (3.24) we deduce that

$$\frac{1}{h^2}\sum_{j=1}^{N} \mid \varphi_j \mid^2 - \frac{N+1}{2h^2} \mid \varphi_N \mid^2 - \frac{1}{h^2}\varphi_j\varphi_{j+1}$$

$$\lambda\frac{(N+1)}{12} \mid \varphi_N \mid^2 - \frac{\lambda}{3}\sum_{j=1}^{N} j = 1\varphi_j\varphi_{j+1} - \frac{\lambda}{6}\sum_{j=1}^{N} \mid \varphi_j \mid^2,$$

or, in other words,

$$(N+1)\left[\frac{1}{2h^2} + \frac{\lambda}{12}\right] \mid \varphi_N \mid^2 = \left(\frac{1}{h^2} + \frac{\lambda}{6}\right)\sum_{j=1}^{N} \left[\mid \varphi_j \mid^2 - \varphi_j\varphi_{j+1}\right] + \frac{\lambda}{2}\sum_{j=1}^{N} \varphi_j\varphi_{j+1} \tag{3.25}$$

$$= \left(\frac{1}{2h^2} + \frac{\lambda}{12}\right)\sum_{j=0}^{N} \mid \varphi_j - \varphi_{j+1} \mid^2 + \frac{\lambda}{2}\sum_{j=1}^{N} \varphi_j\varphi_{j+1}.$$

We now multiply in (3.9) by φ_j. It follows that

$$-\sum_{j=1}^{N} \frac{\left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right]}{h^2}\varphi_j = \lambda\sum_{j=1}^{N} \left[\frac{2}{3}\varphi_j + \frac{1}{6}\varphi_{j+1} + \frac{1}{6}\varphi_{j-1}\right]\varphi_j. \tag{3.26}$$

We also have

$$\sum_{j=1}^{N} \left[\frac{2}{3}\varphi_j + \frac{1}{6}\varphi_{j+1} + \frac{1}{6}\varphi_{j-1}\right]\varphi_j = \frac{2}{3}\sum_{j=1}^{N} \mid \varphi_j \mid^2 + \frac{1}{3}\sum_{j=1}^{N} \varphi_j\varphi_{j+1} \tag{3.27}$$

and

$$\sum_{j=1}^{N}\left[\varphi_{j+1}+\varphi_{j-1}-2\varphi_{j}\right]\varphi_{j}=2\sum_{j=1}^{N}\left[\varphi_{j}\varphi_{j+1}-\mid\varphi_{j}\mid^{2}\right]=-\sum_{j=0}^{N}\mid\varphi_{j}-\varphi_{j+1}\mid^{2}. \tag{3.28}$$

Combining (3.26)-(3.28) we obtain

$$\begin{aligned}
\frac{1}{h^{2}}\sum_{j=0}^{N}|\varphi_{j}-\varphi_{j+1}|^{2} &= \frac{2\lambda}{3}\sum_{j=1}^{N}\mid\varphi_{j}\mid^{2}+\frac{\lambda}{3}\sum_{j=1}^{N}\varphi_{j}\varphi_{j+1}\\
&= -\frac{\lambda}{6}\sum_{j=0}^{N}|\varphi_{j}-\varphi_{j+1}|^{2}+\lambda\sum_{j=1}^{N}\mid\varphi_{j}\mid^{2}
\end{aligned}$$

i.e.,

$$\left(\frac{1}{h^{2}}+\frac{\lambda}{6}\right)\sum_{j=0}^{N}|\varphi_{j}-\varphi_{j+1}|^{2}=\lambda\sum_{j=1}^{N}\mid\varphi_{j}\mid^{2},$$

or, equivalently,

$$\left(\frac{1}{h^{2}}-\frac{\lambda}{3}\right)\sum_{j=1}^{N}\mid\varphi_{j}\mid^{2}=\left(\frac{1}{h^{2}}+\frac{\lambda}{6}\right)\sum_{j=1}^{N}\varphi_{j}\varphi_{j+1}. \tag{3.29}$$

We normalize the eigenvalues so that

$$h\sum_{j=1}^{N}\mid\varphi_{j}\mid^{2}=1.$$

Then

$$\sum_{j=1}^{N}\varphi_{j}\varphi_{j+1}=\frac{6h}{6+\lambda h^{2}}\left(\frac{1}{h^{2}}-\frac{\lambda}{3}\right);\left(\frac{6+\lambda h^{2}}{6}\right)h\sum_{j=0}^{N}\left|\frac{\varphi_{j}-\varphi_{j+1}}{h}\right|^{2}=\lambda \tag{3.30}$$

which, combined with (3.25), gives

$$\begin{aligned}
L\left[\frac{1}{2}+\frac{\lambda h^{2}}{12}\right]\left|\frac{\varphi_{N}}{h}\right|^{2} &= \left(\frac{1}{2}+\frac{\lambda h^{2}}{12}\right)h\sum_{j=0}^{N}\left|\frac{\varphi_{j}-\varphi_{j+1}}{h}\right|^{2}+\frac{\lambda h}{2}\sum_{j=1}^{N}\varphi_{j}\varphi_{j+1}\\
&= \left(\frac{1}{2}+\frac{\lambda h^{2}}{12}\right)h\sum_{j=0}^{N}\left|\frac{\varphi_{j}-\varphi_{j+1}}{h}\right|^{2}+\frac{3\lambda h^{2}}{6+\lambda h^{2}}\left(\frac{1}{h^{2}}-\frac{\lambda}{3}\right)
\end{aligned}$$

or, equivalently, in view of (3.30),

$$L\left|\frac{\varphi_{N}}{h}\right|^{2} = h\sum_{j=0}^{N}\left|\frac{\varphi_{j}-\varphi_{j+1}}{h}\right|^{2}+\frac{36\lambda h^{2}}{(6+\lambda h^{2})^{2}}\left(\frac{1}{h^{2}}-\frac{\lambda}{3}\right)$$

$$= h \sum_{j=0}^{N} \left| \frac{\varphi_j - \varphi_{j+1}}{h} \right|^2 \left(1 + \frac{6h^2}{(6 + \lambda h^2)} \left(\frac{1}{h^2} - \frac{\lambda}{3} \right) \right)$$

$$= \left(\frac{12 - \lambda h^2}{6 + \lambda h^2} \right) h \sum_{j=0}^{N} \left| \frac{\varphi_j - \varphi_{j+1}}{h} \right|^2,$$

which is equivalent to (3.10).

∎

As an inmediate consequence of the identity (3.10) and (3.18) we deduce that Theorem 3.1 holds.

3.4 Boundary observability of the discrete wave equation

The main goal of this section is to prove Theorem 3.2. To do that we first prove some basic identities.

Lemma 3.3 (Conservation of energy) *For any $h > 0$ and any solution of (3.1) it follows that*

$$E_h(t) = E_h(0), \quad \forall 0 < t < T. \tag{3.31}$$

Proof. Multiplying in (3.1) by u'_j we deduce that

$$\sum_{j=1}^{N} \left(\frac{2}{3} u''_j + \frac{1}{6} u''_{j+1} + \frac{1}{6} u''_{j-1} \right) u'_j = \sum_{j=1}^{N} \frac{(u_{j+1} + u_{j-1} - 2u_j)}{h^2} u'_j. \tag{3.32}$$

The right hand side term of (3.32) can be treated as in section 2. This yields

$$\sum_{j=1}^{N} \left(\frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} \right) u'_j = \frac{1}{2} \frac{d}{dt} \sum_{j=0}^{N} \left| \frac{u_j - u_{j+1}}{h} \right|^2. \tag{3.33}$$

The left hand side term can be handeled as follows:

$$\sum_{j=1}^{N} \left(\frac{2}{3} u''_j + \frac{1}{6} u''_{j+1} + \frac{1}{6} u''_{j-1} \right) u'_j = \frac{2}{3} \sum_{j=1}^{N} u''_j u'_j + \frac{1}{6} \sum_{j=1}^{N} \left(u''_{j+1} u'_j + u''_{j-1} u'_j \right) \tag{3.34}$$

$$= \frac{1}{3} \frac{d}{dt} \sum_{j=1}^{N} |u'_j|^2 + \frac{1}{6} \frac{d}{dt} \sum_{j=1}^{N} u'_j u'_{j+1} = \frac{1}{6} \frac{d}{dt} \sum_{j=1}^{N} |u'_j|^2$$

$$+ \frac{1}{12} \frac{d}{dt} \sum_{j=1}^{N} \left| u'_j + u'_{j+1} \right|^2.$$

Combining (3.32)-(3.34) we deduce that

$$\frac{dE_h(t)}{dt} = 0$$

which is equivalent to (3.31).

■

Lemma 3.4 *For any $h > 0$ and any solution u of (3.1) the following identity holds:*

$$TE_h(0) - \frac{h}{12} \sum_{j=0}^{N} \int_0^T \left| u_j' - u_{j+1}' \right|^2 dt + X_h(t) \Big|_0^T = \frac{L}{2} \int_0^T \left[\left| \frac{u_N(t)}{h} \right|^2 + \frac{|u_N'(t)|^2}{6} \right] dt \qquad (3.35)$$

with

$$X_h(t) = h \sum_{j=1}^{N} j \left(u_{j+1} - u_{j-1} \right) \left(\frac{1}{3} u_j' + \frac{1}{12} u_{j+1}' + \frac{1}{12} u_{j-1}' \right). \qquad (3.36)$$

Proof. We multiply in (3.1) by $j \left(u_{j+1} - u_{j-1} \right) / 2$. It follows that

$$\int_{j=1}^{N} \int_0^T \left[\frac{2}{3} u_j'' + \frac{1}{6} u_{j+1}'' + \frac{1}{6} u_{j-1}'' \right] j \left(\frac{u_{j+1} - u_{j-1}}{2} \right) dt \qquad (3.37)$$

$$- \sum_{j=1}^{N} \int_0^T \left[\frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} \right] j \left(\frac{u_{j+1} - u_{j-1}}{2} \right) dt = I_1 - I_2 = 0.$$

Let us develop the two terms in (3.37). The second one I_2 can be treated as in the proof of Lemma 2.2. It follows that

$$I_2 = -\frac{1}{2} \sum_{j=0}^{N} \int_0^T \left| \frac{u_j - u_{j+1}}{h} \right|^2 dt + \frac{(N+1)}{2} \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt. \qquad (3.38)$$

On the other hand, in view of (2.33),

$$\begin{aligned}
I_1 &= \sum_{j=1}^{N} \int_0^T \left[\frac{2}{3} u_j'' + \frac{1}{6} u_{j+1}'' + \frac{1}{6} u_{j-1}'' \right] j \frac{(u_{j+1} - u_{j-1})}{2} dt \qquad (3.39) \\
&= \frac{1}{3} \sum_{j=1}^{N} \int_0^T u_j' u_{j+1}' dt + \frac{2}{3} \sum_{j=1}^{N} u_j' j \frac{(u_{j+1} - u_{j-1})}{2} \Big|_0^T \\
&\quad + \frac{1}{12} \int_{j=1}^{N} \int_0^T \left(u_{j+1}'' + u_{j-1}'' \right) j \left(u_{j+1} - u_{j-1} \right) dt \\
&= \frac{1}{3} \sum_{j=1}^{N} \int_0^T u_j' u_{j+1}' dt + \frac{1}{3} \sum_{j=1}^{N} u_j' j \left(u_{j+1} - u_{j-1} \right) \Big|_0^t \\
&\quad - \frac{1}{12} \sum_{j=1}^{N} \int_0^T j \left(\left| u_{j+1}' \right|^2 - \left| u_{j-1}' \right|^2 \right) dt + \frac{1}{12} \sum_{j=1}^{N} \left(u_{j+1}' + u_{j-1}' \right) j \left(u_{j+1} - u_{j-1} \right) \Big|_0^T \\
&= \frac{1}{3} \sum_{j=1}^{N} \int_0^T u_j' u_{j+1}' dt + \frac{1}{6} \sum_{j=1}^{N} \int_0^T \left| u_j' \right|^2 dt - \frac{N+1}{12} \int_0^T \left| u_N'(t) \right|^2 dt
\end{aligned}$$

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$$+ \sum_{j=1}^{N} j\left(u_{j+1} - u_{j-1}\right)\left[\frac{1}{3}u_j' + \frac{1}{12}u_{j+1}' + \frac{1}{12}u_{j-1}'\right]\Bigg|_0^T.$$

Combining (3.37)-(3.39) we deduce that

$$\frac{L}{2}\int_0^T\left[\left|\frac{u_N(t)}{h}\right|^2 + \frac{|u_N'(t)|^2}{6}\right]dt = \frac{h}{2}\sum_{j=0}^{N}\int_0^T\left|\frac{u_j - u_{j+1}}{h}\right|^2 dt + \frac{h}{3}\sum_{j=1}^{N}\int_0^T u_j' u_{j+1}' dt \qquad (3.40)$$

$$+ \frac{h}{6}\int_{j=1}^{N}\int_0^T\left|u_j'\right|^2 dt + X_n(t)\Bigg|_0^T$$

$$= TE_h(0) + \frac{h}{3}\sum_{j=0}^{N}\int_0^T\left[u_j' u_{j+1}' - \frac{|u_j' + u_{j+1}'|^2}{4}\right]dt + X_h(t)\Bigg|_0^T.$$

Finally we observe that

$$\sum_{j=0}^{N}\int_0^T\left[u_j' u_{j+1}' - \frac{|u_j' + u_{j+1}'|^2}{4}\right]dt = -\sum_{j=0}^{N}\int_0^T\frac{|u_j' - u_{j+1}'|^2}{4}dt. \qquad (3.41)$$

Combining (3.40) and (3.41) we obtain (3.35).

∎

Lemma 3.5 (Equipartition of energy) *For any $h > 0$ and any solution u of (3.1) the following identity holds:*

$$h\int_0^T\sum_{j=0}^{N}\left|\frac{u_{j+1} - u_j}{h}\right|^2 dt = h\int_0^T\sum_{j=0}^{N}\left[\frac{1}{3}\left|u_j'\right|^2 + \frac{1}{6}\left|u_j' + u_{j+1}'\right|^2\right]dt - Y_h(t)\Bigg|_0^T \qquad (3.42)$$

with

$$Y_h(t) = h\sum_{j=1}^{N}\left(\frac{2}{3}u_j' + \frac{1}{6}u_{j+1}' + \frac{1}{6}u_{j-1}'\right)u_j. \qquad (3.43)$$

Proof. We multiply in (3.1) by u_j. It follows that

$$h\sum_{j=1}^{N}\int_0^T\left(\frac{2}{3}u_j'' + \frac{1}{6}u_{j+1}'' + \frac{1}{6}u_{j-1}''\right)u_j dt = h\sum_{j=1}^{N}\int_0^T\left[\frac{u_{j+1} + u_{j-1} - 2u_j}{h^2}\right]u_j dt. \qquad (3.44)$$

We have

$$h \sum_{j=1}^{N} \int_0^T \left(\frac{2}{3} u_j'' + \frac{1}{6} u_{j+1}'' + \frac{1}{6} u_{j-1}'' \right) u_j dt = h \sum_{j=1}^{N} \left. \left(\frac{2}{3} u_j'' + \frac{1}{6} u_{j+1} + \frac{1}{6} u_{j-1} \right) u_j \right|_0^T \qquad (3.45)$$

$$- h \sum_{j=1}^{N} \int_0^T \left(\frac{2}{3} u_j' + \frac{1}{6} u_{j+1}' + \frac{1}{6} u_{j-1}' \right) u_j' dt$$

$$= Y_h(t)|_0^T - h \sum_{j=1}^{N} \int_0^T \left(\frac{2}{3} \mid u_j' \mid^2 + \frac{1}{3} u_j' u_{j+1}' \right) dt = Y_h(t) \Big|_0^T - h \sum_{j=1}^{N} \int_0^T \left(\frac{1}{3} \mid u_j' \mid^2 + \frac{1}{6} \mid u_j' + u_{j+1}' \mid^2 \right) dt$$

On the other hand,

$$h \sum_{j=1}^{N} \int_0^T \left(\frac{u_{j+1} - u_{j-1} - 2u_j}{h^2} \right) u_j dt = -h \sum_{j=0}^{N} \int_0^T \left| \frac{u_j - u_{j+1}}{h} \right|^2 dt. \qquad (3.46)$$

Combining (3.44)-(3.46) we deduce (3.42).

∎

Let us now estimate the term $\sum_{j=0}^{N} \int_0^T \left| u_j' - u_{j+1}' \right|^2 dt$ in (3.35). Let Λ be the largest eigenvalue entering in the Fourier development of the solution u of (3.1), i.e.,

$$u = \sum_{|\mu_k| \leq \sqrt{\Lambda}} a_k e^{i\mu_k t} \varphi_k \qquad (3.47)$$

with $\mu_k = \sqrt{\lambda_k}$ for $k > 0$ and $\mu_k = -\mu_{-k}$ when $k < 0$. Therefore

$$u' = i \sum_{|\mu_k| \leq \Lambda} a_k \mu_k e^{i\mu_k t} \varphi_k.$$

Thus

$$\sum_{j=0}^{N} \left| u_j' - u_{j+1}' \right|^2 = \sum_{j=0}^{N} \left| \sum_{|\mu_k| \leq \sqrt{\Lambda}} a_k \mu_k e^{i\mu_k t} \left(\varphi_{k,j} - \varphi_{k,j+1} \right) \right|^2 \qquad (3.48)$$

$$= \sum_{j=0}^{n} \sum_{|\mu_k| \leq \sqrt{\Lambda}} \mid a_k \mid^2 \mu_k^2 \left(\varphi_{k,j} - \varphi_{k,j+1} \right)^2 + \sum_{j=0}^{N} \sum_{\substack{|\mu_k| \leq \sqrt{\Lambda} \\ |\mu_\ell| \leq \sqrt{\Lambda} \\ \mu_k \neq \mu_\ell}} \mu_k \mu_\ell a_k \bar{a}_\ell e^{i(\mu_k - \mu_\ell)t} \left(\varphi_{k,j} - \varphi_{k,j+1} \right) \left(\varphi_{\ell,j} - \varphi_{\ell,j+1} \right).$$

The following identities hold:

Lemma 3.6 *For any eigenvectors φ with eigenvalue λ of (3.9) the following identity holds:*

$$\sum_{j=0}^{N} \left| \frac{\varphi_j - \varphi_{j+1}}{h} \right|^2 = \lambda \sum_{j=1}^{N} \left(\frac{2}{3} \mid \varphi_j \mid^2 + \frac{1}{3} \varphi_j \varphi_{j+1} \right) = \lambda \sum_{j=1}^{N} \left[\frac{1}{3} \mid \varphi_j \mid^2 + \frac{1}{6} \mid \varphi_j + \varphi_{j+1} \mid^2 \right] \quad (3.49)$$

Moreover, if φ_k and φ_ℓ are eigenvectors associated to eigenvalues $\lambda_k \neq \lambda_\ell$ it follows that

$$\sum_{j=0}^{N} \left(\varphi_{k,j} - \varphi_{k,j+1} \right) \left(\varphi_{\ell,j} - \varphi_{\ell,j+1} \right) = 0. \quad (3.50)$$

Proof. Identity (3.49) can be derived easily multiplying in (3.9) by φ_j and adding for $j = 1, \cdots, N$.

To prove (3.50) we multiply by $\varphi_{\ell,j}$ the equation satisfied by $\varphi_{k,j}$. It follows that

$$-\sum_{j=1}^{N} \left(\varphi_{k,j+1} - \varphi_{k,j-1} - 2\varphi_{k,j} \right) \varphi_{\ell,j} = \lambda_k h^2 \sum_{j=1}^{N} \left(\frac{2}{3} \varphi_{k,j} + \frac{1}{6} \varphi_{k,j+1} + \frac{1}{6} \varphi_{k,j-1} \right) \varphi_{\ell,j}. \quad (3.51)$$

On the other hand,

$$\sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} - 2\varphi_{k,j} \right) \varphi_{\ell,j} = \sum_{j=1}^{N} \left(\varphi_{\ell,j+1} + \varphi_{\ell,j-1} - 2\varphi_{\ell,j} \right) \varphi_{k,j}. \quad (3.52)$$

Reversing the role of φ_k and φ_ℓ in (3.51) we also have:

$$-\sum_{j=1}^{N} \left(\varphi_{\ell,j+1} + \varphi_{\ell,j-1} - 2\varphi_{\ell,j} \right) \varphi_{k,j} = \lambda_\ell h^2 \sum_{j=1}^{N} \left(\frac{2}{3} \varphi_{\ell,j} + \frac{1}{6} \varphi_{\ell,j-1} + \frac{1}{6} \varphi_{\ell,j+1} \right) \varphi_{k,j}. \quad (3.53)$$

On the other hand,

$$\sum_{j=1}^{N} \left(\frac{2}{3} \varphi_{k,j} + \frac{1}{6} \varphi_{k,j-1} + \frac{1}{6} \varphi_{k,j+1} \right) \varphi_{\ell,j} = \sum_{j=1}^{N} \left(\frac{2}{3} \varphi_{\ell,j} + \frac{1}{6} \varphi_{\ell,j+1} + \frac{1}{6} \varphi_{\ell,j-1} \right) \varphi_{k,j}. \quad (3.54)$$

Combining (3.51)-(3.54) and taking into account that $\lambda_k \neq \lambda_\ell$ we deduce that all the terms entering in these identities vanish. In particular,

$$\sum_{j=1}^{N} \left(\frac{2}{3} \varphi_{k,j} + \frac{1}{6} \varphi_{k,j+1} + \frac{1}{6} \varphi_{k,j-1} \right) \varphi_{\ell,j} = 0; \sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} - 2\varphi_{k,j} \right) \varphi_{\ell,j} = 0. \quad (3.55)$$

Combining the two identities in (3.55) it is easy to see that

$$\sum_{j=1}^{N} \varphi_{k,j} \varphi_{\ell,j} = 0; \sum_{j=1}^{N} \left(\varphi_{k,j+1} + \varphi_{k,j-1} \right) \varphi_{\ell,j} = 0. \quad (3.56)$$

From (3.56) we easily deduce that (3.50) holds.

In view of Lemma 3.6 the term in (3.48) can be estimated as follows:

$$\sum_{j=0}^{N}\left|u_j'-u_{j+1}'\right|^2 = \sum_{j=0}^{N}\sum_{\lambda_k\leq\Lambda}\lambda_k\mid a_k\mid^2 |\varphi_{k,j}-\varphi_{k,j+1}|^2 \tag{3.57}$$

$$= h^2\sum_{j=1}^{N}\sum_{|\mu_k|\leq\sqrt{\Lambda}}\lambda_k^2\mid a_k\mid^2\left(\frac{2}{3}\mid\varphi_{k,j}\mid^2+\frac{1}{3}\varphi_{k,j}\varphi_{k,j+1}\right)$$

$$\leq h^2\Lambda\sum_{j=1}^{N}\sum_{|\mu_k|\leq\sqrt{\Lambda}}\lambda_k\mid a_k\mid^2\left(\frac{2}{3}\mid\varphi_{k,j}\mid^2+\frac{1}{3}\varphi_{k,j}\varphi_{k,j+1}\right)$$

$$= h^2\Lambda\sum_{j=1}^{N}\sum_{|\mu_k|\leq\sqrt{\Lambda}}\lambda_k\mid a_k\mid^2\left(\frac{1}{3}\mid\varphi_{k,j}\mid^2+\frac{1}{6}\left(\varphi_{k,j}+\varphi_{k,j+1}\right)^2\right)$$

$$= h^2\Lambda\sum_{j=0}^{N}\left[\frac{1}{3}\mid u_j'\mid^2+\frac{1}{6}\left|u_j'+u_{j+1}'\right|^2\right].$$

In the last step we have used the fact that

$$\sum_{j=0}^{N}\left[\frac{1}{3}\varphi_{k,j}\varphi_{\ell,j}+\frac{1}{6}\left(\varphi_{k,j}+\varphi_{k,j+1}\right)\left(\varphi_{\ell,j}+\varphi_{\ell,j+1}\right)=0\right.$$

if φ_k and φ_ℓ are eigenvectors associated to eigenvalues $\lambda_k\neq\lambda_\ell$, which is a consequence of (3.56).

On the other hand, combining the conservation of energy identity (3.31) and the equipartition of energy (3.42) we deduce that

$$h\int_0^T\sum_{j=0}^{N}\left(\frac{1}{3}\left|u_j'\right|^2+\frac{1}{6}\left|u_j'+u_{j+1}'\right|^2\right)dt = TE(0)+\left.\frac{Y_h(t)}{2}\right|_0^T. \tag{3.58}$$

Combining (3.40)-(3.41) with (3.57)-(3.58) we deduce that

$$T\left(1-\frac{\Lambda h^2}{12}\right)E(0)+\left.Z_h(t)\right|_0^T \leq \frac{L}{2}\int_0^T\left[\left|\frac{u_N(t)}{h}\right|^2+\frac{|u_N'(t)|^2}{6}\right]dt \tag{3.59}$$

with

$$Z_h(t) = X_h(t)-\frac{\Lambda h^2}{24}Y_h(t). \tag{3.60}$$

The following holds:

Lemma 3.7 *For any $h>0$ and any solution u of (3.1) it follows that*

$$|Z_h(t)| \leq \sqrt{L^2-\frac{\Lambda h^4}{48}-\frac{3\Lambda h^2}{16\lambda_1}}, \quad \forall' < t < T. \tag{3.61}$$

Proof. To simplify the notation we do not make explicit the dependence on t. We have

$$Z_h 8t) = X_h(t) - \frac{\Lambda h^2}{24} Y_h(t) = h \sum_{j=1}^{N} \left(j \frac{(u_{j+1} - u_j)}{2} - \frac{\Lambda h^2}{24} u_j \right) \left(\frac{2}{3} u'_j + \frac{1}{6} u'_{j+1} + \frac{1}{6} u'_{j-1} \right) = \sum_{i,j=1}^{N} m_{ij} a_i b_j$$

with

$$a_i = i \frac{(u_{i+1} - u_i)}{2} - \frac{\Lambda h^2}{24} u_i; \; b_i = u'_i$$

and

$$m_{ii} = \frac{2h}{3}, m_{ij} = \frac{h}{6} \text{ if } j = i+1 \text{ or } i-1, m_{ij} = 0 \text{ if } \mid i - j \mid \geq 2.$$

Therefore

$$\mid Z_h \mid \leq \left(\sum_{i,j=1}^{N} m_{ij} a_i a_j \right)^{1/2} \left(\sum_{i,j=1}^{N} m_{ij} b_i b_j \right)^{1/2}. \tag{3.62}$$

On the other hand,

$$\sum_{i,j=1}^{N} m_{ij} b_i b_j = h \sum_{j=1}^{N} \left[\frac{2}{3} \left| u'_j \right|^2 + \frac{1}{6} \left(u'_{j+1} - u'_{j-1} \right) u'_j \right] = h \left[\sum_{j=0}^{N} \frac{1}{3} \left| u'_j \right|^2 + \frac{1}{6} \left| u'_j + u'_{j+1} \right|^2 \right] \tag{3.63}$$

and

$$\sum_{i,j=1}^{N} m_{ij} a_i a_j \leq h \sum_{j=1}^{N} \mid a_i \mid^2 = \frac{h}{4} \sum_{j=1}^{N} \left[j \left(u_{j+1} - u_{j-1} \right) - \frac{\Lambda h^2}{12} u_j \right]^2 \tag{3.64}$$

$$= \frac{h}{4} \sum_{j=1}^{N} \left[2j^2 \left| u_{j+1} - u_j \right|^2 + 2j^2 \mid u_j - u_{j-1} \mid^2 + \frac{\Lambda^2 h^4}{144} u_j^2 - \frac{\Lambda h^2}{6} j u_j \left(u_{j+1} - u_{j-1} \right) \right]$$

$$\leq L^2 h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2 + \frac{\Lambda^2 h^5}{576} \sum_{j=1}^{N} \mid u_j \mid^2 + \frac{\Lambda h^3}{24} \sum_{j=1}^{N} u_j u_{j+1}$$

$$= L^2 h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2 - \frac{\eta h^2}{2} h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2 + \left(\eta^2 + \eta \right) \sum_{j=1}^{N} \mid u_j \mid^2$$

$$\leq \left(L^2 - \frac{\eta h^2}{2} + \frac{3(\eta^2 + \eta)}{\lambda_1} \right) h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2 \leq \left(L^2 - \frac{\eta h^2}{2} + \frac{9\eta}{2\lambda_1} \right) h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2.$$

We have used the notation $\eta = \Lambda h^2 / 24$, the fact that $\eta \leq 1/2$ and also that

$$\frac{1}{3} h \sum_{j=1}^{N} \mid u_j \mid^2 \leq \frac{1}{\lambda_1} h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2$$

which is a consequence of (3.49).

Combining (3.62)-(3.64) we deduce that

$$
\begin{aligned}
\mid Z_h \mid &\leq \left[h \sum_{j=1}^{N} \left(\frac{1}{3} \mid u'_j \mid^2 + \frac{1}{6} \left| u'_j + u'_{j+1} \right|^2 \right) \right]^{1/2} \left[\left(L^2 - \frac{\eta h^2}{2} + \frac{9\eta}{2\lambda_1} \right) h \sum_{j=0}^{N} \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right]^{1/2} \\
&\leq \sqrt{L^2 - \frac{\eta h^2}{2} + \frac{9\eta}{2\lambda_1}} \, E_h(0).
\end{aligned}
$$

which is equivalent to (3.61).

∎

Let us now set $\Lambda = \gamma/h^2$. Combining (3.59) and (3.61) we obtain

$$
\left[T(1 - \gamma/12) - 2\sqrt{L^2 - \frac{\gamma h^2}{48} + \frac{3\gamma}{16\lambda_1}} \right] E_h(0) \leq \frac{L}{2} \int_0^T \left[\left| \frac{u_N(t)}{h} \right|^2 + \frac{|u'_N(t)|^2}{6} \right] dt. \qquad (3.65)
$$

Therefore, the statement of Theorem 3.2 holds with

$$
T(\gamma) = \frac{2\sqrt{L^2 + \frac{\gamma}{16}\left(\frac{3}{\lambda_1} - \frac{h^2}{3}\right)}}{1 - \gamma/12}
$$

and

$$
C(T, \gamma) = \frac{L}{2\left(T(1 - \gamma/12) - 2\sqrt{L^2 + \frac{\gamma}{16}\left(\frac{3}{\lambda_1} - \frac{h^2}{3}\right)}\right)}.
$$

3.5 Boundary observability of the discrete wave equation: improved estimates

The goal of this section is to get rid of the term $\int_0^T \mid u'_N(t) \mid^2 dt$ on the right hand side of (3.14).

Note that the solution u of (3.1) can be written as

$$
u = \sum a_k e^{i\mu_k t} \varphi_k
$$

and therefore

$$
u'_N(t) = i \sum a_k \mu_k e^{i\mu_k t} \varphi_k.
$$

Thus

$$
\int_0^T |u'_N(t)|^2 \, dt = \int_0^T \left| \sum a_k \mu_k e^{i\mu_k t} \varphi_{k,N} \right|^2 dt. \qquad (3.66)
$$

We now restrict our analysis to the solutions in the class $\mathcal{C}_h(\gamma)$ that are generated by the eigenvectors associated to eigenvalues

$$
\lambda \leq \gamma/h^2 \qquad (3.67)
$$

with $0 < \gamma < 12$. Restriction (3.67) is equivalent to

$$(j + 1)h \leq \varepsilon(\gamma)L \tag{3.68}$$

for a suitable $0 < \varepsilon(\gamma) < 1$. We shall return later to the explicit computation of $\varepsilon(\gamma)$.

Then, according (3.22) the gap between the roots of consecutive eigenvalues in the range (3.67) is given by

$$\frac{\pi}{L} \left(\frac{1 + \cos(\varepsilon(\gamma)\pi)}{2} \right)^{1/2}. \tag{3.69}$$

Then, according to Ingham's inequality [I] we deduce that, provided,

$$T > \frac{2L}{((1 + \cos(\varepsilon(\gamma)\pi))/2)^{1/2}}, \tag{3.70}$$

it follows that

$$
\int_0^T \left| \sum_{|\mu_k| \leq \sqrt{\gamma}/h} a_k \mu_k e^{i\mu_k t} \varphi_{k,N} \right|^2 \tag{3.71}
$$

$$
\leq \quad C_2(T, \varepsilon(\gamma)) \sum_{|\mu_k| \leq \sqrt{\gamma}/h} | a_k |^2 \lambda_k | \varphi_{k,N} |^2
$$

$$
\leq \quad C_2(T, \varepsilon(\gamma))\gamma \sum_{|\mu_k| \leq \sqrt{\gamma}/h} | a_k |^2 \left| \frac{\varphi_{k,N}}{h} \right|^2
$$

$$
\leq \quad \frac{C_2(T, \varepsilon(\gamma))\gamma}{C_1(T, \varepsilon(\gamma))} \int_0^T \left| \sum_{|\mu_k| \leq \sqrt{\gamma}/h} a_k e^{i\mu_k t} \frac{\varphi_{k,N}}{h} \right|^2 dt
$$

$$
= \quad \frac{\gamma C_2(T, \varepsilon(\gamma))}{C_1(T, \varepsilon(\gamma))} \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt.
$$

Combining Theorem 3.2 and (3.71) it follows that:

Theorem 3.3 *For any $0 < \gamma < 12$ and*

$$T > \max \left(T(\gamma), \frac{2L}{((1 + \cos(\varepsilon(\gamma)\pi))/2)^{1/2}} \right) \tag{3.72}$$

it follows that

$$E_h(0) \leq \left[C(T, \gamma) + \frac{\gamma C_2(T, \varepsilon(\gamma))}{6C_1(T, \varepsilon(\gamma))} \right] \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{3.73}$$

for any solution u of (3.1) inthe class $\mathcal{C}_h(\gamma)$.

Remark. The time of observability in Theorem 3.3 is the maximum between the time $T(\gamma)$ in Theorem 3.2 and the time (3.70) needed to apply Ingham's inequality to the eigenvalues corresponding to the class of solutions $\mathcal{C}_h(\gamma)$. The observability constant in (3.73) is the addition of the constant $C(T,\gamma)$ of Theorem 3.2 and the constant needed to absorve the term $\int_0^T |\ u'_N(t)\ |^2\ dt$ by Ingham's inequality as in (3.71).

Note that $\varepsilon(\gamma) \to 0$ as $\gamma \to 0$ while the constants $C_1(T,\varepsilon)$ and $C_2(T,\varepsilon)$ in Ingham's inequality converge to 1. Thus, according to Theorem 3.2, the observability constant in (3.73) converges to $L/2(T-2L)$ as $\gamma \to 0$.

Observe that an inequality of the form (3.73) can also be obtained directly by Ingham's inequality. However we have preferred to obtain the weaker form of observability inequality in Theorem 3.2 by multipliers since this method can be more easily adapted to other situations.

∎

Let us now compute $\varepsilon(\gamma)$. According to the explicit value of the eigenvalues obtained in (3.17) we have to estimate j so that

$$6\left(1 - \cos\left((j+1)\pi h/L\right)\right) \leq \gamma\left(2 + \cos((j+1)\pi h/L)\right),$$

or, taking into account that $(j+1)h \leq \varepsilon L$, we have to estimate $\varepsilon > 0$ such that

$$6(1 - \cos(\varepsilon\pi)) \leq \gamma(2 + \cos(\varepsilon\pi)),$$

or, equivalently,

$$6 - 2\gamma \leq (\gamma + 6)\cos(\varepsilon\pi),$$

i.e.,

$$\varepsilon(\gamma) = \frac{1}{\pi}\arccos\left(\frac{6 - 2\gamma}{\gamma + 6}\right). \tag{3.74}$$

Obviously $\varepsilon(\gamma) \to 0$ as $\gamma \to 0$ and $\varepsilon(\gamma) \to 1$ as $\gamma \to 12$.

According to (3.70) and (3.74) the time needed to apply Ingham's inequality is:

$$\frac{2L}{((1 + \cos(\varepsilon(\gamma)\pi))/2)^{1/2}} = 2L\left(\frac{12 + 2\gamma}{12 - \gamma}\right)^{1/2}.$$

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