NULL CONTROLLABILITY OF A SYSTEM OF LINEAR THERMOELASTICITY

by

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Abstract

We consider a linear system of thermoelasticity in a compact, C^∞, n-dimensional connected Riemannian manifold. This system consists of a wave equation coupled to a heat equation. When the boundary of the manifold is non-empty Dirichlet boundary conditions are considered. We study the controllability properties of this system when the control acts in the hyperbolic equation (and not in the parabolic one) and has its support restricted to an open subset of the manifold. We show that, if the control time and the support of the control satisfy the geometric control condition for the wave equation, this system of thermoelasticity is null-controllable. More precisely, any finite energy solution can be driven to zero at the control time. An analogous result is proved when the control acts on the parabolic equation. Finally, when the manifold has no boundary, the null-controllability of the linear system of three-dimensional thermoelasticity is proved.

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1 Introduction and main results

Let M be a n-dimensional, compact, connected and C^∞ Riemannian manifold. Let us consider the following system of thermoelasticity in M:

$$\begin{cases} u_{tt} - c^2\Delta u + \alpha\Delta\theta = 0 & \text{in} \quad M \times (0,\infty) \\ \theta_t - \nu\Delta\theta + \beta u_t = 0 & \text{in} \quad M \times (0,\infty) \\ u = \theta = 0 & \text{on} \quad \partial M \times (0,\infty) \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{cases} u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & \text{in} \quad M \\ \theta(x,0) = \theta^0(x) & \text{in} \quad M. \end{cases} \tag{1.2}$$

The coupling parameters α, β and the viscosity ν are assumed to be positive constants.

System (1.1) represents the coupling between a wave equation for the *displacement* u and a parabolic equation for the *temperature* θ. We assume a non trivial coupling: $\alpha, \beta > 0$.

The boundary of M may be empty. In that case, the Dirichlet boundary conditions in (1.1) are meaningless.

This system arises naturally in three space dimensions when analyzing the linear system of thermoelasticity:

$$\begin{cases} w_{tt} - \mu\Delta w - (\lambda+\mu)\nabla \text{ div } w + \alpha\nabla\theta = 0 & \text{in} \quad M \times (0,\infty) \\ \theta_t - \nu\Delta\theta + \beta \text{ div } w_t = 0 & \text{in} \quad M \times (0,\infty) \end{cases} \tag{1.3}$$

When the manifold M has no boundary system (1.3) can be decoupled in a diagonal system of wave equations for $v = \text{ curl } w$:

$$v_{tt} - \mu\Delta v = 0 \text{ in } M \times (0,\infty) \tag{1.4}$$

and system (1.1) for the pair $(u = \operatorname{div} w, \theta)$ with $c^2 = \lambda + 2\mu$.

From a control theoretical point of view the wave equation (1.4) is by now well understood (see C. Bardos, G. Lebeau and J. Rauch [BLR] and J. L. Lions [Li1,2]). Thus we shall mainly focus in (1.1)-(1.2).

System (1.1)-(1.2) is well-posed in the energy space $H = H_0^1(M) \times L^2(M) \times H_0^1(M)$ and the energy

$$E(t) = \frac{1}{2}\int_M [|u_t(x,t)|^2 + c^2|\nabla u(x,t)|^2 + \frac{\alpha}{\beta}|\nabla\theta(x,t)|^2]dx \tag{1.5}$$

decreases along trajectories. More precisely

$$\frac{dE}{dt}(t) = -\frac{\alpha\nu}{\beta}\int_M |\Delta\theta(x,t)|^2 dx. \tag{1.6}$$

The problem of the controllability of system (1.1)-(1.2) consists, roughly, on the following: *Given* $T > 0$, *find sufficient conditions on the control region* Ω (*an open and non-empty subset of* M) *such that for "any" initial and final conditions* (u^0, u^1, θ^0) , (v^0, v^1, η^0) *there exists a control* $f = f(x,t) \in L^2(M \times (0,T))$ *such that the solution of*

$$\begin{cases} u_{tt} - c^2\Delta u + \alpha\Delta\theta = f\chi_\Omega & \text{in} \quad M \times (0,T) \\ \theta_t - \nu\Delta\theta + \beta u_t = 0 & \text{in} \quad M \times (0,T) \\ u = \theta = 0 & \text{on} \quad \partial M \times (0,T) \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad M \\ \theta(x,0) = \theta^0(x) & \text{in} \quad M \end{cases} \tag{1.7}$$

satisfies, at the final time $t = T$, *the condition*

$$\begin{cases} u(x,T) = v^0(x), u_t(x,T) = v^1(x) & \text{in} \quad M \\ \theta(x,T) = \eta^0(x) & \text{in} \quad M. \end{cases} \tag{1.8}$$

In (1.7), χ_Ω denotes the characteristic function of the set Ω.

Notice that we act on the equation of displacement through the force f concentrated on $\Omega \times (0,T)$ but we do not introduce any control (heat source) in the parabolic equation. Thus, we are trying to control the full system by a control acting only on the first equation.

As we shall see later on (section 2) equation (1.1) has strong regularizing effects on an infinite dimensional subspace of H. Thus it is not natural to expect exact controllability results, i.e. the property above to hold for every initial and final data (u^0, u^1, θ^0), (v^0, v^1, η^0) in H.

However we can expect to be able to reach any sufficiently smooth final state. We thus focus on the *null controllability* problem: *given* $T > 0$, *find sufficient conditions on* Ω *such that for every* (u^0, u^1, θ^0) *in* H *there exists a control* $f \in L^2(M \times (0,T))$ *such that the solutions of* (1.7) *satisfies*:

$$\begin{cases} u(x,T) = u_t(x,T) = 0 & \text{in } M \\ \theta(x,T) = 0 & \text{in } M. \end{cases} \tag{1.9}$$

Our main result is as follows.

Theorem 1. *Assume that* $T > 0$ *and* $\Omega \subset M$ *satisfy the geometric control condition for the wave equation in* M. *More precisely, suppose that any ray of geometric optics of length* cT *intersects* Ω. *Then,*

a) If $\partial M \neq \emptyset$, *for any* (u^0, u^1, θ^0) *in* H *there exists* $f \in L^2(M \times (0,T))$ *such that* (1.9) *holds.*

b) If $\partial M = \emptyset$ *the same result holds for all initial data such that* $\int_M (\theta^0 + \beta u^0) = 0$.

Moreover, there exists a constant $C > 0$ *such that*

$$\| f \|_{L^2(M\times(0,T))} \leq C \| (u^0, u^1, \theta^0) \|_H \tag{1.10}$$

for every $(u^0, u^1, \theta^0) \in H$ *as above.*

Remarks: 1.- Theorem 1 shows that the geometric control condition, introduced in [BLR] which is, roughly, a necessary and sufficient condition for the exact controllability of the wave equation in the energy space with L^2 -controls, is also sufficient for the null control of the system of thermoelasticity. When M has no boundary the geometric control condition reduces to the following: Any geodesic curve in M of length cT intersects Ω.

2.- The methods of [Z] applied to system (1.7) would provide the exact controllability of the displacement and simultaneously the approximate controllability of the temperature. More precisely, proceeding as in [Z] one can show that given any (u^0, u^1, θ^0) and (v^0, v^1, η^0) in H and $\varepsilon > 0$ there exists $f \in L^2(0,T;H)$ such that

$$\begin{cases} u(x,t) = v^0(x), u_t(x,T) = v^1(x) & \text{in } M \\ \| \theta(T) - \eta^0 \|_{H^1(M)} \leq \varepsilon. \end{cases} \tag{1.11}$$

However this type of result does not provide the reachability of any specific final state. In this sense Theorem 1 is much sharper.

Theorem 1 is a higher dimensional version of the one-dimensional result by S. Hansen [H]. The one-dimensional results in [H] are however stronger than ours since they apply in the context of boundary control.

3.- In the case of the heat equation the null controllability was recently proved by G. Lebeau and L. Robbiano [LRo] without restriction on the time of control and the control region. It was known from D. Russell [R] that the exact controllability of the wave equation implies the null controllability of the heat equation. But Russell's argument introduces unnecessary restrictions on Ω and T for the control of the heat equation (those that are needed for the exact controllability of the wave equation).

4.- There are various results in the literature showing that thermoelastic systems are partially controllable in the sense that, when the coupling parameters (α and β in (1.7)) are small enough the displacement is exactly controllable. However these results do not provide any information about the controllability of the temperature (see K. Narukawa [N], J. Lagnese [L] and J. L. Lions [Li2]).

5.- When $\partial M = \emptyset$, the quantity $\int_M (\theta(x,t) + \beta u(x,t))dx$ is preserved along trajectories. Therefore, $\int_M (\theta^0 + \beta u^0)dx = 0$ is a necessary condition for the null-controllability of system (1.7). Theorem 1 shows that, under the geometric control condition, this constraint is also sufficient.

6.- When $\partial M = \emptyset$ we can prove the null-controllability of any initial data by adding a second control in the parabolic equation of (1.7), i. e. if we replace the first two equations of (1.7) by

$$\begin{cases} u_{tt} - c^2\Delta u + \alpha\Delta\theta = f\chi_\Omega & \text{in} \quad M \times (0,\infty) \\ \theta_t - \nu\Delta\theta + \beta u_t = g & \text{in} \quad M \times (0,\infty). \end{cases}$$

The parabolic control g can be taken to be of the form $g = c(t)h(x)$ where h is any smooth function such that $\int_M h \neq 0$ and $c(t)$ is a control that depends in the initial data. To do this we may proceed in two steps. First, during an arbitrarily small time interval, we use the control g (with $f = 0$) to drive the quantity $\int_M (\theta(x,t) + \beta u(x,t))dx$ to zero. Then, applying part b) of the Theorem above we choose the control f with $g = 0$ to drive both components of the solution to zero.

■

Theorem 1 is completely equivalent to the following observability inequality for the adjoint system of thermoelasticity:

$$\begin{cases} v_{tt} - (c^2 + \alpha\beta)\Delta v + \nu\beta\Delta\psi = 0 & \text{in} \quad M \times (0,T) \\ -\psi_t - \nu\Delta\psi + \alpha\Delta v = 0 & \text{in} \quad M \times (0,T) \\ v = \psi = 0 & \text{on} \quad \partial M \times (0,T) \\ v(T) = v^0, v_t(T) = v^1, \psi(T) = \psi^0 & \text{in} \quad M. \end{cases} \tag{1.12}$$

Theorem 2. *Under the assumptions of Theorem 1 there exists $C > 0$ such that*

a) If $\partial M \neq \emptyset$

$$\| v^0 \|^2_{L^2(M)} + \| v^1 - \beta\psi^0 \|^2_{H^{-1}(M)} + \| \psi^0 \|^2_{H^{-1}(M)} \leq C \int_0^T \int_\Omega |v|^2 dxdt \tag{1.13}$$

for every solution of (1.12) *with data* $(v^0, v^1, \psi^0) \in H' = L^2(M) \times H^{-1}(M) \times H^{-1}(M)$.

b) If $\partial M = \emptyset$ the same holds for all initial data such that $\int_M \psi^0 = 0$.

The constant C depends on M, Ω, T and the parameters of the system but it is independent of the initial data.

Remark: 1.- Notice that (1.13), by backward uniqueness, implies the following unique-continuation property: If (v,ψ) solve (1.12) and $v = 0$ in $\Omega \times (0,T)$ with Ω and T satisfying the geometric control condition, then $v \equiv \psi \equiv 0$ in $M \times (0,T)$ if $\partial M \neq \emptyset$. If $\partial M = \emptyset$ this implies that $v \equiv 0$ and $\psi \equiv c$, c being a constant.

We do not use any analiticity assumption on M since in order to prove (1.13) and therefore the unique continuation property above, we only use the unique continuation of the eigenfunctions of the Laplacian.

2.- Condition $\int_M \psi^0 = 0$ is necessary for (1.13) if $\partial M = \emptyset$. Indeed, it is sufficient to observe that $(v, \psi) \equiv (0, 1)$ solve (1.12).

■

As an immediate consequence of Theorem 1 we get the reachability of the range, at time $t = T$, of the semigroup $S(t) : H \to H$ generated by system (1.1).

Corollary 1. *Under the assumptions of Theorem 1,*

a) If $\partial M \neq \emptyset$ for every (u^0, u^1, θ^0) and $(v^0, v^1, \eta^0) \in S(T)H$ there exists a control $f \in L^2(M \times (0,T))$ such that the solution of (1.7) satisfies (1.8).

b) If $\partial M = \emptyset$ the same result holds provided $\int_M (\theta^0 + \beta u^0)dx = \int_M (\eta^0 + \beta v^0)dx$.

Moreover, there exists a constant $C > 0$ such that

$$\| f \|_{L^2(M\times(0,T))} \leq C(\| (u^0, u^1, \theta^0) \|_H + \| S(T)^{-1}(v^0, v^1, \eta^0) \|_H). \tag{1.14}$$

Remark: By backward uniqueness and duality it is easy to see that for any $(v^0, v^1) \in H_0^1(M) \times L^2(M)$ the set $\{\eta^0 \in H : (v^0, v^1, \eta^0) \in S(T)H\}$ is dense in $H_0^1(M)$. Therefore, Corollary 1 implies the exact-approximate controllability results of the form (1.11) obtained in [Z].

■

The method of proof is based on a spectral decomposition of system (1.1) and its adjoint (1.12) on the basis generated by the eigenfunctions of the laplacian in $H_0^1(M)$. It is proved that, for high frequencies, the spectrum splits into a parabolic and a hyperbolic part. Projecting the dynamics of the system into the parabolic and hyperbolic subspaces (those generated by the eigenfunctions associated to the parabolic and hyperbolic eigenvalues respectively) the system is decomposed into two weakly coupled systems, the first one behaving as a heat equation and the second one as a wave equation. The wave component is easily handeled applying the results in [BLR]. The parabolic part is treated by using the method developed in [LRo]. At this level the following observability inequality for the eigenfunctions of the laplacian is essential:

Theorem 3. *Let M be a compact, connected Riemannian manifold of class C^∞. Let $e_j \in H_0^1(M)$ be an orthonormal basis of $L^2(M)$ consisting on the eigenvalues of the Dirichlet Laplacian*:

$$\begin{cases} -\Delta e_j = \omega_j^2 e_j & \text{in} \quad M \\ e_j = 0 & \text{on} \quad \partial M. \end{cases}$$

Then, for any open an non-empty subset Ω of M there exist two positive constants $C_j > 0, j = 1, 2$ such that

$$\int_\Omega |\sum_{\omega_j \leq \mu} a_j e_j(x)|^2 dx \geq C_1 e^{-C_2 \mu} \sum_{\omega_j \leq \mu} |a_j|^2 \tag{1.15}$$

for every $\mu > 0$ and every $\{a_j\}_{\omega_j \leq \mu}$.

■

This theorem is proved by means of suitable Carleman type inequalities.

The rest of the paper is organized as follows. Section 2 is devoted to the spectral analysis of system (1.1). In particular a spectral decomposition of the energy space H into its parabolic and hyperbolic components is given. In Section 3 we describe the fixed point algorithm that will be used to prove the null-controllability. Section 4 is devoted to the controllability of the hyperbolic component. In Section 5, following [LRo] we analyze the controllability of the parabolic component. In Section 6 we complete the proof of Theorem 1. In Section 7 we prove that the null-controllability also holds when the control acts in the heat equation of system (1.1). In Section 8 we discuss the controllability of the linear system of three-dimensional thermoelasticity (1.3). Finally, in an Appendix we sketch the proof of an interpolation inequality for the Laplacian which is an essential technical step in Section 5.

2 Spectral Analysis

Let us denote by $\{e_j\}_{j\in N}$ an orthonormal basis of $L^2(M)$ constituted by the eigenfunctions of the Laplacian in M:

$$\begin{cases} -\Delta e_j = \omega_j^2 e_j & \text{in } M \\ e_j = 0 & \text{on } \partial M. \end{cases} \tag{2.1}$$

We note that $\omega_0 = 0$ and $e_0 = 1$ if $\partial M = \emptyset$. When $\partial M \neq \emptyset, \omega_0 = 0$ is not an eigenvalue and the first one $\omega_1 > 0$ is simple and its corresponding eigenfunction is of constant sign. The rest of the eigenvalues ω_j constitute a non-decreasing sequence of strictly positive real numbers

$$0 = \omega_0 < \omega_1 \le \omega_2 \le \dots$$

We can rewrite system (1.1) as a first order system for the new unknown

$$U(t) = \begin{pmatrix} u(t) \\ u_t(t) \\ \theta(t) \end{pmatrix} = (u(t), u_t(t), \theta(t))^t.$$

We then have

$$\frac{dU}{dt}(t) = AU(t) \tag{2.2}$$

where

$$A = \begin{pmatrix} 0 & I & 0 \\ c^2\Delta & 0 & -\alpha\Delta \\ 0 & -\beta I & \nu\Delta \end{pmatrix} \tag{2.3}$$

is an unbounded operator in H with domain

$$D(A) = \left(H^2(M) \cap H_0^1(M)\right) \times H_0^1(M) \times \left\{\theta \in H^3(M) : \theta = \Delta\theta = 0 \text{ on } \partial M\right\}.$$

The operator A generates a strongly continuous semigroup in $H, e^{At} : H \to H$. Thus, given $U^0 = (u^0, u^1, \theta^0)^t \in H, U(t) = e^{At}U^0$ solves (1.15) with initial data U^0 and therefore its first and third components $(u(t), \theta(t))$ solve (1.1)-(1.2).

We can decompose $U(t)$ in Fourier series:

$$U(t) = \sum_{j=0}^{\infty} U_j(t) e_j \tag{2.4}$$

where each $U_j(t) : [0,\infty) \to I\!\!R^3$ satisfies the linear system of differential equations

$$\frac{dU_j(t)}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ -c^2\omega_j^2 & 0 & \alpha\omega_j^2 \\ 0 & -\beta & -\nu\omega_j^2 \end{pmatrix} U_j(t) = A_j U_j(t). \tag{2.5}$$

The eigenvalues $\lambda \in I\!\!C$ of the matrix A_j are the roots of the cubic equation

$$det(\lambda I - A_j) = \left(\lambda^2 + c^2\omega_j^2\right)\left(\lambda + \nu\omega_j^2\right) + \alpha\beta\lambda\omega_j^2 = 0. \tag{2.6}$$

When $\partial M = \emptyset$, for $j = 0$, since $\omega_0 = 0$, the equation becomes $\lambda^3 = 0$,

$$e^{tA_0} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & -\beta t & 1 \end{pmatrix}$$

and the corresponding solution of (2.5) is of the form

$$U_j(t) = (u_0^0 + tu_0^1, u_0^1, \theta_0^0 - \beta u_1^0 t)^t$$

where $(u_0^0, u_0^1, \theta_0^0)^t$ are the initial data.

For $j > 0$, we have $Re\lambda < 0$ for every root of (2.6) since the energy $E_j(t)$ of the solution $U_j(t) = (u_j(t), u_{j,t}(t), \theta_j(t))$ given by

$$E_j(t) = \frac{1}{2}\left[|u_{j,t}(t)|^2 + \omega_j^2(c^2|u_j(t)|^2 + \frac{\alpha}{\beta}|\theta_j(t)|^2)\right]$$

satisfies

$$\frac{dE_j(t)}{dt} = -\frac{\alpha}{\beta}\nu\omega_j^4|\theta_j(t)|^2.$$

and $\theta_j(t) \not\equiv 0$. Indeed, since $U_j(t) = e^{\lambda t}(u_j^0, u_j^1, \theta_j^0)^t$ and $(\lambda I - A_j)(u_j^0, u_j^1, \theta_j^0)^t = 0$, $\theta_j \equiv 0$ implies $U_j \equiv 0$.

Let us now analyze the behavior of the roots of (2.6) for ω_j large. The roots λ of (2.6) split in two types.

Let us first analyze the *parabolic roots* λ_j^p. We write $\lambda_j = -\nu\omega_j^2 + \varepsilon_j$. Equation (2.6) becomes

$$\left[(\nu\omega_j^2 - \varepsilon_j)^2 + c^2\omega_j^2\right]\varepsilon_j + \alpha\beta\omega_j^2(\varepsilon_j - \nu\omega_j^2) = 0$$

and therefore

$$\varepsilon_j = \frac{\alpha\beta(\nu - \varepsilon_j/\omega_j^2)}{(\nu - \varepsilon_j/\omega_j^2)^2 + c^2/\omega_j^2}$$

Thus, for $\dfrac{1}{\omega_j^2}$ small enough, by the implicit function theorem, $\varepsilon_j = Z(\frac{1}{\omega_j^2})$, where Z is analytic at $z = 0$, $Z(0) = \frac{\alpha\beta}{\nu}$ and $Z(z)$ is real for real z.

We conclude that, for j large enough

$$\lambda_j^p = -\nu\omega_j^2 + \frac{\alpha\beta}{\nu} + O\left(\frac{1}{\omega_j^2}\right), \lambda_j \in (-\infty, 0). \tag{2.7}$$

Let us analyze now the *conjugate hyperbolic roots* of (2.6). We denote them by $\lambda_j^{h,+}$ and $\lambda_j^{h,-}$ with $\lambda_j^{h,-} = \overline{\lambda_j^{h,+}}$ and $Im(\lambda_j^{h,+}) > 0$. We set $\lambda_j^{h,+} = ic\omega_j + \eta_j$ and equation (2.6) reads as follows:

$$(2ic\omega_j\eta_j + \eta_j^2)(\nu\omega_j^2 + ic\omega_j + \eta_j) + \alpha\beta\omega_j^2(ic\omega_j + \eta_j) = 0.$$

Thus,

$$\eta_j = \frac{-\alpha\beta(ic + \eta_j/\omega_j)}{(2ic + \eta_j/\omega_j)(\nu + ic/\omega_j + \eta_j/\omega_j^2)}.$$

The arguments above show that

$$\lambda_j^{h,+} = ic\omega_j - \frac{\alpha\beta}{2\nu} + O\left(\frac{1}{\omega_j}\right),\ as\ j\ \to \infty. \tag{2.8}$$

We set

$$H_j^p = ker(\lambda_j^p I - A_j),\ H_j^h = ker(\lambda_j^{h,+} I - A_j) \oplus ker(\lambda_j^{h,-} I - A_j)$$

and

$$H^p = \bigoplus_{|\lambda_j^p| > R_0} H_j^p\ ,\ H^h = \bigoplus_{|\lambda_j^{h,\pm}| > R_0} H_j^h$$

for R_0 large enough so that $\lambda_j^p, \lambda_j^{h,\pm}$ are all distintic. This is feasible in view of (2.7) and (2.8).

We also introduce the subspace of H generated by the low frecuencies:

$$H^0 = \bigoplus_{|\lambda| \leq R_0} ker(\lambda I - A)^m$$

where m denotes the algebraic multiplicity of the eigenvalue λ.

We denote by Π^p, Π^h and Π^0 the corresponding spectral projectors. Note that $\Pi = \Pi^p + \Pi^h = I - \Pi^0$ is the projector over the subspace generated by the high frecuencies $|\lambda| > R_0$.

Let us study the continuity of Π^p and Π^h.

For ω_j large the hyperbolic eigenvectors (those that correspond to the eigenvalues $\lambda_j^{h,\pm}$) are of the form:

$$U_j^{h,\pm} = \begin{pmatrix} (\lambda_j^{h,\pm})^{-1} \\ 1 \\ \dfrac{-\beta}{\nu\omega_j^2 + \lambda_j^{h,\pm}} \end{pmatrix} e_j \tag{2.9}$$

Its third component (that corresponds to the temperature) is of the order of $\frac{1}{\omega_j}$ in $H^1(M)$, while the first two are the order of 1 in $H^1(M) \times L^2(M)$. Thus, the behavior of $U_j^{h,\pm}$ for j large is determined by its first two components.

The parabolic eigenvectors are of the form

$$U_j^p = \begin{pmatrix} \dfrac{\alpha\omega_j}{(\lambda_j^p)^2 + c^2\omega_j^2} \\ \dfrac{\alpha\lambda_j^p\omega_j}{(\lambda_j^p)^2 + c^2\omega_j^2} \\ \dfrac{1}{\omega_j} \end{pmatrix} e_j \tag{2.10}$$

The first two components (the hyperbolic ones) tend to zero in $H_0^1(M) \times L^2(M)$ as $j \to \infty$, while the third one is of unit norm in $H_0^1(M)$.

In view of (2.9) and (2.10) we deduce that the eigenvectors of A, incorporating the generalized eigenvectors for the low frequencies $|\lambda| \le R_0$ form a Riesz basis of the energy space $H = H_0^1(M) \times L^2(M) \times H_0^1(M)$ and the spectral projectors Π^p and Π^h are continuous. We have

$$H = H^p \oplus H^h \oplus H^0.$$

We can define the semigroups $e^{tA^h} : H^h \to H^h$, $e^{tA^p} : H^p \to H^p$ and $e^{tA^0} : H^0 \to H^0$, for $t \ge 0$, where A^h, A^p and A^0 are respectively the restrictions of A to H^h, H^p and H^0.

On the other hand, since H^0 is of finite dimension e^{tA^0} is well defined for any $t \in \mathbb{R}$. In view of the asymptotic expansion (2.8), e^{tA^h} is also well defined in H^h for all $t \in \mathbb{R}$.

3 The control strategy

Given $T > 0$ and $\Omega \subset M$ satisfying the geometric control condition, we take any open non-empty subset ω of M and $T' < T$ such that the pair (Ω, T') still satisfies the geometric control condition.

We will first consider two controls, f_h with support in $\Omega \times (0, T')$, f_p with support in $\omega \times (T', T)$ devoted to control the hyperbolic and parabolic part of the system respectively. Notice that, since f_p is a parabolic control no restriction is needed on the length of the time interval (T', T) or on the support of the control ω (see [LRo]).

A posteriori we will see that the parabolic control is not really needed to control the system.

The controls f_h and f_p are imposed to be in the class

$$\begin{cases} f_h \in L^2(\Omega \times (0, T')) \\ f_p \in L^2(T', T; H_0^1(\omega)). \end{cases} \tag{3.1}$$

Given initial data $U^0 \in H$ and controls (f_h, f_p) as in (3.1), the controlled system

$$\begin{cases} \frac{dU}{dt} = AU + (0, f_h, f_p)^t \quad , \quad 0 < t < T \\ U(0) = U^0 \end{cases} \tag{3.2}$$

has a unique solution in $C([0, T]; H)$ given by the variation of constants formula:

$$U(t) = e^{tA} U^0 + \int_0^t e^{(t-s)A} (0, f_h(s), f_p(s))^t ds = \mathcal{L}(t; U^0, (f_h, f_p)) \tag{3.3}$$

Note that system (3.2) is the abstract version of:

$$\begin{cases} u_{tt} - c^2 \Delta u + \alpha \Delta \theta = f_h \chi_\Omega & \text{in} \quad M \times (0, T) \\ \theta_t - \nu \Delta \theta + \beta u_t = f_p \chi_\omega & \text{in} \quad M \times (0, T) \\ u = \theta = 0 & \text{on} \quad \partial M \times (0, T) \\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) & \text{in} \quad M \\ \theta(x, 0) = \theta^0(x) & \text{in} \quad M. \end{cases}$$

We will prove the following proposition:

Proposition 1. *There exists a subspace $\mathcal{V} \subset H$ of finite codimension and a linear and continuous operator*

$$K : \mathcal{V} \to L^2(\Omega \times (0, T')) \times \mathcal{D}(\omega \times (T', T)) \tag{3.4}$$

that to each $U^0 \in \mathcal{V}$ associates the pair of controls $KU^0 = (f_h, f_p)$ such that

$$\Pi\mathcal{L}(T; U^0, KU^0) = 0 \ , \ \forall U^0 \in \mathcal{V}. \tag{3.5}$$

Remark: By $\mathcal{D}$ we denote the space of C^∞ and compactly supported test functions. Proposition 1 states that, at time $t = T$, we may control all the high frequencies so that the final state lives in H^0. Moreover, the parabolic control f_p may be chosen to be arbitrarily smooth and with arbitrarily small support.

■

Proposition 1 is a consequence of the following two lemmas:

Lemma 1. *There exists a linear and continuous operator*

$$A : H \times L^2((T', T); H_0^1(\omega)) \to L^2(\Omega \times (0, T')) \tag{3.6}$$

such that

$$\Pi^h\mathcal{L}(T; U^0, (A(U^0, f_p), f_p)) = 0 \ , \ \forall U^0, f_p. \tag{3.7}$$

Lemma 2: *There exists a linear and continuous operator*

$$B : H \times L^2(\Omega \times (0, T')) \to \mathcal{D}(\omega \times (T', T)) \tag{3.8}$$

such that

$$\Pi^p\mathcal{L}(T; U^0, (f_h, B(U^0, f_h))) = 0 \ , \ \forall U^0, f_h. \tag{3.9}$$

Remark: Operator A is constructed such that given any initial data and parabolic control, it provides the hyperbolic control such that the hyperbolic high frequencies are driven to zero at the final time. The operator B provides the parabolic control (which is arbitrarilly smooth) driving to zero the parabolic high frequencies for any initial data and hyperbolic control.

■

Let us see that these lemmas imply Proposition 1.

We observe that

$$\Pi\mathcal{L}(T; U^0, (f_h, f_p)) = 0 \tag{3.10}$$

as soon as the following two equations are satisfied simultaneously:

$$\begin{cases} f_h = A(U^0, f_p) &= A_1(U^0) + A_2(f_p) \\ f_p = B(U^0, f_h) &= B_1(U^0) + B_2(f_h). \end{cases} \tag{3.11}$$

If we set

$$C = B_1 + B_2 A_1 : H \to \mathcal{D}(\omega \times (T', T)) \tag{3.12}$$

then, solving system (3.11) is equivalent to finding a solution $f_p \in L^2((T', T); H_0^1(\omega))$ of

$$CU^0 = (I - B_2 A_2) f_p. \tag{3.13}$$

Operator B_2A_2 is compact in $L^2((T',T);H_0^1(\omega))$ because of the regularizing effect of B_2. Thus, by Fredholm's alternative there exist finitely many linear, continuous forms $\{\ell_j\}_{j=1,\dots,N}$ in $L^2((T',T);H_0^1(\omega))$ such that equation (3.13) has a solution $f_p \in L^2((T',T);H_0^1(\omega))$ if and only if

$$\ell_j(C(U^0)) = 0 \ , \ j = 1, \dots, N. \tag{3.14}$$

Under these conditions, (3.13) admits a solution $f_p = DC(U^0)$ with D continuous from $\{f \in L^2(T',T;H_0^1(\omega)) : \ell_j(f) = 0, j = 1, \dots, N\}$ into $L^2(T',T;H_0^1(\omega))$.

We have

$$f_p = B_2A_2f_p + CU^0 = B_2A_2DC(U^0) + CU^0$$

and therefore $f_p \in \mathcal{D}(\omega \times (T',T))$.

■

Once Proposition 1 is proved we have to deal with the possibility of controlling the low frequencies involved in H^0. This will be done by a rather classical uniqueness argument that applies in this case since H^0 is of finite dimension.

Therefore, in order to prove Theorem 1 we have to overcome the following steps:

1.- Proof of Lemma 1;
2.- Proof of Lemma 2;
3.- Control of the low frequencies;
4.- Getting rid of the parabolic control.

The first two steps are the object of section 4 and 5 respectively. The last two steps will be completed in Section 6.

4 Control of the hyperbolic high frequencies.

Observe that $\Pi^h\mathcal{L}(T;U^0,(f_h,f_p)) = 0$ if and only if

$$\Pi^h\mathcal{L}(T;0,(f_h,0)) = -\Pi^h\mathcal{L}(T;U^0,(0,f_p)). \tag{4.1}$$

On the other hand, the linear operator

$$(U^0,f_p) \longrightarrow -\Pi^h\mathcal{L}(T;U^0,(0,f_p))$$

is continuous from $H \times L^2((T',T);H_0^1(\omega))$ into H^h. Since the support of f_h is contained in $\Omega \times (0,T')$ and more precisely $f_h(t) = 0$ for $T' < t < T$, we have

$$\Pi^h\mathcal{L}(T;0,(f_h,0)) = e^{(T-T')A^h}\Pi^h\mathcal{L}(T';0,(f_h,0)).$$

As pointed out in section 2, e^{tA^h} is well defined in H^h for all $t \in \mathbb{R}$. Therefore, (4.1) is equivalent to

$$\Pi^h\mathcal{L}(T';0,(f_h,0)) = -e^{-(T-T')A^h}\Pi^h\mathcal{L}(T;U^0,(0,f_p)) = U^h. \tag{4.2}$$

Solving (4.2) for a given $U^h \in H^h$ is equivalent to show that the hyperbolic component of the system can be driven from zero to U^h in time $t = T'$. This is an exact controllability result for the evolution system generated by e^{tA^h} in H^h.

It is well known that exact controllability for hyperbolic equations can be obtained as a consequence of a suitable observability inequality (see [Li1]). Let us see what is the observability inequality we need.

Let us consider the dual of H:

$$H' = L^2(M) \times H^{-1}(M) \times H^{-1}(M) \tag{4.3}$$

with the duality pairing

$$< U, V >= (u^1, v^0) - (u^0, v^1) + (\theta^0, \psi^0) + \beta(u^0, \psi^0) \tag{4.4}$$

for $U = \begin{pmatrix} u^0 \\ u^1 \\ \theta^0 \end{pmatrix} \in H$ and $V = \begin{pmatrix} v^0 \\ v^1 \\ \psi^0 \end{pmatrix} \in H'$. In (4.4), by $(.,.)$ we denote both, the scalar product in $L^2(M)$ and the duality pairing between $H_0^1(M)$ and $H^{-1}(M)$.

We have

$$< AU, V >=< U, \widetilde{A}V > \tag{4.5}$$

with

$$\widetilde{A} = \begin{pmatrix} 0 & -I & 0 \\ -(c^2 + \alpha\beta)\Delta & 0 & \nu\beta\Delta \\ -\alpha\Delta & 0 & \nu\Delta \end{pmatrix}. \tag{4.6}$$

Given $U \in C([0,T]; H)$ solution of

$$\begin{cases} \dfrac{dU(t)}{dt} - AU(t) = \begin{pmatrix} 0 \\ f_h \\ f_p \end{pmatrix} \quad , \quad t \in (0,T) \\ U(0) = U^0 \in H \end{cases} \tag{4.7}$$

and $V \in C([0,T]; H')$ solution of

$$\begin{cases} \dfrac{dV(t)}{dt} + \widetilde{A}V(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad t \in (0,T) \\ V(T) = V^0 \in H' \end{cases} \tag{4.8}$$

we have

$$< U, V > |_0^T = \int_0^T \int_M (f_h v + f_p \psi) dx dt \tag{4.9}$$

with $V = \begin{pmatrix} v \\ v_t \\ \psi \end{pmatrix}$.

Notice that system (4.8) coincides with

$$\begin{cases} v_{tt} - (c^2 + \alpha\beta)\Delta v + \nu\beta\Delta\psi = 0 & \text{in} \quad M \times (0,T) \\ -\psi_t - \nu\Delta\psi + \alpha\Delta v = 0 & \text{in} \quad M \times (0,T) \\ v = \psi = 0 & \text{on} \quad \partial M \times (0,T) \\ v(T) = v^0, v_t(T) = v^1, \psi(T) = \psi^0 & \text{in} \quad M. \end{cases} \tag{4.10}$$

In (4.9) by $h|_0^T$ we denote $h(T)-h(0)$ for the time dependent function $h(t)=<U(t),V(t)>$.

We will denote by $\widetilde{\Pi}^h, \widetilde{\Pi}^p, \widetilde{\Pi}^0$ the adjoints of the spectral projectors Π^h, Π^p and Π^0 and we will use the dual decomposition:

$$H' = \widetilde{H}^h \oplus \widetilde{H}^p \oplus \widetilde{H}^0.$$

Let us assume now that for every $U^h \in H^h$ there exists $f_h \in L^2(0,T';L^2(\Omega))$ such that (4.2) holds with a linear and continuous dependence

$$\| f_h \|_{L^2(0,T';L^2(\Omega))} \leq C \| U^h \|_H . \tag{4.11}$$

The solution U of (4.7) with $U^0=0$ and this control $f_h (f_p = 0)$ is such that $\Pi^h(U(T')) = U^h$ and

$$< U^h, V^0 > = \int_0^{T'} \int_M f_h v dxdt, \ \forall V^0 \in \widetilde{H}^h. \tag{4.12}$$

Combining (4.11) and (4.12) we deduce that

$$\| V^0 \|_{H'}^2 \leq C^2 \int_0^{T'} \int_\Omega v^2 dxdt, \ \forall V^0 \in \widetilde{H}^h \tag{4.13}$$

where here and in what follows $\| \, . \, \|_{H'}$ is the norm given by

$$\| V^0 \|_{H'} = \left(\| v^0 \|_{L^2(M)}^2 + \| v^1 - \beta \psi^0 \|_{H^{-1}(M)}^2 + \| \psi^0 \|_{H^{-1}(M)}^2 \right)^{1/2}$$

It is also easy to see that, if (4.13) holds, then (4.2) is solvable with the continuous dependence estimate (4.11). This can be done by using Lions' HUM ([Li1]).

Let us establish (4.13) in $\widetilde{H}^h = \bigoplus_{|\lambda_j^p| > R_0} \widetilde{H}_j^h$ where $\widetilde{H}_j^h$ is generated by

$$V_j^{h,\pm} = \begin{pmatrix} 1 \\ -\lambda_j^{h,\pm} \\ \dfrac{\alpha \omega_j^2}{\lambda_j^{h,\pm} + \nu \omega_j^2} \end{pmatrix} e_j$$

which is a hyperbolic eigenfunction of $\widetilde{A}$.

For any $V^0 \in \widetilde{H}^h$ we have

$$V^0 = \sum_{|\lambda_j| > R_0} \left(a_j^+ V_j^{h,+} + a_j^- V_j^{h,-} \right)$$

with $\{a_j^\pm\} \in \ell^2$, the norm $\| V^0 \|_{\widetilde{H}^h}$ being equivalent to $\left(\displaystyle\sum_{|\lambda_j| > R_0} \left(|a_j^+|^2 + |a_j^-|^2 \right) \right)^{1/2}$.

In view of (2.8) we have

$$\lambda_j^{h,+} \lambda_j^{h,-} = |\lambda_j^{h,+}|^2 = c^2 \omega_j^2 + O(1)$$

and

$$\lambda_j^{h,+} + \lambda_j^{h,-} = 2Re \lambda_j^{h,+} = -\frac{\alpha\beta}{\nu} + O\left(\frac{1}{\omega_j}\right).$$

Then

$$V(t)=\begin{pmatrix} v(t)\\ v_t(t)\\ \psi(t)\end{pmatrix}=\sum_{|\lambda_j|>R_0}\left(a_j^+e^{\lambda_j^{h,+}(T'-t)}V_j^{h,+}+a_j^-e^{\lambda_j^{h,-}(T'-t)}V_j^{h,-}\right)$$

is such that

$$v_{tt}-c^2\Delta v+\frac{\alpha\beta}{\nu}v_t=\varphi(x,t)\text{ in }M\times(0,T') \tag{4.14}$$

with

$$\varphi=\sum_{|\lambda_j|>R_0}\left(b_j^+e^{\lambda_j^{h,+}(T'-t)}+b_j^-e^{\lambda_j^{h,-}(T'-t)}\right)e_j,\ \{b_j^\pm\}\in\ell^2,\ ||b_j^\pm||_{\ell^2}\le C||V^0||_{\widetilde{H}^h}. \tag{4.15}$$

Estimate (4.15) states that v satisfies the wave equation

$$v_{tt}-c^2\Delta v+\frac{\alpha\beta}{\nu}v_t=0\text{ in }M\times(0,T');\quad v=0\text{ on }\partial M\times(0,T') \tag{4.16}$$

up to an additive compact term. Indeed, if we are dealing with solutions of (4.14) or (4.16) in the class $v\in C([0,T'];L^2(M))\times C^1([0,T'];H^{-1}(M))$ we may allow perturbations φ in $L^2(0,T';H^{-1}(M))$. Inequality (4.15) ensures that φ has in fact one more degree of regularity.

By [BLR] it is well known that if (Ω,T') satisfy the geometric control condition in M, then there exists a constant $C>0$ such that

$$\|\,(v(T'),v_t(T'))\,\|^2_{L^2(M)\times H^{-1}(M)}\le C\int_0^{T'}\int_\Omega|v|^2dxdt. \tag{4.17}$$

Going back to the solutions of (4.14) and taking (4.15) into account we deduce that

$$\|\,V^0\,\|^2_{\widetilde{H}^h}\sim\|\,(v(T'),v_t(T'))\,\|^2_{L^2(M)\times H^{-1}(M)}\le C\left(\int_0^T\int_\Omega|v|^2dxdt+\|\,(v(T'),v_t(T'))\,\|^2_{H^{-1}(M)\times H^{-2}(M)}\right). \tag{4.18}$$

Inequality (4.13) can now be proved easily by a compactness-uniqueness argument. In view of (4.18), we are reduced to show that

$$N_{T'}=\{V^0\in\widetilde{H}^h:v=0\text{ in }\Omega\times(0,T')\}$$

is reduced to zero. In view of (4.18) we know that $N_{T'}$ is a finite-dimensional subspace of $\widetilde{H}^h$, decreasing in T'. By a small perturbation of T', then we may therefore suppose that $N_T=N_{T'}$ for $T-T'$ small, and then, $N_{T'}$ is stable by the action of $\frac{d}{dt}=-\widetilde{A}^h$. Then, if $N_{T'}$ is not reduced to zero it contains an eigenfunction of $\widetilde{A}^h$. But, by construction of $N_{T'}$, the first component of that eigenfunction would vanish in Ω. Therefore, $e_j=0$ in Ω for some j and this contradicts the unique continuation of the Laplace operator in the manifold M.

This concludes the proof of the observability inequality (4.13) which, in its turn, implies the solvability of (4.1) and/or (4.2), with continuous dependence of the solutions.

Lemma 1 is now proved.

5 Control of the parabolic high frequencies

Given $U^0\in H$ and $f_h\in L^2(\Omega\times(0,T'))$ we have to find $f_p\in L^2((T',T);H_0^1(\omega))$ such that

$$\Pi^p\mathcal{L}(T;U^0;(f_h,f_p))=0. \tag{5.1}$$

We observe that (5.1) is equivalent to

$$\Pi^p\mathcal{L}(T;0,(0,f_p)) = -\Pi^p\mathcal{L}(T;U^0,(f_h,0)).$$

In view of the support of the controls this is equivalent to:

$$\Pi^p\mathcal{L}(T-T';0,(0,f_p(\cdot - T'))) = -e^{(T-T')A^p}(\Pi^p\mathcal{L})(T';U^0,(f_h,0)). \tag{5.2}$$

Since the mapping $(U^0,f_h)\to\Pi^p\mathcal{L}(T';U^0,(f_h,0))$ is continuous from $H\times L^2(\Omega\times(0,T'))$ into H^p equation (5.2) reduces to the following problem: For $T>0$ fixed, given $U^0\in H^p$ find $f_p\in\mathcal{D}(\omega\times(0,T))$ such that

$$\Pi^p\mathcal{L}(T;U^0,(0,f_p)) = 0 \tag{5.3}$$

with f_p depending continuously on U^0.

Thus, we have reduced the proof of Lemma 2 to solving (5.3). This is a null-controllability problem for the parabolic projection of our system of thermoelasticity. This problem was solved by G. Lebeau and L. Robbiano [LRo] in the context of the heat equation. We are going to adapt their arguments to solve (5.3).

First of all, we need an estimate of the size of the control that we need to control a given finite number of frequencies in the parabolic component of our system.

To do that we need the Theorem 3 stated in the introduction:

Proof of Theorem 3.

Given any $T>0$ we consider the cylinder $X_T=M\times(0,T)$.

We set

$$f(x,t)=\sum_{\omega_j\le\mu}\frac{\mathrm{sh}(t\omega_j)}{\omega_j}a_je_j(x) \tag{5.4}$$

with $\mathrm{sh}(t\omega)/\omega=t$ if $\omega=0$.

Then, for $P=\partial_t^2+\Delta$, we have

$$\begin{cases} Pf=0 & \text{in} \quad X_T\\ f(x,0)=0 & \text{in} \quad M\\ f(x,t)=0 & \text{on} \quad \partial M\times(0,T)\end{cases} \tag{5.5}$$

As a consequence of the interpolation inequalities obtained in [LRo] (section 3) we have that: For any $\alpha\in(0,T/2)$ and Ω open non-empty subset of M there exists $C>0$ and $\delta\in(0,1)$ such that

$$\|\varphi\|_{H^1(M\times(\alpha,T-\alpha))}\le C\|\varphi\|^\delta_{H^1(X_T)}\left(\|P\varphi\|_{L^2(X_T)}+\|\partial_t\varphi(x,0)\|_{L^2(\Omega)}\right)^{1-\delta} \tag{5.6}$$

for any $\varphi\in H^2(X_T)$ such that $\varphi=0$ on $(\partial M\times(0,T))\cup M\times\{0\}$. We refer to the Appendix at the end of this paper for a sketch of the proof.

Applying (5.6) to f we deduce that

$$\|f\|_{H^1(M\times(\alpha,T-\alpha))}\le C\|f\|^\delta_{H^1(X_T)}\|\partial_tf(x,0)\|^{1-\delta}_{L^2(\Omega)}. \tag{5.7}$$

We observe that

$$\|f\|^2_{H^1(M\times(\alpha,T-\alpha))}\ge\|f\|^2_{L^2(M\times(\alpha,T-\alpha))}=\int_\alpha^{T-\alpha}\int_M|\sum_{\omega_j\le\mu}\frac{sh(t\omega_j)}{\omega_j}a_je_j(x)|^2dxdt$$

$$= \sum_{\omega_j \leq \mu} |a_j|^2 \int_{\alpha}^{T-\alpha} |\frac{\mathrm{sh}(t\omega_j)}{\omega_j}|^2 dt \geq \sum_{\omega_j \leq \mu} |a_j|^2 \int_{\alpha}^{T-\alpha} t^2 dt = C \sum_{\omega_j \leq \mu} |a_j|^2. \tag{5.8}$$

On the other hand $\partial_t f(x,0) = \sum_{\omega_j \leq \mu} a_j e_j(x)$ and

$$\| f \|^2_{H^1(X_T)} \leq C e^{2T\mu}(1+\mu^2) \sum_{\omega_j \leq \mu} |a_j|^2. \tag{5.9}$$

Thus, by (5.7) we conclude that

$$\sum_{\omega_j \leq \mu} |a_j|^2 \leq C e^{2T\delta\mu}(1+\mu^2)^{\delta} \left(\sum_{\omega_j \leq \mu} |a_j|^2 \right)^{\delta} \left(\int_{\Omega} | \sum_{\omega_j \leq \mu} a_j e_j(x)|^2 dx \right)^{1-\delta} \tag{5.10}$$

and

$$\sum_{\omega_j \leq \mu} |a_j|^2 \leq C e^{2T\delta\mu/(1-\delta)}(1+\mu^2)^{\delta/(1-\delta)} \int_{\Omega} | \sum_{\omega_j \leq \mu} a_j e_j(x)|^2 dx. \tag{5.11}$$

This concludes the proof of Theorem 3.

■

Let us go back to the control of the parabolic component of our system.

Proposition 2. *For any open and non-empty subset ω of M and for every $T \in (0,1], \mu > 0$ there exists a linear map*

$$K_{T,\mu} : H^p \longrightarrow L^2(0,T; H_0^1(\omega)) \tag{5.12}$$

such that

$$\Pi^p_{\mu} \mathcal{L}(T; U^0, (0, K_{T,\mu}(U^0)) = 0. \tag{5.13}$$

In fact, the operator of $K_{T,\mu}$ can be constructed such that,

a) *The support of $K_{T,\mu}(U^0)$ is contained in a fixed compact set of $\omega \times [0,T]$ for every $U^0 \in H^p$;*

b)$K_{T,\mu}(U^0) \in C^{\infty}(\omega \times [0,T])$ *and*

$$\| K_{T,\mu}(U^0) \|_{H^k(M\times(0,T))} \leq \frac{C}{\sqrt{T}} \left(\frac{1+\mu^{2k}}{T^{2k}} \right) e^{C\mu} \| U^0 \|_{H^p} \tag{5.14}$$

with $C > 0$ large enough independent of μ, T and k.

Proof: Let us denote by $\widetilde{H}^p_{\mu}$ the subspace of $\widetilde{H}^p$ corresponding to the frequencies $\omega_j \leq \mu$.

Let us consider solutions V of the adjoint system (4.8) with data V^0 in $\widetilde{H}^p_{\mu}$. The solution V of (4.8) can be written in Fourier series:

$$V(t) = \sum_{\omega_j \leq_{\mu}} a_j e^{\lambda^p_j (T-t)} V^p_j \tag{5.15}$$

where V_j^p are the parabolic eigenfunctions of $\widetilde{A}$, i.e.

$$V_j^p = \frac{\beta}{\nu\omega_j}\begin{pmatrix} 1 \\ -\lambda_j^p \\ \frac{\alpha\omega_j^2}{\lambda_j^p+\omega_j^2\nu}\end{pmatrix} e_j = \frac{1}{\omega_j}\begin{pmatrix} \beta/\nu \\ -\lambda_j^p\beta/\nu \\ \omega_j^2 + O(1)\end{pmatrix} e_j = \begin{pmatrix} v_{j,1}^p \\ v_{j,2}^p \\ v_{j,3}^p\end{pmatrix} e_j. \tag{5.16}$$

It is clear that

$$\| V(0) \|_{\widetilde{H}^p}^2 \sim \| \sum_{\omega_j\leq\mu} a_j\omega_j e^{\lambda_j^p T} e_j \|_{H^{-1}(M)}^2 = \| \psi(0) \|_{H^{-1}(M)}^2 . \tag{5.17}$$

Let us apply the duality identity (4.9) with null hyperbolic control $f_h = 0$ and f_p of the form $f_p = \rho g_p$. In fact, ρ will be taken to be of the form $\rho(x,t) = \rho_1((T-t)/T)\rho_2(x)$ with $\rho_2 \in \mathcal{D}(\omega)$ such that $\rho_2 = 1$ in $\widehat{\omega}$, $\widehat{\omega}$ being a non-empty and open subset strictly contained in ω and $\rho_1 \in \mathcal{D}(\mathbf{R})$ that we will define below.

We have

$$< U, V > |_0^T = \int_0^T \int_M \rho g_p \psi dxdt. \tag{5.18}$$

The fact that $\Pi_\mu^p(U(T)) = 0$ is then equivalent to:

$$- < U^0, V(0) > = \int_0^T \int_M \rho g_p \psi dxdt, \quad \forall V^0 \in \widetilde{H}_\mu^p,$$

or,

$$\int_0^T \int_M \rho g_p e^{\lambda_j^p (T-t)} v_{j,3}^p e_j dxdt = - < U^0, V_j^p > e^{\lambda_j^p T}, \quad \forall j : \omega_j \leq \mu. \tag{5.19}$$

We can take

$$g_p = -\sum_k < U^0, V_k^p > e^{\lambda_k^p T} G_p^k \tag{5.20}$$

provided $G_p^k = G_p^k(x,t)$ is such that

$$\int_0^T \int_M \rho G_p^k e^{\lambda_j^p (T-t)} v_{j,3}^p e_j dxdt = \delta_{j,k}, \quad \forall \omega_j \leq \mu, \omega_k \leq \mu. \tag{5.21}$$

To solve (5.21) we set

$$G_p^k = \sum_l G_p^{k,l} e^{\lambda_l^p (T-t)} e_l(x) v_{l,3}^p \tag{5.22}$$

and then (5.21) is equivalent to

$$\sum_l G_p^{k,l} \int_0^T \int_M \rho e_j(x) e_l(x) v_{j,3}^p v_{l,3}^p e^{((\lambda_j^p+\lambda_l^p)(T-t))} dxdt = \delta_{j,k}. \tag{5.23}$$

Solving (5.23) is equivalent to inverting the symmetric matrix:

$$A_{j,l} = \int_0^T \int_M \rho e_l e_j v_{j,3}^p v_{l,3}^p e^{((\lambda_j^p+\lambda_l^p)(T-t))} dxdt. \tag{5.24}$$

However, by Thorem 3, the quadratic form

$$(Au \mid u) = \int_0^T \int_M \rho \mid \sum u_j v_{j,3}^p e^{\lambda_j^p (T-t)} e_j \mid^2 dxdt \tag{5.25}$$

satisfies

$$(Au \mid u) \geq \int_0^T \rho_1((T-t)/T) \sum C_1 e^{-C_2\mu} e^{2\lambda_j^p (T-t)} \mid u_j \mid^2 dt \tag{5.26}$$

since $v_{j,3}^p \simeq \omega_j >> 1$.

On the other hand,

$$\int_0^T \rho_1((T-t)/T) e^{2\lambda_j^p (T-t)} dt = T \int_0^1 \rho_1(s) e^{2(\lambda_j^p s)T} ds = T \int_0^1 \rho_1(s) e^{-2s|\lambda_j^p|T} ds. \tag{5.27}$$

We have to take ρ_1 so that its support contains $s = 0$ in order to guarantee that the quantity (5.27) remains bounded away from zero. We can indeed take ρ_1 in a Gevrey class such that $\rho_1(0) = \rho(1) = 0$ and $\rho(s) > 0$ for $s \in (0,1)$, $\rho(s) = 1$ for $s \in (1/3, 2/3)$ and with

$$\int_0^1 \rho_1(s) e^{-\gamma s} ds \geq C e^{-\sqrt{\gamma}}, \quad \forall \gamma > 0$$

for some $C > 0$. With this choice of ρ_1 we have, for $\mid T \mid \leq 1$,

$$(Au \mid u) \geq C_1 T e^{-C_2\mu} \sum \mid u_j \mid^2 \tag{5.28}$$

since $\sqrt{\mid \lambda_j \mid} \leq C\mu$. From (5.28) we deduce the bound

$$\sup_k \left(\sum_l \mid G_p^{k,l} \mid^2 \right)^{1/2} \leq C e^{C\mu}/T. \tag{5.29}$$

The norms of the control $f_p = \rho g_p$ can be estimated easily. We have

$$f_p = -\sum_{k,l} < U^0, V_k^p > e^{\lambda_k^p T} G_p^{k,l} e^{\lambda_l^p (T-t)} e_l(x) v_{l,3}^p \rho_2(x) \rho_1((T-t)/T).$$

So using (5.29), $\lambda_l^p \sim -\nu\omega_l^2$, $\parallel e_l \parallel_{H^k} \sim C\omega_l^k$ and Weyls formula (that guarantees that $\sum_{\omega_j \leq \mu} 1 \leq C\mu^{dim(M)} \leq e^{C'\mu}$) one obtains

$$\sup_{0 \leq t \leq T} \parallel \partial_t^r f_p \parallel_{H^k(M)}^2 \leq C(k,r) \frac{(1+\mu)^{2r+k}}{T^{1+r}} e^{C\mu} \parallel U^0 \parallel_{H^p} . \tag{5.30}$$

In particular one has with constants $C, D > 0$ independent of U^0, $\mu \geq 1$ and $T \in (0,1)$:

$$\parallel f_p \parallel_{l^2(0,T;H_0^1(M))}^2 \leq \frac{D}{\sqrt{T}} e^{C\mu} \parallel U^0 \parallel_{H^p} . \tag{5.31}$$

This concludes the proof of Proposition 2.

Now, we are in a position to show the solvability of (5.3) and to conclude the proof of Lemma 2. To do this we follow the control strategy introduced in [LRo].

We fix $\delta \in (0, T/2), \rho \in (0,1)$ and $\widehat{\omega}$ an open and non-empty set strictly contained in ω.

For $\ell = 1, 2 \ldots$ we set $\mu_\ell = 2^\ell$ and $T_\ell = A 2^{-\rho \ell}$ where $A > 0$ is such that $2 \sum_{\ell=1}^{\infty} T_\ell = T - 2\delta$. Given $U^0 \in H^p$ we define

$$\begin{cases} U^1 = e^{\delta A^p} U^0 \\ U^{\ell+1} = e^{T_\ell A^p} W^\ell \\ W^\ell = \Pi^p \mathcal{L}(T_\ell; U^\ell, (0, K_{T_\ell, \mu_\ell}(U^\ell))) \end{cases} \tag{5.32}$$

where $K_{T_{\ell,\mu_\ell}}$ is the control operator introduced in Proposition 1 such that the support (in x) of the control is contained in $\widehat{\omega}$.

By construction $\Pi^p_{\mu_\ell}(W^\ell) = 0$ and therefore

$$\| e^{T_\ell A^p} W^\ell \|_H \leq e^{-cT_\ell \mu_\ell^2} \| W^\ell \|_H \ .$$

On the other hand, by classical estimates we have that

$$\| W^\ell \|_H \leq \| U^\ell \|_H + \sqrt{T_\ell} \, \| K_{T_{\ell,\mu_\ell}}(U^\ell) \|_{L^2(0,T;H^1_0(M))} \ .$$

Therefore, by (5.31) we obtain

$$\| W^\ell \|_H \leq \| U^\ell \|_H \left(1 + D e^{C \mu_\ell}\right).$$

Thus, setting $m_\ell = e^{-cT_\ell \mu_\ell^2} \left(1 + D e^{C\mu_\ell}\right)$ we get

$$\varepsilon_{\ell+1} = \| U^{\ell+1} \|_H \leq m_\ell \| U^\ell \|_H = m_\ell \varepsilon_\ell.$$

It is easy to see that there exist $C_1, C_2 > 0$ such that $m_\ell \leq C_1 e^{-C_2 2^{(2-\rho)\ell}}$ for every ℓ. Thus $\varepsilon_\ell \to 0$ as $\ell \to \infty$ and more precisely there exist positive constants $C_3, C_4 > 0$ such that

$$\| U^\ell \|_H \leq C_3 \exp\left(-C_4 2^{(2-\rho)\ell}\right) \| U^0 \|_H \ . \tag{5.33}$$

If we set

$$a_0 = \delta, a_1 = \delta + 2T_1, \ldots, a_\ell = a_{\ell-1} + 2T_\ell$$

we have that $a_\ell \to T - \delta$ as $\ell \to \infty$.

Then, for any $U^0 \in H^p$ we define the control

$$K_p(U^0)(x,t) = \begin{cases} K_{T_\ell, \mu_\ell}(U^\ell)(x, t - a_{\ell-1}), & \text{for} \quad a_{\ell-1} \leq t \leq a_{\ell-1} + T_\ell \\ 0, & \text{for} \quad a_{\ell-1} + T_\ell \leq t \leq a_{\ell-1} + 2T_\ell = a_\ell \end{cases}$$

We claim that

$$\Pi^p \mathcal{L}\left(T; U^0, \left(0, K_p\left(U^0\right)\right)\right) = 0 \ , \ \forall U^0 \in H^p.$$

Indeed, by construction

$$\Pi^p \mathcal{L}\left(a_\ell; U^0, \left(0, K_p\left(U^0\right)\right)\right) = U^{\ell+1}$$

and, in view of (5.33),

$$\lim_{\ell \to \infty} \Pi^p \mathcal{L}(a_\ell; U^0, (0, K_p(U^0))) = \Pi^p \mathcal{L}(T - \delta; U^0, (0, K_p(U^0))) = 0$$

and since $K(U^0) = 0$ for $T - \delta \leq t \leq T$ we conclude that

$$\Pi^p \mathcal{L}(T; U^0, K_p(U^0))) = e^{\delta A p} \Pi^p \mathcal{L}(T - \delta; U^0, (0, K_p(U^0))) = 0.$$

It is clear that the support of $K_p(U^0)$ is contained in $\widehat{\omega} \times [\delta, T-\delta]$ and therefore the control has compact support in $\omega \times (0,T)$.

On the other hand, by (5.14) and (5.33),

$$\| K_p(U^0) \|_{H^k(M\times(0,T))} = \sum_{\ell=1}^{\infty} \| K_{T_\ell,\mu_\ell}(U^\ell) \|_{H^k(M\times(a_{\ell-1},a_{\ell-1}+T_\ell))}$$

$$\leq C_3 \sum_{\ell=1}^{\infty} \left(\frac{1}{T_\ell}\right)^{2k+1} \left(1+\mu_\ell^{2k}\right) e^{C\mu_\ell} \exp\left(-C_4 2^{(2-\rho)\ell}\right) \| U^0 \|_{H^p} .$$

which is finite for every $k \in I\!N$. Thus the control $K_p(U^0)$ is also of class C^∞.

This concludes the proof of the solvability of (5.3).

The operator B of Lemma 2 can be constructed as indicated in the begining of this section:

$$B(U^0, f_h)(x,t) = K_p\left(\Pi^p \mathcal{L}(T'; U^0, (f_h, 0))\right)(x, t-T')$$

where K_p is the operator above in the time interval $t \in (0, T-T')$.

Clearly B solves (3.10) and satisfies (3.9).

This concludes the proof of Lemma 2.

■

Remark: As a consequence of Proposition 2, and by the duality arguments above we deduce that for any $\omega \subset M$, $T>0$ and $k \in I\!N$ there exists $C_k > 0$ such that

$$\| V(0) \|_{H'}^2 \leq C_k \| \psi \|_{H^{-k}(\omega\times(0,T))}^2 \tag{5.34}$$

for every solution of (4.10) with data $V^0 \in \widetilde{H}^p$.

It is worth mentioning that (5.34) does not seem to be a direct consequence of the elliptic estimate of Theorem 3 but rather a byproduct of the controllability result of the parabolic component of the system that needs the iterative argument above. The same happens in the context of the heat equation ([LRo]).

■

6 Proof of the main results.

In section 3 we have proved Proposition 1 assuming that Lemma 1 and 2 hold. In section 4 and 5 we have proved these two Lemmas. Thus, as mentioned in section 3, Theorems 1 and 2 will be proved if we show the controllability of the low frequencies and then we get rid of the parabolic control.

We proceed in two steps.

6.1 Control of the low frequencies.

For any $T>0$ satisfying the assumption of Theorem 1 we introduce the space of initial data that are controllable at time $t=T$:

$$F_T = \{U^0 \in H : \exists (f_h, f_p) \in L^2(\Omega \times (0,T)) \times \mathcal{D}(\omega \times (0,T)) \text{ s. t. } \mathcal{L}\left(T; U^0, (f_h, f_p)\right) = 0\}. \tag{6.1}$$

Let us introduce its orthogonal subspace in H':

$$F_T^\perp = \{V \in H' :< U, V >= 0 \quad , \quad \forall U \in F_T\}. \tag{6.2}$$

By Proposition 1 we know that $\dim\,(F_T^\perp) < \infty$. On the other hand, $F_T \subset F_{T'}$ for every $T' > T$. Indeed, if U^0 is controllable at time $t = T$, extending the controls by zero for $T \le t \le T'$, we see that it also controllable at time T'. Thus $F_T^\perp$ decreases with T. Therefore, by possibly perturbing T, we can suppose that $F_T = F_{\widehat{T}}$ for every $\widehat{T} \in (T, T+\varepsilon)$ for $\varepsilon > 0$ small enough.

We claim that $F_T^\perp$ is stable by the semigroup $e^{t\widetilde{A}}$, i.e. $F_T^\perp = e^{t\widetilde{A}} F_T^\perp$ for every $t > 0$. Obviously, it is sufficient to check that $F_T^\perp = e^{\delta\widetilde{A}} F_T^\perp$ for every $\delta > 0$ sufficiently small. Indeed, if $U \in H$ is such that $< U, e^{\delta\widetilde{A}} V >= 0$ for every $V \in F_T^\perp$, then, $< e^{\delta A} U, V >= 0$. Thus $e^{\delta A} U \in F_T$, thus $U \in F_{T+\delta}$ and therefore $U \in F_T$. Therefore $(e^{\delta\widetilde{A}} F_T^\perp)^\perp \subset F_T$ and $F_T^\perp \subset e^{\delta\widetilde{A}} F_T^\perp$. Since $\dim\,(e^{\delta\widetilde{A}} F_T^\perp) \le \dim\,(F_T^\perp)$ we conclude that $e^{\delta\widetilde{A}} F_T^\perp = F_T^\perp$.

Let us denote by $Z(t)$ the restriction of the semigroup $e^{t\widetilde{A}}$ to $F_T^\perp : Z(t) = e^{t\widetilde{A}}|_{F_T^\perp}$. Then $Z(t) = e^{tB}$ where B is a matrix in $F_T^\perp$ such that $\widetilde{A}V = BV$ for every $V \in F_T^\perp$. But then $\ker\,(B-\lambda)^j \subset \ker\,(\widetilde{A}-\lambda)^j$ and therefore there exists some $N \in \mathbb{N}$ such that every element V of $F_T^\perp$ can be written as $V = \sum_{\omega_j \le N} c_j e_j$ with $c_j \in \mathbb{C}^3$.

We claim that all the elements of H can be driven to F_T in an arbitrarily short time, more precisely, given any $\varepsilon > 0$, for every $U^0 \in H$ there exist $(f_h, f_q) \in L^2(\Omega \times (0,\varepsilon)) \times \mathcal{D}(\omega \times (0,\varepsilon))$ such that

$$\mathcal{L}\left(\varepsilon; U^0, (f_h, f_q)\right) \in F_T.$$

Before proving this claim we note that this implies the controllability of the system in time $T+\varepsilon$ with two controls. Since $\varepsilon > 0$ is arbitrarily small and the property of geometric control is stable with respect to small perturbations of T we conclude that null-controllability at time $t = T$ holds with two controls.

Let us now prove the claim. From the duality identity (4.9) and taking into account that $F_T^\perp$ is finite dimensional, it is easy to see that every element can be driven to F_T in time $t = \varepsilon$, if the following uniqueness property holds: If V is a solution of (4.8) with data $V^0 \in F_T^\perp$ and $v = 0$ in $\Omega \times (0,\varepsilon)$ and $\psi = 0$ in $\omega \times (0,\varepsilon)$, then $V^0 \equiv 0$. But this is obvious since every element V^0 of $F_T^\perp$ can be written as $V^0 = \sum_{\omega_j \le N} c_j e_j$ and the fact that

$$\sum_{\omega_j \le N} c_j e_j = 0 \text{ in } \omega$$

implies that $c_j = 0$ for all j such that $\omega_j \le N$ because of Theorem 3.

■

6.2 Getting rid of the parabolic control.

We have shown that for any $U^0 \in H$ there exist $(f_h, f_p) \in L^2(\Omega \times (0,T)) \times \mathcal{D}(\omega \times (0,T))$ such that

$$\mathcal{L}\left(T; U^0, (f_h, f_p)\right) = 0 \tag{6.3}$$

and that the mapping $U^0 \to (f_h, f_p)$ is continuous in the sense that

$$\| f_h \|_{L^2(\Omega \times (0,T))} \le C_0 \| U^0 \|_H \tag{6.4}$$

and

$$\| f_p \|_{H^k(\omega \times (0,T))} \le C_k \| U^0 \|_H \tag{6.5}$$

for every $k \in I\!N$.

We can actually construct controls (f_h, f_p) such that

$$\mathcal{L}\left(T; U^0, (f_h, \partial_t(f_p))\right) = 0 \tag{6.6}$$

holds with the estimates (6.4)-(6.5). In other words we can take the parabolic control to be the time derivate of a function $f_p \in \mathcal{D}(\omega \times (0,T))$ keeping the same continuity estimates.

To see this it is necessary to prove that we can do it at the level of Proposition 2. This is the object of the following Proposition.

Proposition 3. *Under the assumptions of Proposition 2, the map*

$$\widetilde{K}_{T,\mu} = K_{T,\mu} \circ A^{-1} : H^p \longrightarrow H^1(0,T; H_0^1(\omega)) \tag{6.7}$$

where $K_{T,\mu}$ is the operator of Proposition 2, is such that

$$\widetilde{\widetilde{K}}_{T,\mu} = \partial_t \widetilde{K}_{T,\mu} \tag{6.8}$$

satisfies all the conditions of Proposition 1, excepting the fact that in the estimate (5.14) one has now an extra multiplicative factor $1/T$.

Proof: Given $U^0 \in H^p$ we introduce $W^0 \in H^p$ such that $AW^0 = U^0$. Obviously, W^0 is in fact more regular that U^0 :

$$\| W^0 \|_{H^2(M)\times H^1(M)\times H^2(M)} \leq C \| U^0 \|_H .$$

We set

$$\widetilde{K}_{T,\mu}(U^0) = \partial_t K_{T,\mu}(W^0)$$

where $K_{T,\mu}$ is as in Proposition 2. Let us see that $\widetilde{K}_{T,\mu}$ answers to the question. Indeed, $f_p = K_{T,\mu}(W^0)$ is such that the solution of

$$\begin{cases} W_t - AW = \begin{pmatrix} 0 \\ 0 \\ f_p \end{pmatrix} \\ W(0) = W^0 \end{cases} \tag{6.9}$$

satisfies

$$\Pi_\mu^p(W(T)) = 0. \tag{6.10}$$

Then $U = W_t$ solves

$$\begin{cases} U_t - AU = \begin{pmatrix} 0 \\ 0 \\ \partial_t f_p \end{pmatrix} \\ U(0) = W_t(0) = AW^0 = U^0. \end{cases} \tag{6.11}$$

We have used in particular the fact that $f_p(x,0) \equiv 0$.

On the other hand, since $f_p(x,T) \equiv 0$,

$$\Pi_\mu^p(U(T)) = \Pi_\mu^p(W_t(T)) = \Pi_\mu^p(AW(T)) = 0$$

due to the fact that Π_μ^p and A commute.

■

Once Proposition 3 is proved we need to complete the arguments of Section 6.1 to get rid of the uncontrolled low frequencies. The same arguments as above show that it is sufficient to prove the following uniqueness result: If $V^0 \in F_T^{\perp}$ and $v = 0$ in $\Omega \times (0, \varepsilon)$ and $\psi_t = 0$ in $\omega \times (0, \varepsilon)$, then can we guarantee that $V^0 \equiv 0$?

When the boundary of M is non-empty, it is easy to prove that this holds. However, when $\partial M = \emptyset$ it is easy to see that this result is not true since $(v, \psi) \equiv (0, 1)$ solves (4.10) and lies in the subspace generated by the first eigenfunctions.

Thus, let us distinguish these two cases.

Case 1: $\partial M \neq \emptyset$

From (6.6),(6.4) and (6.5) and using the duality identity (4.9) it is easy to deduce that for every $k > 0$ there exists a constant $C_k > 0$ such that the following inequality holds for any solution of (4.8):

$$\| V(0) \|_{H'}^2 \leq C_k \left[\int_0^T \int_\Omega v^2 dxdt + \| \psi_t \|_{H^{-k}(\omega \times (0,T))}^2 \right]. \tag{6.12}$$

Indeed, applying identity (4.9) with the control $(f_h, \partial_t f_p)$ such that (6.4)-(6.6) hold we deduce that

$$- < U^0, V(0) >= \int_0^T \int_\Omega f_h v + \int_0^T \int_\Omega \partial_t f_p \psi \ , \ \forall (U^0, V^0) \in H \times H'$$

Then, from (6.4) and (6.5) we obtain that

$$| < U^0, V(0) > | \leq C_k \left[\| v \|_{L^2(\Omega \times (0,T))} + \| \partial_t \psi \|_{H^{-k}(\omega \times (0,T))} \right] \| U^0 \|_H, \forall U^0 \in H$$

which is equivalent to (6.12).

On the other hand, analyzing the first equation on (4.10),

$$v_{tt} - (c^2 + \alpha\beta)\Delta v + \nu\beta\Delta\psi = 0$$

we see that, if $\omega \subset \Omega$:

$$\| \Delta\psi \|_{H^{-2}(\omega \times (0,T))} \leq C \| v \|_{L^2(\Omega \times (0,T))} . \tag{6.13}$$

From the second equation of (4.10),

$$-\psi_t - \nu\Delta\psi + \alpha\Delta v = 0$$

we deduce that

$$\| \psi_t + \nu\Delta\psi \|_{H^{-2}(\omega \times (0,T))} \leq C \| v \|_{L^2(\Omega \times (0,T))} . \tag{6.14}$$

Combining (6.13) and (6.14) we conclude that

$$\| \psi_t \|_{H^{-2}(\omega \times (0,T))} \leq C \| v \|_{L^2(\Omega \times (0,T))} . \tag{6.15}$$

From (6.12) and (6.15) we immediately deduce the existence of a positive constant C such that

$$\| V(0) \|_{H'}^2 \leq C \int_0^T \int_\Omega v^2 dxdt \ , \forall V^0 \in H'. \tag{6.16}$$

By duality it is easy to see that (6.16) and the statement a) of Theorem 1 are equivalent.

Case 2: $\partial M = \emptyset$

In this case (6.12) might not be true because of the counterexample above. However, it can be seen that the inequality holds provided V^0 is orthogonal in H' to $(0, 0, 1)$. By duality using (4.4) it is easy to see that this inequality is equivalent to the statement b) of Theorem 1.

■

7 Control on the temperature.

This section is devoted to prove the null-controllability of system (1.1) when the control acts on the equation of temperature:

$$\begin{cases} u_{tt} - c^2\Delta u + \alpha\Delta\theta = 0 & \text{in} \quad M \times (0,T) \\ \theta_t - \nu\Delta\theta + \beta u_t = f\chi_\Omega & \text{in} \quad M \times (0,T) \\ u = \theta = 0 & \text{on} \quad \partial M \times (0,T) \\ u(x,0) = u^0(x)\,,\; u_t(x,0) = u^1(x) & \text{in} \quad M \\ \theta(x,0) = \theta^0(x) & \text{in} \quad M. \end{cases} \tag{7.1}$$

We have the following result:

Theorem 4. *Under the assumptions of Theorem 1, when $\partial M \neq \emptyset$, for any (u^0, u^1, θ^0) in H there exists $f \in L^2(M \times (0,T))$ such that the solution of* (7.1) *satisfies* (1.9).

If $\partial M = \emptyset$ the same holds provided $\int_M u^0 dx = \int_M u^1 dx = 0$.

Remark.

1.- When $\partial M = \emptyset$, integrating the first equation of (7.1) in M we deduce that $\frac{d^2}{dt^2}\int_M u(x,t)dx = 0$. Therefore $\int_M u^1(x)dx = \int_M u_t(x,T)dx$ and $\int_M u^0(x)dx + T\int_M u^1(x)dx = \int_M u(x,T)dx$. These two identities show that $\int_M u^0 dx = \int_M u^1 dx = 0$ is a necessary condition for null-controllability.

2.- The statement of Theorem 4 is equivalent to the following observability inequality for the adjoint system (4.10):

$$\| V(0) \|_{H'}^2 \leq C \int_0^T \int_\Omega \psi^2 dxdt \tag{7.2}$$

which holds for any data $V^0 \in H'$ when $\partial M \neq \emptyset$ and for any data V^0 such that $\int_M v^0 dx = \int_M v^1 dx = 0$ when $\partial M = \emptyset$. Notice that $(v, \psi) = (a + bt, 0)$ solves (4.10) when $\partial M = \emptyset$ and that consequently, (7.2) may not hold for these solutions.

3.- It is easy to see that for $(u^0, u^1, \theta^0) \in H$ and $f \in L^2(M \times (0,T))$, then $(u, u_t, \theta) \in C([0,T]; H)$. Indeed, multiplying in the first equation of (7.1) by u_t and in the second one by $-\frac{\alpha}{\beta}\Delta\theta$ we deduce that

$$\frac{dE}{dt}(t) + \frac{\nu\alpha}{\beta}\int_M |\Delta\theta|^2 dx = -\frac{\alpha}{\beta}\int_M \chi_\Omega f \Delta\theta dx \leq \frac{\nu\alpha}{2\beta}\int_M |\Delta\theta|^2 dx + \frac{\alpha}{2\beta\nu}\int_\Omega f^2 dx.$$

Thus

$$E(t) \leq E(0) + \frac{\alpha}{2\beta\nu}\int_0^t \int_\Omega f^2 dxdt.$$

This energy inequality allows to deduce that the solution does belong to $C([0,T]; H)$.

■

Proof of Theorem 4.

To prove this result we have to modify a bit the control strategy. The main remark that has to be done is that the effect of the hyperbolic control f_h can also be obtained by means of a $L^2(\Omega \times (0,T'))$-control acting on the heat equation. More precisely, we may replace (3.2) by:

$$\begin{cases} \frac{dU}{dt} = AU + (0,0,f_h + f_p)^t \quad , \quad 0 < t < T \\ U(0) = U^0 \end{cases} \tag{7.3}$$

with f_h and f_p as in (3.1). In the Remark above we have observed that this sytem has an unique solution in $C([0,T];H)$.

When considering (7.3) we have to replace Lemmas 1 and 2 of section 3 by the following ones:

Lemma 1'. *There exists a linear and continuous operator*

$$A : H \times L^2((T',T); H_0^1(\omega)) \to L^2(\Omega \times (0,T')) \tag{7.4}$$

such that

$$\Pi^h \mathcal{L}(T; U^0, (0, f_p + A(U^0, f_p))) = 0 \ , \ \forall U^0, f_p. \tag{7.5}$$

Lemma 2': *There exists a linear and continuous operator*

$$B : H \times L^2(\Omega \times (0,T')) \to \mathcal{D}(\omega \times (T',T)) \tag{7.6}$$

such that

$$\Pi^p \mathcal{L}(T; U^0, (0, B(U^0, f_h) + f_h)) = 0 \ , \ \forall U^0, f_h. \tag{7.7}$$

To prove Lemma 1' we have to observe that, in view of (4.13) and taking into account the structure of the solution (v,ψ) of (4.10) we have

$$\| W^0 \|_{H'}^2 \leq C^2 \int_0^{T'} \int_\Omega \psi^2 dxdt \tag{7.8}$$

with

$$W^0 = \sum_{|\lambda_j|>R_0} \left(\frac{\alpha\omega_j^2}{\lambda_j^{h,+} + \nu\omega_j^2} a_j^+ V_j^{h,+} + \frac{\alpha\omega_j^2}{\lambda_j^{h,-} + \nu\omega_j^2} a_j^- V_j^{h,-} \right) \tag{7.9}$$

instead of the previous initial data V^0.

Since $\| W^0 \|_{H'}$ and $\| V^0 \|_{H'}$ are equivalent norms we deduce that

$$\| V^0 \|_{H'}^2 \leq C' \int_0^{T'} \int_\Omega \psi^2 dxdt \tag{7.10}$$

and this is equivalent to Lemma 1'.

Lemma 2' is an immediate consequence of Proposition 2 of section 5.

To conclude we have to get rid of the low frequencies as in section 6.1. But now we need the following uniqueness result: If $V^0 \in F_T^\perp$ and $\psi = 0$ in $\Omega \times (0,\varepsilon)$, then $V^0 \equiv 0$. From Theorem 3 we deduce easily that $\psi \equiv 0$, i. e. V is of the form $V = (v, v_t, 0)$. Then from (4.10) we deduce that $\Delta v = 0$ in $M \times (0,T)$. When $\partial M \neq \emptyset$ this implies that $v \equiv 0$ and the proof is concluded.

When $\partial M = \emptyset$, $\Delta v = 0$ in $M \times (0,T)$ implies that v is a function of t only (independent of x). From the first equation of (4.10) we deduce that $v_{tt} \equiv 0$ and this implies that v is necessarily of the form $v = a + bt$.

Using the structures of $V_j^{h,\pm}$ and V_j^p given respectively in sections 4 and formula (5.16), it is clear that for v solution of (4.10) with $V^0 \in H'$, one has $\psi \in L^2(0,T)$ with the estimate

$$\int_0^T \int_M |\sum a_j\omega_j e_j e^{-t\lambda_j^p}|^2 = \sum \int_0^T a_j^2\omega_j^2 e^{2t\lambda_j^p} \leq C \sum a_j^2 \qquad (\lambda_j^p \sim \omega_j^2).$$

We can now apply Lions' HUM method (see [Li2]). Assume for instance that $\partial M \neq \emptyset$. If $V^0 \in H'$ is such that the minimum of the functional

$$\frac{1}{2}\int_0^T \int_\Omega |\psi|^2 dxdt + < U^0, V(0) >$$

over H' is achivied at V^0, then the control

$$f = \psi \tag{7.11}$$

where $V = (v, v_t, \psi)^t$ is the solution of (4.10) with data V^0, satisfies the desired null-controllability condition.

This concludes the proof of Theorem 3.

■

8 Null-controllability of the three-dimensional system of thermoelasticity.

Let M be a compact, connected and C^∞ Riemannian manifold without boundary of dimension three.

Let Ω be an open subset of M and consider the controlled system of three-dimensional linear thermoelasticity:

$$\begin{cases} w_{tt} - \mu\Delta w - (\lambda+\mu)\nabla \mathrm{div} w + \alpha\nabla\theta = f & \text{in} \quad M \times (0,T) \\ \theta_t - \nu\Delta\theta + \beta \text{ div } w_t = 0 & \text{in} \quad M \times (0,T) \\ w(x,0) = w^0(x), w_t(x,0) = w^1(x) & \text{in} \quad M \\ \theta(x,0) = \theta^0(x) & \text{in} \quad M. \end{cases} \tag{8.1}$$

with Lamé constants $\lambda, \mu > 0$.

Here the temperature θ is a scalar function, while the displacement w is a vector-field, i. e. a section of the tangent bundle TM of M. We shall denote by $\widetilde{H}^s(M)$ the Hilbert space of vector-fields with H^s regularity.

We introduce the Hilbert space

$$H = \widetilde{H}^0(M) \times \widetilde{H}^{-1}(M) \times H^{-1}(M). \tag{8.2}$$

It is easy to see that for any $(w^0, w^1, \theta^0) \in H$ and $f \in L^2(0,T; \widetilde{H}^{-1}(M))$ there exists an unique solution of (9.1) in the class $C(0,T;H)$.

We have the following result of null-controllability:

Theorem 5. *Suppose that any geodesic curve in M of length $\sqrt{\mu}T$ intersects Ω. Then, for any (w^0, w^1, θ^0) in H such that $\int_M \theta^0 = 0$ there exists $f \in L^2(0,T; \widetilde{H}^{-1}(M))$* with support in $\bar{\Omega} \times [0,T]$ *such that the solution of* (8.1) *satisfies*

$$w(T) = w_t(T) = 0, \theta(T) = 0 \text{ in } M. \tag{8.3}$$

Moreover, there exists $C > 0$ such that

$$\| f \|^2_{L^2(0,T;\widetilde{H}^{-1}(M))} \leq C \| (w^0, w^1, \theta^0) \|^2_H \tag{8.4}$$

for any $(w^0, w^1, \theta^0) \in H$ as above.

Proof:

One can suppose that Ω has a regular boundary, so the dual space $(H^1(\Omega))'$ is identified with $\{g \in H^{-1}(M) : \text{supp}g \subset \bar{\Omega}\}$ by the L^2 scalar product.

Let us consider the adjoint system

$$\left\{\begin{array}{lll} \varphi_{tt} - \mu\Delta\varphi - (\lambda+\mu)\nabla\text{div}\varphi + \beta\nabla\psi_t = 0 & \text{in} & M\times(0,T) \\ -\psi_t - \nu\Delta\psi - \alpha \text{ div } \varphi = 0 & \text{in} & M\times(0,T) \\ \varphi(x,T) = \varphi^0(x), \varphi_t(x,T) = \varphi^1(x) & \text{in} & M \\ \psi(x,T) = \psi^0(x) & \text{in} & M. \end{array}\right. \tag{8.5}$$

It is easy to see that the null-controllability result above is equivalent to the following observability inequality:

$$\| \varphi(0) \|^2_{(H^1(M))^3} + \| \varphi_t(0) + \beta\nabla\psi(0) \|^2_{(L^2(M))^3} + \| \psi(0) \|^2_{H^1(M)} \leq C \int_0^T \int_\Omega [| \nabla\varphi |^2 + | \varphi |^2] dxdt \tag{8.6}$$

for every solution of (8.5) with $\int_M \psi^0 dx = 0$.

In order to prove (8.6) we first consider the vector field $\sigma = \text{curl}\varphi$ that satisfies:

$$\left\{\begin{array}{ll} \sigma_{tt} - \mu\Delta\sigma = 0 & \text{in } M\times(0,T) \\ \sigma(x,T) = \text{ curl } \varphi^0(x) = \sigma^0(x),\ \sigma_t(x,T) = \text{ curl } \varphi^1(x) = \sigma^1(x) & \text{in } M. \end{array}\right. \tag{8.7}$$

As we mentioned in section 4, by [BLR] we know that, under the geometric control assumption of Theorem 5, there exists a positive constant $C > 0$ such that

$$\| \sigma(0) \|^2_{\widetilde{H}^0(M)} + \| \sigma_t(0) \|^2_{\widetilde{H}^{-1}(M)} \leq C \int_0^T \int_\Omega | \sigma |^2 \, dxdt. \tag{8.8}$$

On the other hand the pair (ρ, ψ) with $\rho = \text{div}\varphi$ satisfies

$$\left\{\begin{array}{lll} \rho_{tt} - (\lambda+2\mu)\Delta\rho + \beta\Delta\psi_t = 0 & \text{in} & M\times(0,T) \\ -\psi_t - \nu\Delta\psi - \alpha\rho = 0 & \text{in} & M\times(0,T) \\ \rho(x,T) = \rho^0(x) = \text{div}\varphi^0, \rho_t(x,T) = \rho^1(x) = \text{div}\varphi^1, \psi(x,T) = \psi^0 & \text{in} & M \end{array}\right. \tag{8.9}$$

or, equivalently,

$$\left\{\begin{array}{lll} \rho_{tt} - (\lambda+2\mu+\alpha\beta)\Delta\rho - \beta\nu\Delta^2\psi = 0 & \text{in} & M\times(0,T) \\ -\psi_t - \nu\Delta\psi - \alpha\rho = 0 & \text{in} & M\times(0,T) \\ \rho(x,T) = \rho^0(x) = \text{div}\varphi^0, \rho_t(x,T) = \rho^1(x) = \text{div}\varphi^1, \psi(x,T) = \psi^0 & \text{in} & M. \end{array}\right. \tag{8.10}$$

Introducing the new unknown $\xi = -\Delta\psi$, the system (8.10) can be written as

$$\left\{\begin{array}{lll} \rho_{tt} - (\lambda+2\mu+\alpha\beta)\Delta\rho + \beta\nu\Delta\xi = 0 & \text{in} & M\times(0,T) \\ -\xi_t - \nu\Delta\xi + \alpha\Delta\rho = 0 & \text{in} & M\times(0,T) \\ \rho(x,T) = \rho^0(x) = \text{div}\varphi^0, \rho_t(x,T) = \rho^1(x) = \text{div}\varphi^1, \xi(x,T) = -\Delta\psi^0 & \text{in} & M. \end{array}\right. \tag{8.11}$$

Applying the observability inequality (6.16) to this system we conclude that

$$\| \rho(0) \|^2_{L^2(M)} + \| \rho_t(0) - \beta\xi(0) \|^2_{H^{-1}(M)} + \| \xi(0) \|^2_{H^{-1}(M)} \leq C \int_0^T \int_\Omega \rho^2 dxdt. \tag{8.12}$$

It is easy to see that (8.8) and (8.12) imply (8.6) (because the space N of harmonic solutions of (8.5) such that $\psi \equiv 0$, $\text{div}\varphi \equiv 0$, $\text{curl}\varphi \equiv 0$ is finite dimensional and for $\varphi \in N$, if φ vanishes on $\Omega \times (0,T)$ then necessarily $\varphi \equiv 0$.

This concludes the proof of Theorem 5.

■

Remark: 1.- The results of this section apply, in particular, to the system of linear thermoelasticity in $\mathbf{R}^3$ with periodic boundary conditions.

2.- If we control the system of thermoelasticity by means of a heat source acting in the equation of temperature

$$\left\{\begin{array}{lll} w_{tt} - \mu\Delta w - (\lambda+\mu)\nabla \mathrm{div} w + \alpha\nabla\theta = 0 & \text{in} & M\times(0,T) \\ \theta_t - \nu\Delta\theta + \beta \mathrm{div} w_t = f\chi_\Omega & \text{in} & M\times(0,T) \\ w(0) = w^0 \ , \ w_t(0) = w^1 & \text{in} & M \\ \theta(0) = \theta^0 & \text{in} & M \end{array}\right. \tag{8.13}$$

then the null-controllability does not hold.

This is obvious since $v =$ curl w solves the uncontrolled wave equation

$$v_{tt} - \mu\Delta v = 0.$$

However, in this situation one can control to zero $\mathrm{div} w$ and θ.

■

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Appendix.

In this Section we give a sketch of the proof of the following interpolation inequality for the Laplacian $P = \Delta_x + \partial_t^2$:

Lemma A. *Let M be a C^∞ connected Riemannian manifold. Let Ω be an open non-empty subset of M. For any $T > 0$ and $\alpha \in (0, T/2)$ there exist $\delta \in (0,1)$ and $C > 0$ such that*

$$\| \varphi \|_{H^1(M\times(\alpha,T-\alpha))} \leq C \| \varphi \|^\delta_{H^1(M\times(0,T))} \left(\| P\varphi \|_{L^2(M\times(0,T))} + \| \partial_t\varphi(x,0) \|_{L^2(\Omega)} \right)^{1-\delta} \qquad (A.1)$$

for any $\varphi \in H^2(M \times (0,T))$ such that $\varphi = 0$ on $(\partial M \times (0,T)) \cup (M \times 0)$.

Proof: We proceed in several steps.

Step 1. First of all we apply the following Carleman inequality that was proved in [LRo, Proposition 1]:

$$\int_{\mathbb{R}^{n+1}_+} |h^2 Pg|^2 e^{2\psi/h} dxdt + h \int_{\mathbb{R}^n} \left\{ |g(x,0)|^2 + |h\nabla_x g(x,0)|^2 + |h\partial_t g(x,0)|^2 \right\} e^{2\psi/h} dx$$

$$\geq Ch \int_{\mathbb{R}^{n+1}_+} \left\{ |g|^2 + |h\nabla_{x,t} g|^2 \right\} e^{2\psi/h} dxdt \qquad (A.2)$$

for any $h > 0$ sufficiently small and any $g \in C_0^\infty(K)$ with $K = \{(x,t) \in \mathbb{R}^{n+1} : t \geq 0, |(x-x_0,t)| \leq \tau_0\}$.

Inequality (A.2) holds for any ψ satisfying Hörmander's hypoellipticity condition such that $\psi_t \neq 0$ for all $(x,t) \in K$. Let x_0 be in Ω. We will apply (A.2) with $\psi(x,t) = \gamma(-t - |x-x_0)|^2)$, γ being such that it guarantees the hypoellipticity, $\gamma(0) = 0$ and γ strictly increasing near zero. We also choose $g = \chi\varphi$ where $\chi = \chi(x,t)$ is a smooth function such that $0 \leq \chi \leq 1$ and

$$\chi(x,t) = \begin{cases} 1 \text{ , if } (x,t) \in M \times (0,T) : \psi(x,t) \geq -\varepsilon \\ 0 \text{ , if } (x,t) \in M \times (0,T) : \psi(x,t) \leq -2\varepsilon \end{cases}$$

with $\varepsilon > 0$ small enough (see figure 1).

We have $Pg = \chi P\varphi + (P\chi)\varphi + 2\nabla_{x,t}\chi.\nabla_{x,t}\varphi = \chi P\varphi + [P,\chi]\varphi$. By (A.2):

$$\int |h^2 \chi P\varphi|^2 e^{2\psi/h} dxdt + \int |h^2 [P,\chi]\varphi|^2 e^{2\psi/h} dxdt + h \int |h\chi(x,0)\partial_t\varphi(x,0)|^2 e^{2\psi/h} dx$$

$$\geq Ch \int \{ |\chi\varphi|^2 + |h\nabla_{x,t}(\chi\varphi)|^2 \} e^{2\psi/h} dxdt.$$

We observe that $[P,\chi]\varphi$ is supported in the set $\{-\varepsilon \geq \psi \geq -2\varepsilon\}$ and $\psi \leq 0$ for $t \geq 0$. On the other hand, $\partial_t(\chi\varphi)(x,0) = \chi\partial_t\varphi(x,0)$ and $\nabla_{x,t}(\chi\varphi) = \nabla_{x,t}\varphi$ on $\{\psi \geq -\varepsilon\}$. Therefore,

$$he^{-2\varepsilon/h} \| \varphi \|^2_{H^1(\psi\geq-2\varepsilon)} + h \| P\varphi \|^2_{L^2(\psi\geq-2\varepsilon)} + \| \partial_t\varphi(x,0) \|^2_{L^2(\Omega)} \geq C \| \varphi \|^2_{H^1(\psi\geq-\varepsilon/2)} e^{-\varepsilon/h} \qquad (A.3)$$

for $h > 0$ small enough.

On the other hand, by possibly modifying the constant C in (A.3) it is easy to see that this holds for $h > 0$ large too. Thus, optimizing the choice of h in (A.3) as in [Ro] we deduce the existence of $C > 0$ and $\delta \in (0,1)$ such that

$$\| \varphi \|_{H^1(\psi\geq-\varepsilon/2)} \leq C \| \varphi \|^\delta_{H^1(\psi\geq-2\varepsilon)} \left(\| P\varphi \|_{L^2(\psi\geq-2\varepsilon)} + \| \partial_t\varphi(x,0) \|_{L^2(\Omega)} \right)^{1-\delta}. \qquad (A.4)$$

Step 2. Using classical local interpolation inequalities for the laplacian, from (A.4) we deduce that for any compact set K of M and $\varepsilon > 0$, there exist $C, \delta > 0$ such that

$$\| \varphi \|_{H^1(K\times(\varepsilon,T-\varepsilon))} \leq C \| \varphi \|^{\delta}_{H^1(M\times(0,T))} \left(\| P\varphi \|_{L^2(M\times(0,T))} + \| \partial_t \varphi(x,0) \|_{L^2(\Omega)}\right)^{1-\delta} \qquad (A.5)$$

with $C > 0$ depending on K and ε.

Step 3. Finally we have to prove an estimate on a neighborhood of the lateral boundary $\partial M \times (\varepsilon, T-\varepsilon)$. This can be done as in the last section of [LRo].