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E. ZUAZUA Exact boundary controllability for the semilinear wave equation

Résumé. Nous étudions la contrôlabilité exacte frontière de l'équation des ondes semilinéaire avec des conditions aux limites de Dirichlet. Nous démontrons la contrôlabilité exacte lorsque la nonlinéarité est globalement Lipschitzienne, le temps de contrôlabilité étant celui qui correspond au problème linéaire. La méthode de démonstration combine HUM (Hilbert uniqueness method) et un argument de point fixe. Par ailleurs, en utilisant une méthode de pénalisation, on prouve l'existence d'un ensemble de contrôles optimaux vérifiant un système d'optimalité constitué par deux équations des ondes semilinéaires couplées.

Abstract. The exact boundary controllability for the semilinear wave equation with Dirichlet boundary conditions is studied. The exact controllability is proven when the nonlinearity is globally Lipschitz continuous. The controllability time is the one of the linear wave equation. The method of proof combines HUM (Hilbert uniqueness method) and a fixed point argument. On the other hand, by using a penalization method, we prove the existence of a set of optimal controls that satisfy an optimality system. This optimality system consists of two coupled semilinear wave equations.

1. Introduction

The aim of this paper is to study the exact boundary controllability for the wave equation perturbed by a globally Lipschitz continuous nonlinear term. For the sake of brevity and simplicity we shall focus our attention on the particular case where the control enters in the Dirichlet boundary condition, since it is complex enough to generate interesting techniques and results. However, most of the ideas and results of this paper may be easily adapted and generalized to different boundary conditions, e.g. Neumann or mixed boundary conditions.

Let Ω be a bounded domain of \mathbb{R}^n with smooth boundary $\Gamma = \partial\Omega$, $f : \mathbb{R} \rightarrow \mathbb{R}$

a (globally) Lipschitz continuous function, that is, such that $f' \in L^\infty(\mathbb{R})$, $T > 0$ and $\Gamma_0 \subset \Gamma$ a non-empty open subset of Γ . Let us consider the following semilinear wave equation with Dirichlet boundary control:

$$y'' - \Delta y + f(y) = 0 \quad \text{in } Q = \Omega \times (0, T) \quad (1.1)$$

$$y = \begin{cases} v & \text{on } \Sigma_0 = \Gamma_0 \times (0, T) \\ 0 & \text{on } \Gamma \setminus \Sigma_0 \end{cases} \quad (1.2)$$

$$y(0) = y^0, \quad y'(0) = y^1 \quad \text{in } \Omega. \quad (1.3)$$

In (1.1)-(1.3) we denote by $' = \partial/\partial t$ the derivative with respect to the time variable and $y(0)$ (resp. $y'(0)$) represents the function $x \rightarrow y(x, 0)$ (resp. $x \rightarrow \partial y(x, 0)/\partial t$). On the other hand, we denote by Σ the lateral boundary of Q that is, $\Sigma = \Gamma \times (0, T)$.

The exact controllability problem for (1.1)-(1.3) states as follows: given $T > 0$ large enough, to find, for every initial and final data (y^0, y^1) , (z^0, z^1) (given in a suitable Hilbert space), a control function v driving the system (1.1)-(1.3) to the state (z^0, z^1) at time T ; i.e. such that the solution $y = y(x, t)$ of (1.1)-(1.3) satisfies

$$y(T) = z^0; \quad y'(T) = z^1. \quad (1.4)$$

Concerning the linear case where $f(s) = as$ for some $a \in \mathbb{R}$, research in this problem and other related questions (observability, stabilizability) has been very active during the last years and there is an extensive literature on these topics (see e.g. Lagnese and Lions [8], Lions [10, 11], Russell [15] and the references therein). Always in the linear framework, we may say that the exact controllability problem is by now well understood. However, concerning the semilinear problem (1.1)-(1.3) or, more generally, in the nonlinear context, very little is known.

To our knowledge, the first work on exact controllability of nonlinear distributed systems was done by Markus [13]. In this paper, an implicit function theorem approach was introduced for the study of the exact controllability problem of nonlinear finite dimensional distributed systems. Subsequently, this approach was adapted and generalized to the

nonlinear wave equation by Chewning [2], Fattorini [3], Russell [15] and other authors. In this way, local controllability results were proven. That is, it was proven that "small" initial data lying in some neighbourhood of $(0, 0)$ may be driven to the rest $\{0, 0\}$ (when $f(0) = 0$) for T large enough. It is important to note that this approach does not provide exact controllability results in the sense formulated above.

More recently [17, 18], we have proven the exact controllability of (1.1)-(1.3) for Lipschitz continuous perturbations f satisfying the additional condition

$$\exists \lim_{|s| \rightarrow +\infty} \frac{f(s)}{s} \quad (1.5)$$

i.e. for asymptotically linear perturbations. The exact controllability was proven in the space $L^2(\Omega) \times H^{-1}(\Omega)$ (the largest Hilbert space where the nonlinear problem (1.1) has some sense) with controls in $L^2(\Sigma_0)$ and for T and Γ_0 "large enough" (this will be made precise in Remark 2.1). The method of proof was based on HUM (Hilbert uniqueness method), recently introduced by Lions [9, 10, 11] in the linear framework, and on a fixed point argument.

The main purpose of the present paper is to improve this result showing that the restriction (1.5) is unnecessary. Therefore, we shall prove the exact controllability of (1.1)-(1.3) for every (globally) Lipschitz continuous perturbations. The method of proof is once again based on HUM and Schauder's fixed point theorem, but the scheme of the proof is different and slightly more involved. The exact controllability will be proven in $\bigcup_{\epsilon > 0} \{H_0^1(\Omega) \times H^{-1}(\Omega)\}$ but not in $L^2(\Omega) \times H^{-1}(\Omega)$ (as was the case when (1.5) was assumed) because of the lack of compactness. Concerning the pair (Γ_0, T) we shall only assume that the exact controllability of the linear system with $f = 0$ holds in $L^2(\Omega) \times H^{-1}(\Omega)$ with controls $L^2(\Sigma_0)$ at time T as well as a unique-continuation property for the wave equation with a zero order potential. Therefore, we shall generalize to the semilinear context (under the assumption $f' \in L^\infty(\mathbb{R})$) most of the results that are by now well known in the linear framework.

The plan of the paper is as follows. Section 2 is devoted to the linear wave equation perturbed by a bounded potential

$$y'' - \Delta y + W(x,t)y = 0 \quad \text{in } Q \quad (1.6)$$

with $W \in L^\infty(Q)$. In section 2.1 we shall prove its exact controllability. In section 2.2 we shall prove an "uniform exact controllability" result. In other words, we shall establish that the control function v (associated to some fixed initial and final data) remains bounded (in some norm that will be made precise below) when the potential W is bounded in $L^\infty(Q)$. This result will be crucial for the study of the semilinear problem (1.1)-(1.3).

In section 3 we shall prove the exact controllability of (1.1)-(1.3). The initial and final data $\{y^0, y^1\}, \{z^0, z^1\}$ being fixed in $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ with $\gamma > 0$, we shall construct a nonlinear operator.

$$N : L^2(Q) + L^2(Q)$$

mapping $\xi \in L^2(Q)$ into $y = N(\xi)$, the solution of the problem

$$y'' - \Delta y + \left(\frac{f(\xi) - f(0)}{\xi} \right) y = -f(0) \quad \text{in } Q$$

$$y = \begin{cases} v & \text{on } \Sigma_0 \\ 0 & \text{on } \Gamma \setminus \Sigma_0 \end{cases} \quad (1.7)$$

$$y(0) = y^0, \quad y'(0) = y^1; \quad y(T) = z^0; \quad y'(T) = z^1$$

the function v being the control constructed in section 2 for the potential

$$W = (f(\xi) - f(0))/\xi.$$

We shall prove the compactness of the operator N (at this level the fact that $\gamma > 0$ will be crucial). On the other hand, the uniform controllability result of section 2.2 will show that N maps $L^2(Q)$ into a bounded set of $L^2(Q)$. We shall conclude the exact controllability of (1.1)-(1.3) applying Schauder's fixed point theorem. Concerning the regularity of the control function we will prove that $v \in H_0^\gamma(0, T; L^2(\Gamma_0))$ if $\gamma \in (0, 1)$.

Section 4 is devoted to the study of the existence of optimal controls as well as to their characterization by means of an optimality system. This

will be done using a penalty method inspired by [11]. The optimality system consists of two coupled semilinear wave equations. This type of system seems to be new, and, to our knowledge, it has not been treated in the literature. In section 4.2 some comments are given on the existence and uniqueness of solutions of this system.

The techniques of this paper are general and may be applied in other situations. In section 5 we give some remarks concerning some possible extensions of our results. We also pay attention to a model of plates

$$y'' + \Delta^2 y + f(y) = 0.$$

This problem presents an additional difficulty concerning the unique continuation of weak solutions and leads us to a question that seems to be open.

Some of the results of this paper were briefly announced in [19].

2. Linear wave equation perturbed by a bounded potential

The aim of this section is to study the exact controllability of the following wave equation perturbed by a bounded potential

$$y'' - \Delta y + W y = 0 \quad \text{in } Q$$

$$y = \begin{cases} v & \text{on } \Sigma_0 = \Gamma_0 \cdot (0, T) \\ 0 & \text{on } \Gamma \setminus \Sigma_0 \end{cases} \quad (2.1)$$

$$y(0) = y^0; \quad y'(0) = y^1$$

where $W \in L^\infty(Q)$, $\Gamma_0 \subset \Gamma$ and $T > 0$.

Our main assumptions concern the pair (Γ_0, T) and are as follows:

the pair (Γ_0, T) is such that the linear wave equation (2.1) is exactly controllable for $W = 0$ at time T , in the space $L^2(\Omega) \times H^{-1}(\Omega)$ and with controls in $L^2(\Sigma_0)$ the pair (Γ_0, T) is such that the following unique continuation property holds: if $W \in L^\infty(Q)$ and $\xi \in H^1(Q)$ satisfies

$$\begin{aligned} \theta'' - \Delta\theta + W\theta &= 0 & \text{in } Q \\ \theta &= 0 & \text{on } \Sigma \\ \frac{\partial\theta}{\partial\nu} &= 0 & \text{on } \Sigma_0 \end{aligned} \quad (2.3)$$

then $\theta \equiv 0$.

It is now well known (it is a consequence of HUM, see [11]) that the hypothesis (2.2) is equivalent to the following one: there exists a positive constant $c = c(\Gamma_0, T) > 0$ such that for every smooth solution $\varphi = \varphi(x, t)$ of

$$\begin{aligned} \varphi'' - \Delta\varphi &= 0 & \text{in } Q \\ \varphi &= 0 & \text{on } \Sigma \\ \varphi(0) = \varphi^0, \varphi'(0) = \varphi^1 & \text{in } \Omega \end{aligned}$$

the following estimate holds:

$$\|\varphi^0\|_{H_0^1(\Omega)}^2 + \|\varphi^1\|_{L^2(\Omega)}^2 \leq c \left\| \frac{\partial\varphi}{\partial\nu} \right\|_{L^2(\Sigma_0)}^2 \quad (2.4)$$

(In (2.4) ν denotes the unit outward pointing normal to Σ and $\partial\varphi/\partial\nu$ the normal derivative with respect to ν .)

REMARK 2.1. There are several sufficient conditions on the pair (Γ_0, T) in order for (2.2) to be satisfied:

(a) Given any $x^0 \in \mathbb{R}^N$, (2.2) holds true for $\Gamma_0 = \{x \in \Gamma; (x - x^0) \cdot \nu(x) > 0\}$ and $T > 2\|x - x^0\|_{L^2(\Omega)}$ (we denote by \cdot the scalar product in \mathbb{R}^N). This can be proven by using a multiplier technique (see Ho[4] and Lions [10,11]).

(b) The property is also satisfied when the pair (Γ_0, T) is such that every ray of geometric optics intersects the set $\Sigma_0 = \Gamma_0 \times (0, T)$ in a nondiffractive point. This condition turns out to be also necessary, except for some very special cases. These results have been recently proven by Bardos, Lebeau and Rauch [1] by using microlocal analysis techniques.

From [11, chapter 1, Theorem 6.4] we know that (2.4) implies the estimate

$$\|\varphi^0\|_{L^2(\Omega)}^2 + \|\varphi^1\|_{H^{-1}(\Omega)}^2 \leq c \left\| \frac{\partial\varphi}{\partial\nu} \right\|_{H^{-1}(0, T; L^2(\Gamma_0))}^2 \quad (2.5)$$

and then, by interpolation, it follows that

$$\|\varphi^0\|_{H_0^{1-\gamma}(\Omega)}^2 + \|\varphi^1\|_{H^{-\gamma}(\Omega)}^2 \leq c_\gamma \left\| \frac{\partial\varphi}{\partial\nu} \right\|_{H^{-\gamma}(0, T; L^2(\Gamma_0))}^2 \quad (2.6)$$

for every $\gamma \in [0, 1]$, $\gamma \neq \frac{1}{2}$.

REMARK 2.2. The estimate corresponding to the case $\gamma = \frac{1}{2}$ is

$$\|\varphi^0\|_{H^{1/2}(\Omega)}^2 + \|\varphi^1\|_{(H_{00}^{1/2}(\Omega))}^2 \leq c \left\| \frac{\partial\varphi}{\partial\nu} \right\|_{(H_{00}^{1/2}(0, T; L^2(\Gamma_0)))}^2 \quad (2.7)$$

In that which follows we shall skip this case for the sake of simplicity, since all the results of this paper may be adapted, with trivial modifications, to this case. It suffices to replace $H_0^{1-\gamma}$ by $H_{00}^{1/2}$ and $H^{-\gamma}$ by $(H_{00}^{1/2})'$.

On the other hand, in order to simplify the notation, we shall denote by $\|\cdot\|_S$ the norm in $H^S(\Omega)$ and by $\|\cdot\|_{S, \Gamma_0}$ the norm in $H^S(0, T; L^2(\Gamma_0))$. The norms in $L^2(\Omega)$ and $L^2(\Gamma_0)$ will be respectively denoted by $|\cdot|$ and $|\cdot|_{L_0}$.

REMARK 2.3. From Ruiz [14] we know that hypothesis (2.3) holds true if $\Gamma_0 = \Gamma$ and $T > \text{diameter of } \Omega$.

Hypothesis (2.3) is very probably redundant since (2.2) should imply (2.3). However, this unique-continuation problem seems to be open.

Once (2.6) is proven, from HUM, the following statement holds concerning the relationship between the regularity properties of the controlled states and the control functions:

when $W = 0$ and the initial and final data belong to $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ for some $\gamma \in [0, 1]$, $\gamma \neq 1/2$, then the control function may be chosen in $H_0^\gamma(0, T, L^2(\Gamma_0))$.

In section 2.1, we shall generalize these results to the general case of an arbitrary bounded potential $W \in L^\infty(Q)$ under the assumptions (2.2) and (2.3). That is, we shall prove that when (2.2) (or equivalently, estimate (2.4)) holds for $W \equiv 0$ and (2.3) holds, then (2.2) and (2.8) hold for every bounded potential W . In section 2.2 we shall prove the uniform exact controllability for $\gamma > 0$. This concept, mentioned in the introduction, will be made more precise below.

2.1. Exact controllability

We have the following result.

THEOREM 2.1. Assume that the pair (Γ_0, T) is such that (2.2) and (2.3) hold true. Then:

(a) For every bounded potential $W \in L^\infty(Q)$, the system (2.1) is exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$ with controls in $L^2(\Gamma_0)$ at time T .

(b) Moreover, if $\{y^0, y^1\}, \{z^0, z^1\} \in H_0^1(\Omega) \times H^{-1}(\Omega)$ for some $\gamma \in (0, 1/2)$, $\gamma \neq 1/2$, then the control may be chosen in $H_0^1(0, T; L^2(\Gamma_0))$.

Proof. We proceed in three steps.

Step 1. From HUM, in order to prove the statement (a) of our theorem it suffices to establish the estimate (2.4) for solutions of

$$\begin{aligned} \theta'' - \Delta \theta + W\theta &= 0 & \text{in } Q \\ \theta &= 0 & \text{on } \Sigma \\ \theta(0) &= \theta^0; \theta'(0) = \theta^1 & \text{in } \Omega. \end{aligned} \quad (2.9)$$

That is, it is enough to prove the existence of a positive constant $C = C(\Gamma_0, T, W) > 0$ such that for every solutions $\theta = \theta(x, t)$ of (2.9) we have

$$\|\theta^0\|_1^2 + |\theta^1|^2 \leq C \left| \frac{\partial \theta}{\partial \nu} \right|_{\Gamma_0}^2. \quad (2.10)$$

We write θ as $\theta = \varphi + \eta$, where φ is the solution of (2.9) with $W \equiv 0$, i.e.

$$\begin{aligned} \varphi'' - \Delta \varphi &= 0 & \text{in } Q \\ \varphi &= 0 & \text{on } \Sigma \\ \varphi(0) &= \theta^0, \varphi'(0) = \theta^1 \end{aligned} \quad (2.11)$$

and $\eta = \eta(x, t)$ satisfies

$$\begin{aligned} \eta'' - \Delta \eta + W\eta &= -W\varphi & \text{in } Q \\ \eta &= 0 & \text{on } \Sigma \\ \eta(0) &= \eta'(0) = 0. \end{aligned} \quad (2.12)$$

Applying (2.4) we deduce

$$\|\theta^0\|_1^2 + |\theta^1|^2 \leq C \left\{ \left| \frac{\partial \theta}{\partial \nu} \right|_{\Sigma_0}^2 + \left| \frac{\partial \eta}{\partial \nu} \right|_{\Sigma_0}^2 \right\}. \quad (2.13)$$

On the other hand, from [11, chapter 1, Theorem 4.1.] we know that

$$\left| \frac{\partial \eta}{\partial \nu} \right|_{\Sigma_0} \leq C \|W\varphi\|_{L^1(0, T; L^2(\Omega))} \leq C \|\varphi\|_{L^2(Q)}.$$

Combining this last inequality and (2.13) it follows that

$$\|\theta^0\|_1^2 + |\theta^1|^2 \leq C \left| \frac{\partial \theta}{\partial \nu} \right|_{\Sigma_0}^2 + \|\varphi\|_{L^2(Q)}^2. \quad (2.14)$$

Therefore, it suffices to prove the following estimate:

$$\|\varphi\|_{L^2(Q)} \leq C \left| \frac{\partial \theta}{\partial \nu} \right|_{\Sigma_0}. \quad (2.15)$$

We argue by contradiction. If (2.15) is not satisfied, there exists a sequence of initial data $\{\theta_k^0, \theta_k^1\}$ such that the corresponding sequences of solutions $\{\varphi_k\}, \{\theta_k\}$ satisfy

$$\|\varphi_k\|_{L^2(Q)} = 1 \quad \forall k \in \mathbb{N} \quad (2.16)$$

$$\left| \frac{\partial \theta_k}{\partial \nu} \right|_{\Sigma_0} \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (2.17)$$

Combining (2.15)-(2.17) we deduce that $\{\theta_k^0, \theta_k^1\}$ is bounded in $H_0^1(\Omega) \times L^2(\Omega)$ and thus

$$\{\varphi_k\}, \{\theta_k\} \text{ are bounded in } L^\infty(0, T; H_0^1(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)). \quad (2.18)$$

Therefore (see [16])

$$\{\varphi_k\}, \{\theta_k\} \text{ are relatively compact in } L^2(Q). \quad (2.19)$$

By extracting some subsequences (which we still denote by $\{\varphi_k\}, \{\theta_k\}$) it follows that

$$\begin{aligned} \varphi_k, \theta_k &\rightharpoonup \varphi, \theta \quad \text{in } H^1(Q) \text{ weak} \\ \varphi_k &\rightarrow \varphi \quad \text{in } L^2(Q) \text{ strongly} \end{aligned} \quad (2.20)$$

The limit functions φ, θ are respectively solutions of (2.12), (2.9) with some initial data $(\theta^0, \theta^1) \in H_0^1(\Omega) \times L^2(\Omega)$. In addition, from (2.16), (2.17) and (2.20) we deduce that

$$\|\varphi\|_{L^2(Q)} = 1 \quad (2.21)$$

$$\frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \Sigma_0. \quad (2.22)$$

Let us prove that (2.21) and (2.22) are in contradiction. Obviously, it suffices to prove that (2.22) implies $\varphi = 0$ since then $\varphi = 0$ which contradicts (2.21). But thanks to hypothesis (2.3), (2.9) and (2.22) imply $\theta = 0$.

Step 2. In order to prove the second statement (b) of the theorem we may apply HUM. Then, it suffices to prove the estimate (2.6) for solutions of (2.9), that is,

$$\|\theta^0\|_{1-\gamma}^2 + \|\theta^1\|_{-\gamma}^2 \leq C_\gamma \|\frac{\partial \theta}{\partial \nu}\|_{-\gamma, \Gamma_0}^2 \quad (2.23)$$

for every $\gamma \in (0, 1]$, $\gamma \neq \frac{1}{2}$.

Applying the method of step 1 we may easily prove

$$\|\theta\|_{1-\gamma}^2 + \|\theta^1\|_{-\gamma}^2 \leq C_\gamma \left\{ \|\frac{\partial \theta}{\partial \nu}\|_{-\gamma, \Gamma_0}^2 + \|\frac{\partial \theta}{\partial \nu}\|_{-\gamma, \Gamma_0}^2 \right\}. \quad (2.24)$$

We note that the constant C_γ on (2.24) does not depend on the potential W . Indeed, $C_\gamma = 2c_\gamma$, where c_γ is the one of (2.6) which corresponds to the potential $W \equiv 0$.

Therefore, we have reduced the problem to the proof of the following estimate:

$$\|\frac{\partial \theta}{\partial \nu}\|_{-\gamma, \Gamma_0} \leq C_\gamma \|\frac{\partial \theta}{\partial \nu}\|_{-\gamma, \Gamma_0} \quad (2.25)$$

As in step 1, we may argue by contradiction proving that, if (2.25) does not hold, then there exists a pair of initial data $(\theta^0, \theta^1) \in H_0^{1-\gamma}(\Omega) \times H^{-\gamma}(\Omega)$ such that the corresponding solutions φ, θ, η of (2.12), (2.9), (2.11) satisfy

$$\|\frac{\partial \theta}{\partial \nu}\|_{-\gamma, \Gamma_0} = 1 \quad (2.26)$$

and

$$\frac{\partial \theta}{\partial \nu} = 0 \text{ on } \Sigma_0. \quad (2.27)$$

On order to reach the contradiction between (2.26) and (2.27) we must apply a unique continuation result to θ . However, (2.3) only applies for solutions $\theta \in H^1(Q)$ and we note that $\theta = \theta(x, t)$ is a priori in the weaker class $(, > 0)$

$$\theta \in L^\infty(0, T; H_0^{1-\gamma}(\Omega)) \cap W^{1, \infty}(0, T; H^{-\gamma}(\Omega)).$$

In order to solve this technical problem we need the following lemma.

LEMMA 2.1. Every weak solution $\theta \in L^2(Q)$ of (2.9) such that

$$\frac{\partial \theta}{\partial \nu} \equiv 0 \text{ on } \Gamma_0$$

belongs to the class $H^1(Q)$.

Proof of Lemma 2.1. From the fact that $\theta \in L^2(Q)$ it follows that its initial data satisfy $\{\theta^0, \theta^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$. Then, $\varphi \in L^2(Q)$ and hence $\eta \in H^1(Q)$. On the other hand, from [11], Chapter 1, Theorem 4.1 we deduce that

$$\frac{\partial \eta}{\partial \nu} \in L^2(\Gamma_0).$$

Since $\partial \varphi / \partial \nu = -\partial \eta / \partial \nu$ on Γ_0 it follows that

$$\frac{\partial \varphi}{\partial \nu} \in L^2(\Gamma_0). \quad (2.28)$$

Combining the estimate (2.4) and (2.28) we deduce that $\{\theta^0, \theta^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ and $\theta \in H^1(Q)$. ■

The proof of (2.25) is now completed. Indeed, from Lemma 2.1, the weak solution θ satisfying (2.27) is in the class $H^1(Q)$ and from (2.3) we deduce that $\theta \equiv 0$. Hence $\eta \equiv 0$, which contradicts (2.26).

Step 3. For the sake of completeness we briefly describe how HUM is applied and the exact controllability results of Theorem 2.1 are obtained from the estimates (2.10), (2.23).

In that which follows we denote by L_γ the canonical isomorphism between $H^{-\gamma}(0, T; L^2(\Gamma_0))$ and $H_0^\gamma(0, T; L^2(\Gamma_0))$ for every $\gamma \in [0, 1]$, $\gamma \neq \frac{1}{2}$. It is defined as follows.

For every $v \in H^{-\gamma}(0, T; L^2(\Gamma_0))$ we define $L_\gamma v = \chi \in H_0^\gamma(0, T; L^2(\Gamma_0))$ by

$$\begin{aligned} -\chi'' + \chi &= v \quad \text{on } E_0 \\ \chi(0) = \chi(T) &= 0 \quad \text{on } \Gamma_0. \end{aligned} \quad (2.29)$$

We have

$$\langle L_1 v, v \rangle = \int_{\Gamma_0} (|\chi'|^2 + |\chi|^2) d\Gamma dt = \|v\|_{L^2(\Gamma_0)}^2.$$

We then set, for every $\gamma \in [0, 1]$, $\gamma \neq \frac{1}{2}$,

$$L_\gamma v = (L_1)^\gamma v \quad \forall v \in H^{-\gamma}(0, T; L^2(\Gamma_0)) \quad (2.30)$$

and it follows that $L_\gamma v \in H_0^\gamma(0, T; L^2(\Gamma_0))$ with

$$\langle L_\gamma v, v \rangle = \|v\|_{L^2(\Gamma_0)}^2. \quad (2.31)$$

We now remark that, thanks to the linearity of system (2.1), it is enough to prove the reachability of the rest point $\{z^0, z^1\} = (0, 0)$ (see Remark 2.4 below). Following [11] we construct the operator

$$A_\gamma : H_0^{1-\gamma}(\Omega) \times H^{-\gamma}(\Omega) \rightarrow H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega) \quad (2.32)$$

given by

$$A_\gamma \{\theta^0, \theta^1\} = \{y'(0), -y(0)\} \quad (2.33)$$

where $y = y(x, t)$ satisfies

$$\begin{aligned} y'' - \Delta y + W y &= 0 \quad \text{in } Q \\ y &= \begin{cases} L_\gamma(\partial \theta / \partial \nu) & \text{on } \Gamma_0 \\ 0 & \text{on } E \cup \Gamma_0 \end{cases} \\ y(T) = y'(T) &= 0. \end{aligned} \quad (2.34)$$

In (2.34), $\theta = \theta(x, t)$ is the solution of (2.9) with initial data $\{\theta^0, \theta^1\}$. The solution $y = y(x, t)$ of (2.34) is unique and it is defined by the

transposition method (see [11, chapter 1, Theorem 4.2] for this particular case and Lions and Magenes [12] for a general description of the method).

In Remark 2.5 we give some details on the regularity of y .

On the other hand, it is easy to check that

$$\langle \Lambda_Y \{\theta^0, \theta^1\}, \{\theta^0, \theta^1\} \rangle = \langle L_Y \left(\frac{\partial \theta}{\partial v} \right), \frac{\partial \theta}{\partial v} \rangle = \left\| \frac{\partial \theta}{\partial v} \right\|_{L_Y, \Gamma_0}^2 \quad (2.35)$$

Combining the estimates (2.10), (2.23) with (2.35) we conclude that

$$\Lambda_Y \text{ defines an isomorphism from } H^{1-\gamma}(\Omega) \times H^{-\gamma}(\Omega) \text{ to } H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega). \quad (2.36)$$

The control $v \in H_0^\gamma(0, T; L^2(\Gamma_0))$ driving the system (2.1) from $(y^0, y^1) \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ to $(0, 0)$ at time T is then given by

$$\begin{aligned} \Lambda_Y \{\theta^0, \theta^1\} &= (y^1, -y^0) \\ v &= L_Y \left(\frac{\partial \theta}{\partial v} \right). \end{aligned} \quad (2.37)$$

Obviously, we have the estimate

$$\|v\|_{L_Y, \Gamma_0}^2 \leq C_Y (\|y^0\|_Y^2 + \|y^1\|_{Y-1}^2). \quad (2.38)$$

The proof of Theorem 2.1 is now completed.

REMARK 2.4. We may easily extend Theorem 2.1 to the system

$$\begin{aligned} y'' - \Delta y + W y &= 0 \quad \text{in } Q \\ y &= \begin{cases} v & \text{on } \Gamma_0 \\ 0 & \text{on } \Sigma \cup \Sigma_0 \end{cases} \\ y(0) &= y^0, \quad y'(0) = y^1 \end{aligned} \quad (2.39)$$

where $h \in L^2(Q)$. Defining z by

$$\begin{aligned} z'' - \Delta z + W z &= h \quad \text{in } Q \\ z &= 0 \quad \text{on } \Sigma \\ z(T) &= z^0, \quad z'(T) = z^1 \quad \text{in } \Omega \end{aligned} \quad (2.40)$$

and considering the new dependent variable $y - z$, one reduces the problem of driving the system (2.39) from (y^0, y^1) to (z^0, z^1) , to the problem of driving the system (2.1) from $(y^0 - z(0), y^1 - z'(0))$ to $(0, 0)$.

From (2.38) we deduce that the control associated to (2.39) for initial and final data $(y^0, y^1), (z^0, z^1)$ satisfies

$$\|v\|_{L_Y, \Gamma_0}^2 \leq C_Y (\|y^0\|_Y^2 + \|y^1\|_{Y-1}^2 + \|z^0\|_Y^2 + \|z^1\|_{Y-1}^2 + \|h\|_{L^2(Q)}^2). \quad (2.41)$$

REMARK 2.5. In order to study the regularity of solutions to (2.39) we may write $y = y_1 + y_2$ where

$$\begin{aligned} y_1'' - \Delta y_1 &= 0 \quad \text{in } Q \\ y_1 &= \begin{cases} v & \text{on } \Gamma_0 \\ 0 & \text{on } \Sigma \cup \Sigma_0 \end{cases} \\ y_1(0) &= y_1^0(0) = 0 \end{aligned} \quad (2.42)$$

and

$$\begin{aligned} y_2'' - \Delta y_2 + W y_2 &= -W y_1 + h \quad \text{in } Q \\ y_2 &= 0 \quad \text{on } \Sigma \\ y_2(0) &= y^0, \quad y_2'(0) = y^1. \end{aligned} \quad (2.43)$$

From [11, chapter 1, Theorem 4.2] we know that when $v \in L^2(\Gamma_0)$ then

$$y_1 \in L^\infty(0, T; L^2(\Omega)) \cap W^{1, \infty}(0, T; H^{-1}(\Omega)).$$

It is then easy to prove that if $v \in H_0^\gamma(0, T; L^2(\Gamma_0))$ then

$$y_1 \in W^{\gamma, \infty}(0, T; L^2(\Omega)) \cap W^{1+\gamma, \infty}(0, T; H^{-1}(\Omega)). \quad (2.44)$$

On the other hand

$$-\Delta y_1 = -y_1'' \in W^{\gamma-1, \infty}(0, T; H^{-1}(\Omega))$$

$$y_1|_{\Sigma} \in H^\gamma(0, T; L^2(\Gamma_0))$$

and then

$$y_1 \in H^{\gamma-1}(0, T; H^{1/2}(\Omega)) \quad (2.45)$$

On the other hand, we deduce by standard arguments that when $(y^0, y^1, h) \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega) \times L^2(Q)$ the solution y_2 of (2.43) satisfies

$$y_2 \in L^\infty(0, T; H_0^\gamma(\Omega)) \cap W^{1, \infty}(0, T; H^{\gamma-1}(\Omega)).$$

This implies

$$y_2 \in H^{\gamma-1}(0, T; H_0^1(\Omega)) \cap H^\gamma(0, T; L^2(\Omega)). \quad (2.46)$$

Combining (2.44)-(2.46) it follows that

$$\begin{aligned} & \|y\|_{H^{\gamma-1}(0, T; H^{1/2}(\Omega))}^2 + \|y\|_{H^\gamma(0, T; L^2(\Omega))}^2 \\ & \leq C_Y (\|y\|_{L^\infty(\Gamma_0)}^2 + \|y^0\|_{L^\infty(\Omega)}^2 + \|y^1\|_{L^\infty(\Omega)}^2 + \|h\|_{L^2(Q)}^2) \end{aligned} \quad (2.47)$$

for every solution of (2.39) and for every $\gamma \in [0, 1]$, $\gamma \neq \frac{1}{2}$.

2.2. Uniform exact controllability

In the semilinear context of sections 3 and 4 we will need some information on the dependence of the control function constructed in Theorem 2.1 with respect to the potential W when the initial and final data are fixed.

Concerning this question, we have the following result.

THEOREM 2.2. Let us assume that the pair (Γ_0, T) satisfies (2.2) and (2.3). Let us fix some initial and final data (y^0, y^1) , $(z^0, z^1) \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ with $\gamma \in [0, 1]$, $\gamma \neq \frac{1}{2}$. Then, the control v given by Theorem 2.1 remains bounded in $H_0^\gamma(0, T; L^2(\Gamma_0))$ when the potential V varies in a bounded set of $L^\infty(Q)$.

Proof. Taking into account the construction of the control v made in step 3 of the proof of Theorem 2.1, we deduce that the proof of Theorem 2.2 reduces to prove the following statement (note that the constant C_Y of (2.24) does not depend on the potential W):

the constant $C_Y = C_Y(W)$ of the estimate (2.25) remains bounded when W varies in a bounded set of $L^\infty(Q)$. (2.48)

We argue by contradiction. If (2.48) is not satisfied there exists a sequence of potentials $\{W_k\}$ and a sequence of initial data $\{\theta_k^0, \theta_k^1\}$ in $H_0^{1-\gamma}(\Omega) \times H^{-\gamma}(\Omega)$ such that

$$\|W_k\|_{L^\infty(Q)} \leq C \quad \forall k \in \mathbb{N} \quad (2.49)$$

$$\left\| \frac{\partial \theta_k}{\partial \nu} \right\|_{-\gamma, \Gamma_0} = 1 \quad \forall k \in \mathbb{N} \quad (2.50)$$

$$\left\| \frac{\partial \theta_k}{\partial \nu} \right\|_{-\gamma, \Gamma_0} \rightarrow 0 \quad \text{as } k \rightarrow +\infty. \quad (2.51)$$

Combining (2.24), (2.50) and (2.51) we deduce that $\{\theta_k^0, \theta_k^1\}$ is bounded

in $H_0^{1-\gamma}(\Omega) \times H^{-\gamma}(\Omega)$. Hence

$$\{\varphi_k\}, \{\theta_k\} \text{ are bounded in } L^\infty(0, T; H_0^{1-\gamma}(\Omega)) \cap W^{1, \infty}(0, T; H^{-\gamma}(\Omega)) \quad (2.52)$$

and

$$\{\eta_k\} \text{ is bounded in } L^\infty(0, T; H_0^1(\Omega)) \cap W^{1, \infty}(0, T; L^2(\Omega)). \quad (2.53)$$

From (2.52), (2.53) we deduce that (the fact that $\gamma < 1$ is essential at this level!)

$$\{\varphi_k\}, \{\theta_k\}, \{\eta_k\} \text{ are relatively compact in } L^2(Q) \quad (2.54)$$

$$\left\{ \frac{\partial \eta_k}{\partial \nu} \Big|_{\Gamma_0} \right\} \text{ is relatively compact in } H^{-\gamma}(0, T; L^2(\Gamma_0)). \quad (2.55)$$

By extracting subsequences (which we shall still denote by W_k, φ_k, \dots) we deduce that

$$W_k \rightarrow W \text{ in } L^\infty(\Omega) \text{ weak} \quad (2.56)$$

$$\{\varphi_k, \theta_k, \eta_k\} \rightarrow \{\varphi, \theta, \eta\} \text{ in } L^2(Q) \text{ strongly} \quad (2.57)$$

$$\frac{\partial \eta_k}{\partial \nu} \Big|_{\Gamma_0} \rightarrow \frac{\partial \eta}{\partial \nu} \Big|_{\Gamma_0} \text{ in } H^{-\gamma}(0, T; L^2(\Gamma_0)) \text{ strongly.} \quad (2.58)$$

Obviously (2.56), (2.57) suffice to pass to the limit in the equations corresponding to $\{W_k\}$. At the limit we find that for some initial data $\{\varphi^0, \theta^1\} \in H^{1-\gamma}(\Omega) \times H^{-\gamma}(\Omega)$ (corresponding to the limit solution θ in (2.57)) and the potential $W \in L^\infty(Q)$ (given by (2.56)) the corresponding solutions φ, η satisfy (2.26), (2.27). The argument of the proof of Theorem 2.1 allows us to prove that (2.26), (2.27) are in contradiction.

This concludes the proof of Theorem 2.2. ■

REMARK 2.6. When $\gamma = 1$ we may only prove that $\{\varphi_k\}$ and $\{\theta_k\}$ are bounded in $L^2(Q)$ and this does not suffice to pass to the limit in equations (2.9), (2.11). In this case we may prove that the control v remains bounded in $H_0^1(0, T; L^2(\Gamma_0))$ when the potential goes over a bounded subset of $L^\infty(Q)$ that is relatively compact in $L^2(Q)$.

3. Exact controllability for the semilinear problem

In this section we present and prove the main result of this paper which concerns the exact controllability of the semilinear problem (1.1)-(1.3).

THEOREM 3.1. Assume that the pair (Γ_0, T) is such that the statements (2.2), (2.3) hold true. Then, for every f such that $f' \in L^\infty(\mathbb{R})$ and $\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$ the system (1.1)-(1.3) is exactly controllable in $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ at time T with controls in $H_0^\gamma(0, T; L^2(\Gamma_0))$. In other words, for every initial and final data $\{y^0, y^1\}, \{z^0, z^1\} \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ there exists a control $v \in H_0^\gamma(0, T; L^2(\Gamma_0))$ such that the solution $y = y(x, t)$ of (1.1)-(1.3) satisfies (1.4).

Proof. Let us fix $\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$ and the initial and final data $\{y^0, y^1\}, \{z^0, z^1\} \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$. Let us assume first that $f \in C^1(\mathbb{R})$. As mentioned in the introduction we construct a nonlinear operator

$$N : L^2(Q) \rightarrow L^2(Q) \quad (3.1)$$

as follows.

We set $g(s) = [f(s) - f(0)]/s$ and we note that $g \in L^\infty(\mathbb{R})$. Then, given $\xi \in L^2(Q)$ we consider the problem

$$\begin{aligned} y'' - \Delta y + g(\xi)y &= -f(0) \quad \text{in } Q \\ y &= \begin{cases} v & \text{on } \Gamma_0 \\ 0 & \text{on } \partial\Omega \setminus \Gamma_0 \end{cases} \\ y(0) &= y^0, \quad y'(0) = y^1 \\ y(T) &= z^0, \quad y'(T) = z^1. \end{aligned} \quad (3.2)$$

Applying Theorem 2.1 and Remark 2.4 (with the potential $W = g(\xi)$ and $h = -f(0)$) we deduce the existence of a control $v = v(\xi) \in H_0^1(0, T; L^2(\Gamma_0))$ (uniquely defined by HUM) such that there exists a unique solution $y = y(\xi)$ to (3.2). We then set

$$N\xi = y(\xi) \quad (3.3)$$

Taking into account that

$$\|g(\xi)\|_{L^\infty(Q)} \leq \|f'\|_{L^\infty(R)} \quad \forall \xi \in L^2(Q)$$

and from the uniform exact controllability result of Theorem 2.2 (we use at this level the fact that $\gamma < 1$) we deduce

$$\|v(\xi)\|_{H^1(\Gamma_0)}^2 \leq C_\gamma (\|y^0\|_\gamma^2 + \|z^0\|_\gamma^2 + \|y^1\|_{\gamma-1}^2 + \|z^1\|_{\gamma-1}^2 + |f(0)|^2) \quad (3.4)$$

$$\forall \xi \in L^2(Q).$$

Estimate (3.4) combined with (2.47) yields

$$\|y(\xi)\|_{H^{\gamma-1}(0, T; H^{1/2}(\Omega))}^2 + \|y(\Omega)\|_{H^\gamma(0, T; L^2(\Omega))}^2 \leq C_\gamma \quad (3.5)$$

$$\forall \xi \in L^2(Q).$$

Since $\gamma > 0$, the embedding (cf. [16])

$$H^{\gamma-1}(0, T; H^{1/2}(\Omega)) \cap H^\gamma(0, T; L^2(\Omega)) \subset L^2(Q)$$

is compact. Therefore, from (3.5) we deduce that

- (i) the nonlinear operator N is compact from $L^2(Q)$ into $L^2(Q)$;
- (ii) the operator N maps $L^2(Q)$ into a bounded set of $L^2(Q)$.

On the other hand, taking into account that g is continuous (since f is C^1) it is easy to prove that N is continuous. Thus, we may apply

Schauder's fixed point theorem and conclude that

$$\text{there exists some } \xi \in L^2(Q) \text{ such that } y(\xi) = \xi \quad (3.6)$$

which means that $y = y(\xi)$ satisfies (1.1)-(1.4).

Therefore, the control $v = v(\xi)$ corresponding to the fixed point ξ of (to the potential $W = g(\xi)$) answers to the question. That is, it drives the system (1.1) from $\{y^0, y^1\}$ to $\{z^0, z^1\}$ at time T . Obviously, by construction, $v \in H_0^1(0, T; L^2(\Gamma_0))$.

In the general case where $f' \in L^\infty(R)$ (but f is not necessarily C^1) we introduce a regularizing sequence $\{f_n\} \subset C^1(R)$ such that

$$f_n \rightarrow f \quad \text{in } L^\infty(R), \quad \text{as } n \rightarrow \infty \quad (3.7)$$

$$\|f'_n\|_{L^\infty(R)} \leq \|f'\|_{L^\infty(R)} \quad \forall n \in \mathbb{N}. \quad (3.8)$$

The system (1.1)-(1.3) is exactly controllable for each nonlinear term f_n . On the other hand, from (3.8) we deduce that the sequence of controls $\{v_n\}$ corresponding to $\{f_n\}$ (and to the fixed initial and final data $\{y^0, y^1\}$, $\{z^0, z^1\}$) is bounded in $H_0^1(0, T; L^2(\Gamma_0))$. Thus, we also deduce a uniform bound of type (3.5) on the sequence of solutions $\{y_n\}$. These uniform estimates and (3.7) allow us to pass to the limit in the equations corresponding to $\{f_n, v_n, y_n\}$. At the limit we find a control v driving the system (1.1)-(1.3) for the nonlinearity f from $\{y^0, y^1\}$ to $\{z^0, z^1\}$ at time T .

The proof of Theorem 3.1 is now completed. \blacksquare

REMARK 3.1. The solution $y = y(x, t)$ of the system (1.1)-(1.3) belongs, by construction, to the class

$$H^{\gamma-1}(0, T; H^{1/2}(\Omega)) \cap H^\gamma(0, T; L^2(\Omega)). \quad (3.9)$$

We may also study the regularity of y as follows. We write $y = y_1 + y_2$ where y_1 satisfies (2.42) and y_2 is solution of

$$\begin{aligned}
y_2'' - \Delta y_2 + f(y_2 + y_1) &= 0 \quad \text{in } Q \\
y_2 &= 0 \quad \text{on } \Sigma \\
y_2(0) &= y^0, \quad y_2'(0) = y^1 \quad \text{in } \Omega.
\end{aligned} \tag{3.10}$$

From Remark 2.4 we know that y_1 belongs to the class (3.9). On the other hand, it is easy to prove by standard techniques that

$$y_2 \in L^\infty(0, T; H_0^\gamma(\Omega)) \cap W^{1, \infty}(0, T; H^{\gamma-1}(\Omega)). \tag{3.11}$$

Finally, as in Remark 2.5, (3.11) combined with the fact that y_2 is in (3.9) allows us to prove that y belongs to the class (3.9).

REMARK 3.2. Theorem 3.1 proves the exact controllability of (1.1)-(1.3) in the class

$$\bigcup_{\gamma > 0} \{H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)\}$$

but not in $L^2(Q) \times H^{-1}(\Omega)$. This is because of the lack of compactness of the operator N when $\gamma = 0$.

When f is asymptotically linear the exact controllability in $L^2(Q) \times H^{-1}(\Omega)$ has been proven [17, 18].

On the other hand, when the initial and final data belong to $H_0^1(\Omega) \times L^2(\Omega)$ we have not been able to prove that the control lies in $H_0^1(0, T; L^2(\Gamma_0))$, since the uniform exact controllability of (2.1) has not been established in this case. The $H_0^1(0, T; L^2(\Gamma_0))$ regularity of the control may be proven when f is asymptotically linear by using the methods of [17] and [18].

4. Existence and characterization of optimal controls

In Theorem 3.1 above the exact controllability of problem (1.1)-(1.3) has been proven. It is well known that, when the system is exactly controllable for every $T > T_0$, there exist infinitely many controls driving the system from any initial state to any final state at any time $T > T_0$. It is then

natural to look for optimal controls (controls minimizing the corresponding boundary norm) and to try to characterize them by means of some optimality conditions (the optimality system).

In order to make more precise the functional setting of these problems, let us assume that (Γ_0, T) verify (2.2) and (2.3) and let us fix some initial and final data

$$\{y^0, y^1\}, \{z^0, z^1\} \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega), \text{ with } \gamma \in (0, 1), \gamma \neq \frac{1}{2}. \tag{4.1}$$

We then introduce the set of admissible controls as follows:

$$\begin{aligned}
U_{ad} &= \{v \in H_0^\gamma(0, T; L^2(\Gamma_0)) \text{ such that the solution } y = y(x, t) \text{ of} \\
&\quad (1.1)-(1.3) \text{ also satisfies (1.4)} \tag{4.2}
\end{aligned}$$

and the cost function

$$J(v) = \frac{1}{2} \|v\|_{\gamma, \Gamma_0}^2. \tag{4.3}$$

From Theorem 3.1 we know that the set U_{ad} contains infinitely many controls. Now, we consider the following problem:

$$\text{to find } v \in U_{ad} \text{ such that } J(v) = \min_{u \in U_{ad}} J(u). \tag{P}$$

It is easy to see (since $\gamma > 0$) that at least a solution to problem (P) exists. In fact, any weak $H_0^\gamma(0, T; L^2(\Gamma_0))$ limit of a minimizing sequence leads to an optimal control.

REMARK 4.1. The nonlinear term of (1.1) can be easily handled to prove this statement. The fact that $\gamma > 0$ provides the compactness in $L^2(Q)$ which allows us to pass to the limit in the nonlinear term.

In the linear case where $f(s) = as$ for some $a \in \mathbb{R}$, the set U_{ad} is convex and then the optimal control is unique. This optimal control is in fact the one that HUM provides.

In the general semilinear framework the set U_{ad} is no longer convex and we do not know whether the optimal control is unique or not. In order to obtain additional information on the optimal controls it is interesting to write down a set of equations (an optimality system) that they satisfy.

In section 4.1 we shall use a penalization method inspired by [10] and [11], which will allow us to prove that some of the optimal controls satisfy an optimality system. This system consists of two coupled semilinear wave equations and it is a natural extension of the optimality system of the linear framework that motivated HUM.

This type of system seems to be new and therefore, in section 4.2, we shall give some remarks on the existence and uniqueness of solutions.

4.1. Obtaining the optimality system

For every $\epsilon > 0$ we introduce the penalized cost function

$$J_\epsilon(y, v) = \frac{1}{2} \|v\|_{L^2(\Gamma_0)}^2 + \frac{1}{2\epsilon} \|y'' - \Delta y + f(y)\|_{L^2(Q)}^2. \quad (4.4)$$

and the set of admissible states (y, v) :

$$U_{ad}^P = \{(y, v) \text{ such that } v \in H_0^1(0, T; L^2(\Gamma_0)) \text{ and } y = y(x, t) \text{ satisfies conditions (4.6) below}\} \quad (4.5)$$

$$\begin{aligned} y'' - \Delta y &\in L^2(Q) \\ y &= \begin{cases} v & \text{on } \Gamma_0 \\ 0 & \text{on } \bar{\Omega} \setminus \Gamma_0 \end{cases} \end{aligned} \quad (4.6)$$

$$y(0) = y^0, \quad y'(0) = y^1; \quad y(T) = z^0, \quad y'(T) = z^1.$$

REMARK 4.2. Every function y that verifies (4.6) satisfies $y \in L^2(Q)$ and then $f(y) \in L^2(Q)$. Hence, the function $J_\epsilon(\cdot, \cdot)$ is well defined on U_{ad}^P .

We now consider the penalized problem:

$$\text{to find } (y_\epsilon, v_\epsilon) \in U_{ad}^P \text{ such that } J_\epsilon(y_\epsilon, v_\epsilon) = \min_{(y, u) \in U_{ad}^P} J_\epsilon(y, u). \quad (P_\epsilon)$$

The following lemma is easy to prove taking into account that $\gamma > 0$ provides the compactness in $L^2(Q)$ that allows one to handle the nonlinear term $f(y)$ of J_ϵ .

LEMMA 4.1. For every $\epsilon > 0$ problem (P_ϵ) has at least a solution $(y_\epsilon, v_\epsilon) \in U_{ad}^P$.

The main result of this section is as follows.

THEOREM 4.1. Assume that the pair (Γ_0, T) is such that (2.2) and (2.3) are satisfied and $f \in C^1(\mathbb{R})$. Let $\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$ and $(y^0, y^1), (z^0, z^1) \in H_0^1(\Omega) \times H^{\gamma-1}(\Omega)$. Let $(y_\epsilon, v_\epsilon)_{\epsilon > 0}$ be a sequence of optimal controls associated to the family of penalized problems $(P_\epsilon)_{\epsilon > 0}$.

Then, for some subsequence (which we still denote by $(y_\epsilon, v_\epsilon)_{\epsilon > 0}$) we have

$$(y_\epsilon, v_\epsilon) \rightarrow (y, v) \text{ in } L^2(Q) \times L^2(\Gamma_0) \text{ strongly as } \epsilon \rightarrow 0 \quad (4.7)$$

where $v \in U_{ad}$ is solution to (P) and y satisfies (1.1)-(1.4).

Moreover, we have

$$v = L_\gamma \left(\frac{\partial \theta}{\partial v} \right) \text{ on } \Gamma_0 \quad (4.8)$$

for some $\theta = \theta(x, t)$ such that (y, θ) satisfy:

$$\begin{aligned} y'' - \Delta y + f(y) &= 0 & \text{in } Q \\ \theta'' - \Delta \theta + f'(y)\theta &= 0 & \text{in } Q \\ y &= \begin{cases} L_\gamma (\partial \theta / \partial v) & \text{on } \Gamma_0 \\ 0 & \text{on } \bar{\Omega} \setminus \Gamma_0 \end{cases} & \theta = 0 & \text{on } \Sigma \end{aligned} \quad (4.9)$$

$$y(0) = y^0; \quad y'(0) = y^1; \quad y(T) = z^0, \quad y'(T) = z^1$$

$$\theta \in L^\infty(0, T; H_0^{1-\gamma}(\Omega)) \cap W^{1, \infty}(0, T; H^{-1}(\Omega)).$$

Proof. We proceed in two steps.

Step 1. Optimality conditions for the penalized problem (P_ϵ) .

Let $\{y_\epsilon, v_\epsilon\}$ be a solution of (P_ϵ) . Then for any pair $\{z, u\}$ such that

$$\begin{aligned} z'' - \Delta z &\in L^2(Q) \\ z(0) = z'(0) = z(T) = z'(T) &= 0 \\ z &= \begin{cases} u \in H_0^1(0, T; L^2(\Gamma_0)) & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma \setminus \Sigma_0 \end{cases} \end{aligned} \quad (4.10)$$

we have

$$\{y_\epsilon + \lambda z, v_\epsilon + \lambda u\} \in U_{ad}^P, \forall \lambda \in \mathbb{R} \quad (4.11)$$

and

$$J_\epsilon\{y_\epsilon, v_\epsilon\} \leq J_\epsilon\{y_\epsilon + \lambda z, v_\epsilon + \lambda u\} \quad \forall \lambda \in \mathbb{R}. \quad (4.12)$$

From (4.12) we easily get (in that which follows, for simplicity, we denote by $\{y, v\}$ the pair $\{y_\epsilon, v_\epsilon\}$):

$$\begin{aligned} & \frac{\lambda^2}{2} \|u\|_{L^2(\Gamma_0)}^2 + \lambda \langle v, u \rangle + \frac{1}{2\epsilon} (\|f(y + \lambda z)\|_{L^2(Q)}^2 - \|f(y)\|_{L^2(Q)}^2) \\ & + \frac{\lambda}{\epsilon} (y'' - \Delta y, z'' - \Delta z + \frac{f(y + \lambda z) - f(y)}{\lambda}) \\ & + \frac{\lambda^2}{2\epsilon} \|z'' - \Delta z\|_{L^2(Q)}^2 + \frac{\lambda}{\epsilon} (z'' - \Delta z, f(y + \lambda z)) \geq 0. \end{aligned} \quad (4.13)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $H_0^1(0, T; L^2(\Gamma_0))$ and (\cdot, \cdot) the scalar product in $L^2(Q)$.

We divide (4.13) by $\lambda > 0$ and pass to limit as $\lambda \downarrow 0$. It follows

$$\langle v, u \rangle + \frac{1}{\epsilon} (y'' - \Delta y + f(y), z'' - \Delta z + f'(y)z) \geq 0. \quad (4.14)$$

This argument, applied with $\lambda < 0$ and passing to the limit as $\lambda \uparrow 0$, provides the inverse inequality of (4.14) (≤ 0 instead of ≥ 0). Hence, we have

$$\begin{aligned} \langle v_\epsilon, u_\epsilon \rangle + \frac{1}{\epsilon} (y_\epsilon'' - \Delta y_\epsilon + f(y_\epsilon), z'' - \Delta z + f'(y_\epsilon)z) &= 0 \\ \forall \{z, u\} \text{ satisfying (4.10)} & \end{aligned} \quad (4.15)$$

or equivalently, that the function

$$\theta_\epsilon = \frac{1}{\epsilon} (y_\epsilon'' - \Delta y_\epsilon + f(y_\epsilon)) \quad (4.16)$$

satisfies

$$\begin{aligned} \theta_\epsilon'' - \Delta \theta_\epsilon + f'(y_\epsilon)\theta_\epsilon &= 0 \text{ in } Q \\ \theta_\epsilon &= 0 \text{ on } \Sigma \\ \partial \theta_\epsilon / \partial \nu &= L_Y^{-1}(v_\epsilon) \in H^{-1}(0, T; L^2(\Gamma_0)) \text{ on } \Sigma_0. \end{aligned} \quad (4.17)$$

The estimate (2.23), applied to the potential $W_\epsilon = f'(y_\epsilon)$ yields

$$\|\partial \theta_\epsilon(0)\|_{L^{-1}\Gamma}^2 + \|\theta_\epsilon'(0)\|_{L^{-1}\Gamma}^2 \leq C_{Y, \epsilon} \|v_\epsilon\|_{L^2(\Gamma_0)}^2. \quad (4.18)$$

Step 2. Passing to the limit.

Let us consider a sequence of solutions $\{y_\epsilon, v_\epsilon\}_{\epsilon > 0}$ to the family of problem $(P_\epsilon)_{\epsilon > 0}$.

First of all we remark that

$$\min_{(y, u) \in U_{ad}^P} J_\epsilon(y, u) \leq \min_{u \in U_{ad}} J(u) \quad \forall \epsilon > 0 \quad (4.19)$$

since for every $u \in U_{ad}$, the pair $\{y(u), u\} \in U_{ad}^P$ is admissible for problem (P_ϵ) .

From (4.19) we deduce that

$$\|v_\epsilon\|_{L^\infty(\Gamma_0)} \leq C \quad \forall \epsilon > 0 \quad (4.20)$$

$$\sqrt{\epsilon} \|b_\epsilon\|_{L^2(Q)} \leq C \quad \forall \epsilon > 0. \quad (4.21)$$

We may easily improve the estimate (4.21). Indeed, taking into account that $\gamma > 1$ and

$$\|f'(y_\epsilon)\|_{L^\infty(Q)} \leq \|f'\|_{L^\infty(R)} \quad \forall \epsilon > 0$$

we deduce, from the "uniform controllability property" of Theorem 2.2, that the constant $C_{\gamma, \epsilon}$ of (4.18) is uniformly bounded. Thus, (4.18) and (4.20) imply

$$\|b_\epsilon(0)\|_{L^{1-\gamma}}^2 + \|a'_\epsilon(0)\|_{L^\gamma}^2 \leq C_\gamma \quad \forall \epsilon > 0 \quad (4.22)$$

or equivalently

$$\{b_\epsilon\}_{\epsilon > 0} \text{ is bounded in } L^\infty(0, T; H_0^{1-\gamma}(\Omega)) \cap W^{1, \infty}(0, T; H^{-\gamma}(\Omega)). \quad (4.23)$$

On the other hand, from Remark 3.1 and (4.20) we deduce that

$$\{y_\epsilon\}_{\epsilon > 0} \text{ is bounded in } H^{\gamma-1}(0, T; H^{1/2}(\Omega)) \cap \tilde{H}^1(0, T; L^2(\Omega)). \quad (4.24)$$

From estimates (4.20), (4.23) and (4.24) and the fact that $\gamma \in (0, 1)$, by extracting some subsequences (still denoted by $v_\epsilon, y_\epsilon, a_\epsilon$) we have

$$v_\epsilon \rightharpoonup v \text{ in } H_0^\gamma(0, T; L^2(\Gamma_0)) \text{ weak, } v \in U_{ad} \quad (4.25)$$

$$\{y_\epsilon, a_\epsilon\} \rightarrow \{y, a\} \text{ in } L^2(Q) \text{ strongly} \quad (4.26)$$

Assertions (4.25) and (4.26) allow us to pass to the limit in (4.6) and (4.7). On the other hand, (4.23) implies

$$y'' - \Delta y + f(y) = 0 \text{ in } Q. \quad (4.27)$$

Therefore, (y, a) satisfy (4.9) with

$$v = L_\gamma \left(\frac{\partial a}{\partial \nu} \right) \text{ on } \Gamma_0.$$

On the other hand

$$J(v) \leq \liminf_{\epsilon \rightarrow 0} J_\epsilon(y_\epsilon, v_\epsilon)$$

which combined with (4.19) yields

$$J(v) = \lim_{\epsilon \rightarrow 0} J_\epsilon(y_\epsilon, v_\epsilon) = \min_{u \in U_{ad}} J(u).$$

Then

$$\|v_\epsilon\|_{L^\infty(\Gamma_0)} \rightarrow \|v\|_{L^\infty(\Gamma_0)} \text{ as } \epsilon \rightarrow 0$$

and from (4.25) we conclude that

$$v_\epsilon \rightarrow v \text{ in } H_0^\gamma(0, T; L^2(\Gamma_0)) \text{ strongly.}$$

The proof of Theorem 4.1 is now completed. ■

4.2. Some remarks on the optimality system

4.2.1 First of all we note that the exact controllability of (1.1)-(1.3) in $H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$ ($\gamma \in (0, 1)$, $\gamma \neq \frac{1}{2}$) with controls in $H_0^\gamma(0, T; L^2(\Gamma_0))$ at time T is equivalent to the following statement:

For every $\{y^0, y^1\}, \{z^0, z^1\} \in H_0^1(\Omega) \times H^{\gamma-1}(\Omega)$, the system (4.9) has at least a solution (y, θ) . (4.28)

Indeed, we have proven in Theorem 4.1 that the exact controllability implies the existence of a solution to (4.9). Conversely, it is obvious that if (4.9) has a solution for every initial and final data then the system is exactly controllable.

4.2.2 In the linear case where $f(s) = \alpha s$ for some $\alpha \in \mathbb{R}$, system (4.9) corresponds to the scheme motivating HUM. In this case it is easy to prove that the solution of (4.9) is unique. Indeed, let us assume that $\{y_i, \theta_i\}_{i=1,2}$ are solutions of (4.9). Then $\bar{y} = y_1 - y_2$, $\bar{\theta} = \theta_1 - \theta_2$ satisfy the same equations with initial and final data $(0,0)$. Multiplying by $\bar{\theta}$ in the equation satisfied by \bar{y} and integrating by parts we get

$$\frac{\partial \bar{\theta}}{\partial \nu} = 0 \text{ on } \Sigma_0$$

which imply, by (2.23), $\bar{\theta} \equiv 0$ and hence $\bar{y} \equiv 0$.

We do not know if the solution of (4.9) is unique in the general semilinear case $f' \in L^\infty(\mathbb{R})$.

4.2.3 Let us consider again the linear case $f(s) = \alpha s$, $\alpha \in \mathbb{R}$. As we have seen in the proof of Theorem 2.1, the first step on the application of HUM is the construction of the operator Λ_γ . In the semilinear framework, this construction yields to the study of the existence and uniqueness of solutions to the following system:

$$\begin{aligned} y'' - \Delta y + f(y) &= 0 & \text{in } Q \\ \theta'' - \Delta \theta + f'(y)\theta &= 0 & \text{in } Q \\ \theta &= 0 & \text{on } \Sigma; \quad y = \begin{cases} L_\gamma(\partial\theta/\partial\nu) & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma \setminus \Sigma_0 \end{cases} & (4.29) \\ y(T) = z^0; \quad y'(T) = z^1 \\ \theta(0) = \theta^0; \quad \theta'(0) = \theta^1 \end{aligned}$$

for given initial and final data $(\theta^0, \theta^1) \in H_0^{1-\gamma}(\Omega) \times H^{-1}(\Omega)$, $(z^0, z^1) \in H_0^\gamma(\Omega) \times H^{\gamma-1}(\Omega)$.

When $\gamma \in (0,1)$, $\gamma \neq \frac{1}{2}$, it is easy to see that there exists at least a solution to (4.29) for any $T > 0$. We proceed as follows. Given $\xi \in L^2(Q)$ we solve

$$\theta'' - \Delta \theta + f'(\xi)\theta = 0 \quad \text{in } Q$$

$$\theta = 0 \quad \text{on } \Sigma$$

$$\theta(0) = \theta^0, \quad \theta'(0) = \theta^1$$

and then

$$y'' - \Delta y + f(y) = 0 \quad \text{in } Q$$

$$y = \begin{cases} L_\gamma(\partial\theta/\partial\nu) & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma \setminus \Sigma_0 \end{cases}$$

$$y(t) = z^0, \quad y'(T) = z^1.$$

The nonlinear operator $K : L^2(Q) \rightarrow L^2(Q)$ such that $K(\xi) = y$ is well defined and compact and it maps $L^2(Q)$ into a bounded set of $L^2(Q)$. Therefore we may apply Schauder's fixed point theorem and the existence of a solution to problem (4.29) holds. This argument applies also when $\gamma = 1$ (note that we have not used here any estimate of type (2.23)).

We do not know whether solutions to (4.29) are unique or not under the sole assumption $f \in C^1(\mathbb{R}), f' \in L^\infty(\mathbb{R})$. Of course, uniqueness may be proven under some additional (but not natural) restrictions on f . This question may be interesting even independently of the exact controllability theory.

5. Concluding remarks

5.1. The techniques of this paper are general and apply, in particular, for equations

$$y'' + Ay + f(x, t, y) = 0 \quad (5.1)$$

where A is a second order elliptic operator with smooth coefficients only depending on the space variable x and $f : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally Lipschitz continuous function such that $\partial f/\partial y \in L^\infty(\Omega \times \mathbb{R} \times \mathbb{R})$.

5.2. The same techniques allow us to treat Neumann and Dirichlet/Neumann boundary conditions but for brevity we will not consider these cases here. We refer to [18] for the case where f is asymptotically linear.

5.3. It would be very interesting to have some general unique continuation results ensuring that (2.3) holds for any $\Gamma_0 \subset \Gamma$ when T is large enough.

The results by Ruiz [14] show that when $\Gamma_0 = \Gamma$, then (2.3) holds even under the weaker regularity assumptions $W \in L^{n+1}(\Omega)$ and $\theta \in L^2(\Omega)$. However, the results from [14] do not apply to the case where Γ_0 is a subset of Γ .

The question of the unique continuation of weak solutions seems to be crucial when one tries to extend the results of this paper for models of plates, for instance,

$$y'' + \Delta^2 y + f(y) = 0.$$

In this case, we must establish a priori estimates for solutions of

$$e'' + \Delta^2 \theta + W\theta = 0, \quad \text{with } W \in L^\infty(\Omega) \quad (5.2)$$

which seems to need a unique continuation result for weak solutions $\theta \in H^2(\Omega)$. To our knowledge, such a result has not been proven in the literature.

5.4. The method of this paper does not apply to problems of type (1.1)-(1.3) where f is superlinear, for instance,

$$f(y) = -|y|^{p-1}y, \quad p > 1. \quad (5.3)$$

In [16] we have proven the local controllability under some natural growth assumptions.

The stabilisation results of Komornik and Zuazua [6,7] (only valid for the + sign in (5.3)) allow us to prove that every initial state may be driven to every final state, but a priori in a time that tends to infinity when the norm of the initial and final data goes to infinity.

The exact controllability problem remains open in the superlinear case.

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