

## Spectral boundary controllability of networks of strings \*

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**Abstract.** In this note we give a necessary and sufficient condition for the spectral controllability from one simple node of a general network of strings that undergoes transversal vibrations in a sufficiently large time. This condition asserts that no eigenfunction vanishes identically on the string that contains the controlled node. The proof combines the Beurling-Malliavin's Theorem and an asymptotic formula for the eigenvalues of the network. The optimal control time may be characterized as twice the sum of the lengths of all the strings of the network.

### *Contrôlabilité spectrale de réseaux de cordes vibrantes*

**Résumé.** On considère un réseau général de cordes vibrantes et on étudie le problème du contrôle spectral moyennant des contrôles agissant sur une extrémité libre du réseau. Moyennant une généralisation des Théorèmes de Beurling-Malliavin et à l'aide d'une formule asymptotique des valeurs propres du réseau, on donne une condition nécessaire et suffisante pour la contrôlabilité approchée et spectrale au temps  $T_0 = 2 \sum_{i=1}^M \ell_i$ , où les  $\ell_i$  sont les longueurs des cordes du réseau. Cette condition exige qu'aucune fonction propre ne s'annule identiquement le long de la corde où le contrôle agit.

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### Version française abrégée

On considère un réseau général plat de cordes vibrantes. On suppose que les vibrations sont transversales et que le déplacement de chaque corde est gouverné par l'équation des ondes 1-d. On suppose, pour simplifier la présentation, que tous les coefficients sont identiquement égaux à un. Cependant, les longueurs des cordes sont arbitraires. On suppose aussi qu'un contrôle agit sur une des extrémités du réseau et on étudie le problème de la contrôlabilité spectrale, i. e., celui de savoir si toutes les solutions partant d'une donnée initiale, combinaisons linéaires finies de fonctions propres, peut être ramenée à zéro à l'aide d'un contrôle  $L^2(0, T)$ , en un temps  $T$  fini uniforme. Évidemment, il s'agit d'une propriété de contrôlabilité plus forte que celle de la contrôlabilité approchée où l'on exige seulement que l'espace de données contrôlables soit dense dans l'espace d'énergie naturel.

Ce problème a été étudié et entièrement résolu dans le cas où le réseau est étoilé (voir, e.g., [1]-[3], [5] et [7]). Le cas d'un arbre a été traité dans [6].

Dans cette note nous généralisons ces résultats au cas d'un réseau général.

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Contrairement à [5]-[7] on n'utilise pas des arguments basés sur la formule de d'Alembert et les propriétés de propagation des ondes mais, plutôt, une conséquence du Théorème de Beurling-Malliavin due à Haraux et Jaffard [8] et une estimation asymptotique de la densité des valeurs propres à hautes fréquences basée sur le principe de max-min, inspirée des développements de Courant-Weyl.

On montre que le temps minimal de contrôle est égale à

$$T_0 = 2 \sum_{i=1}^M \ell_i,$$

où  $M$  est le nombre total de cordes dans le réseau et  $\ell_i$  sont ses longueurs. Aussi, lorsque  $T < T_0$  on montre que l'on a jamais de contrôlabilité spectrale.

Ensuite, on donne une condition nécessaire et suffisante pour que le réseau aie le propriété de contrôlabilité spectrale quelque soit  $T > T_0$ . Cette condition impose qu'il n'y aie aucune fonction propre qui s'annule sur la corde où le contrôle agit. Il faut noter que, sous cette condition, le spectre est nécessairement simple ce qui constitue, de manière évidente, une condition nécessaire pour la contrôlabilité spectrale.

Cette condition générale coïncide avec celle que l'on a déjà introduit dans le cas des étoiles et des arbres. Dans le premier cas la condition dit tout simplement que les longueurs des cordes non contrôlées doivent être mutuellement irrationnelles. Dans le cas des arbres la condition exige que les spectres des sous-arbres de l'arbre qui se rejoignent dans un nœud interne quelconque aient une intersection vide.

Dans le cas général considère ici, une interprétation similaire de cette condition dépendrait de la forme particulier du réseau.

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## 1. Introduction

Let  $G$  be a planar, finite, simple graph with  $N$  vertices  $\mathbf{v}_1, \dots, \mathbf{v}_N$  and  $M$  edges  $\mathbf{e}_1, \dots, \mathbf{e}_M$ . Let  $\mathcal{V}_S$  be the set of simple vertices (those that belong to a single edge) and  $\mathcal{V}_M$  the set of the multiple ones and denote, for a vertex  $\mathbf{v}$ ,  $I_{\mathbf{v}} = \{i : \mathbf{v} \in \mathbf{e}_i\}$ , i.e., the set of edges that are incident to  $\mathbf{v}$ .

Further, denote by  $\ell_i$  the length of the edge  $\mathbf{e}_i$  and assume that every edge has been parametrized by its arc length by means of the functions  $x_i : [0, \ell_i] \rightarrow \mathbf{e}_i$  and denote, for  $i = 1, \dots, M$ ,  $j = 1, \dots, N$ ,

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } x_i(0) = \mathbf{v}_j, \\ -1 & \text{if } x_i(\ell_i) = \mathbf{v}_j. \end{cases}$$

Consider a network of elastic homogeneous strings whose rest configuration coincides with the graph  $G$ . The nodes and the strings of the network correspond to the vertices and edges of  $G$ , respectively. We use for them the same notations  $\mathbf{v}_i$  and  $\mathbf{e}_i$ . The strings undergo transversal vibrations, which are modelled according to the 1-d wave equation. At the multiple nodes, we assume that the network remains continuous (i.e. the displacements of all the strings sharing a common multiple node coincide at that node) and that the sum of forces is equal to zero. At some of the simple vertices an external action is applied that determines their motion, the remaining simple vertices being fixed.

Let the function  $u_i : [0, \ell_i] \times \mathbb{R} \rightarrow \mathbb{R}$  describe the deformation of the string  $\mathbf{e}_i$ . Denote  $u_i(\mathbf{v}, t) = u_i(x_i^{-1}(\mathbf{v}), t)$  and  $\frac{\partial u_i}{\partial n}(\mathbf{v}, t) = \varepsilon_{ij} u_{i,x}(x_i^{-1}(\mathbf{v}), t)$  (the normal exterior derivative of  $u_i$  at the node  $\mathbf{v}$ ). We shall also use the notation  $\bar{u}$  for a vector function with components  $u_1, \dots, u_M$ .

Let  $\mathcal{C}$  be a non-empty subset of  $\overline{\mathcal{V}_S}$ . Then the motion of the network is described by the system

$$u_{i,tt} - u_{i,xx} = 0 \quad \text{in } [0, \ell_i] \times \mathbb{R}, \text{ for } i = 1, \dots, M, \quad (1)$$

$$u_i(\mathbf{v}, t) = h_i(t) \quad \text{for } t \in \mathbb{R} \text{ and } \mathbf{v} \in \mathcal{C}, i \in I_{\mathbf{v}}, \quad (2)$$

$$u_i(\mathbf{v}, t) = 0 \quad \text{for } t \in \mathbb{R} \text{ and } \mathbf{v} \in \mathcal{V}_S \setminus \mathcal{C}, i \in I_{\mathbf{v}}, \quad (3)$$

$$u_i(\mathbf{v}, t) = u_j(\mathbf{v}, t) \quad \text{for } t \in \mathbb{R} \text{ and } \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, i, j \in I_{\mathbf{v}}, \quad (4)$$

$$\sum_{i \in I_{\mathbf{v}}} \frac{\partial u_i}{\partial n}(\mathbf{v}, t) = 0 \quad \text{for } t \in \mathbb{R} \text{ and } \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \quad (5)$$

$$u_i(x, 0) = u_i^0(x), \quad u_{i,t}(x, 0) = u_i^1(x) \quad \text{for } x \in [0, \ell_i], \text{ for } i = 1, \dots, M, \quad (6)$$

for  $m = 1, \dots, M$ , where  $u_i^0(x)$  and  $u_i^1(x)$  are the initial deformation and velocity of the  $i$ -th string, respectively.

Define the Hilbert spaces

$$V = \left\{ \bar{u} \in \prod_{i=1}^M H^1(0, \ell_i) : u_i(\mathbf{v}) = u_j(\mathbf{v}) \text{ for } \mathbf{v} \in \mathcal{V}_{\mathcal{M}} \text{ and } u_i(\mathbf{v}) = 0 \text{ for } \mathbf{v} \in \mathcal{V}_S \right\},$$

$$H = \prod_{i=1}^M L^2(0, \ell_i),$$

endowed with the natural product structures.

When all the functions  $h_i$  are equal to zero, problem (1)-(6) (the so-called adjoint system) is well-posed in the space  $V \times H$ .

If all the functions  $h_i \in L^2[0, T]$  for some  $T > 0$  and  $(\bar{u}^0, \bar{u}^1) \in H \times V'$ , then (1)-(6) has a unique weak solution, defined by transposition, that verifies

$$\bar{u} \in C([0, T] : H) \cap C^1([0, T] : V'),$$

where  $V'$  is the dual space of  $V$  (see [9] for details).

Our goal is to find conditions under which, for given initial data  $(\bar{u}^0, \bar{u}^1) \in H \times V'$  and  $T > 0$ , it is possible to choose control functions  $h_i \in L^2[0, T]$  such that the solution of (1)-(6) satisfies

$$u_i(x, T) = u_{i,t}(x, T) = 0, \quad i = 1, \dots, M. \quad (7)$$

When that is possible, we say that the initial data  $(\bar{u}^0, \bar{u}^1)$  are *controllable in time  $T$* . If the whole space  $H \times V'$  is controllable in time  $T$ , then (1)-(6) is said to be *exactly controllable in time  $T$* .

It is known that, if the graph  $G$  is a tree then the problem is exactly controllable (see, e.g. [9], chapter II) if and only if the set  $\mathcal{C}$  of controlled nodes contains all the simple nodes except, at most, one.

When the graph contains circuits, exact controllability is never reached, i.e., there exist initial data, which cannot be driven to rest, regardless how many vertices the set  $\mathcal{C}$  contains. The same happens for a tree with more than one uncontrolled nodes. In those cases, however, the controllability may still hold for a dense subspace of  $H \times V'$ , though that subspace depends, in general, on the structure of the graph and on the lengths of its edges. A detailed study of this fact has been carried out in [1]-[3], [5] and [7] for star-shaped networks. In [6], we developed a method that allows to characterize the space of controllable initial data for tree-shaped networks.

In both cases, the results depend on deep approximation properties of rational numbers applied to the lengths of the strings and involve difficult problems in number theory.

In the case of general networks under consideration, the problem of identifying the space of controllable data seems to be out of reach by now. We focus on the simpler problem of identifying the configurations in which the space of controllable data turns out to contain  $Z \times Z$ , where  $Z$  is the space of all finite linear combinations of the eigenfunctions of the elliptic operator  $\mathbf{A}$  defined by (1)-(5). In this case, the system is said to be *spectrally controllable*. This property is equivalent to the fact that all the initial data  $(\bar{u}^0, \bar{u}^1)$ , where  $\bar{u}^0$  and  $\bar{u}^1$  are eigenfunctions, are controllable. Of course, the space  $Z \times Z$  being dense in  $H \times V'$ , the property of spectral controllability is stronger than the classical property of *approximate controllability*, which is defined by the fact that the set of controllable initial data is dense in that space. In this Note, we prove that in fact these two properties coincide, i.e., they occur simultaneously.

In what follows, in order to make the problem more precise, we concentrate our attention on the extreme situation when the set  $\mathcal{C}$  of controlled nodes consists of one single point, i.e.,  $\mathcal{C} = \{\mathbf{v}^*\}$ . The node  $\mathbf{v}^*$  being simple, the set  $I_{\mathbf{v}^*}$  contains just one index, which we denote by  $i^*$ .

Let  $\{\lambda_n\}_{n \in \mathbb{Z}_+}$  be the increasing sequence of eigenvalues of  $\mathbf{A}$  and  $\{\bar{\varphi}_n\}_{n \in \mathbb{Z}_+}$  the corresponding orthonormal sequence of eigenfunctions, where  $\bar{\varphi}_n = (\varphi_{n1}, \dots, \varphi_{nM})$ . We may assume that the eigenvalues are all simple, since this is a necessary condition for the approximate (and consequently, spectral) controllability to hold. Indeed, if  $\lambda$  is a multiple eigenvalue there exists an eigenfunction  $\bar{\psi}$  such that  $\bar{\psi}_x(\mathbf{v}^*) = 0$ . Then the solution  $\bar{\psi}(x, t) = \cos \sqrt{\lambda} t \bar{\psi}(x)$  of (1)-(5) with  $h_{i^*} = 0$  satisfies  $\bar{\psi}(\mathbf{v}^*, t) = \bar{\psi}_x(\mathbf{v}^*, t) = 0$  for all  $T$ . This implies the lack of approximate controllability in this case.

If  $\bar{u}^0 = \sum_{n \in \mathbb{Z}_+} u_n^0 \bar{\varphi}_n$  and  $\bar{u}^1 = \sum_{n \in \mathbb{Z}_+} u_n^1 \bar{\varphi}_n$ , the solution of (1)-(6) may be expressed as

$$\bar{u}(x, t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \gamma_n e^{i\mu_n t} \bar{\varphi}_n(x), \quad (8)$$

where

$$\mu_n = \operatorname{sgn}(n) \sqrt{\lambda_n} \text{ and } \gamma_n = \frac{1}{2} \left( u_{|n|}^0 + \frac{u_{|n|}^1}{i\mu_n} \right). \quad (9)$$

A simple application of the HUM method of J.-L. Lions (see [10]) allows to ensure that the space  $Z \times Z$  is controllable in time  $T$  if and only if, for any  $n \in \mathbb{Z} \setminus \{0\}$  there exists a constant  $C_n > 0$  such that

$$\int_0^T |u_{i^*,x}(\mathbf{v}^*, t)|^2 dt \geq C_n (|u_n^0|^2 + |u_n^1|^2) \quad (10)$$

for every solution of (1)-(6) with  $h_{i^*} \equiv 0$  and  $(\bar{u}^0, \bar{u}^1) \in V \times H$ .

## 2. The main result

We shall say that the network satisfies the *spectral unique continuation property from  $\mathcal{C}$*  (a set of exterior nodes) if no (non-zero) eigenfunction vanishes identically on the strings containing the nodes from  $\mathcal{C}$ . This property is equivalent to the fact that no non-identically equal to zero eigenfunction has normal derivatives vanishing at  $\mathcal{C}$ .

As it was indicated above, the spectral unique continuation property is a necessary condition for a network to be spectrally controllable in some time  $T > 0$ . The converse assertion is also true as the following theorem shows.

Denote  $T_0 = 2 \sum_{i=1}^M \ell_i$ .

**THEOREM** – *Let the set  $\mathcal{C}$  contains a single point. Then*

a) If the network satisfies the spectral unique continuation property from  $\mathcal{C}$  then it is spectrally controllable from  $\mathcal{C}$  for all time  $T > T_0$ .

b) If  $T < T_0$  the network is not spectrally controllable. Moreover, no element of  $Z \times Z$  is controllable in time  $T$ .

This theorem provides a complete characterization of spectrally controllable networks. Indeed, it takes place

**COROLLARY** – If  $T > T_0$  the properties

- spectral unique continuation,
- spectral controllability,
- approximate controllability,

are equivalent.

*Remark 1.* – Assertion a) of the Theorem remains true if the set  $\mathcal{C}$  of controlled nodes contains more than one point, but the control time  $T_0$  is not necessarily sharp in that case.

*Remark 2.* – If the spectral unique continuation property fails there exists a non-zero eigenfunction of the network that vanishes identically on the controlled string. Then it is possible to construct an eigenfunction for the sub-network obtained from the original one by neglecting the controlled string and replacing the boundary conditions at the other (uncontrolled) node of that string by homogeneous Dirichlet boundary conditions. But such an eigenfunction verifies, in addition, that the sum of the normal derivatives at some nodes is equal to zero. Therefore, due to the analytic character (with respect to the variable  $x$ ) of the eigenfunctions (actually, they are sinusoidal functions over each string of the network), the properties of spectral unique continuation and boundary controllability in time  $T > T_0$  may be shown to hold for almost all networks (in the sense of a naturally defined Borel measure over the set of topologically equivalent networks) having the same topological configuration (i.e., the same topological graph).

*Remark 3.* – When the network is a star, the condition of Theorem 1 is easily proved to be equivalent to the property of mutual irrationality of the lengths of the uncontrolled strings given in [7]. If the graph is a tree, the condition is reduced to the disjointedness of the spectra of some subtrees. In particular, when  $\mathcal{C}$  consists of a single vertex, we obtain the conditions of [6]. Note that in these cases, the spectral controllability has been proved also for the minimal time  $T_0$  with a different, more sophisticated, method based on d’Alembert’s formula and propagation arguments. We do not know, however, whether this fact, i.e., the spectral controllability for  $T = T_0$ , remains true for general networks.

### 3. Proof of the theorem

The proof is based on the following two propositions.

**PROPOSITION 1.** – (Haraux and Jaffard, [8], Corollary 2.3.6) *Let  $\{\mu_n\}_{n \in \mathbb{Z}}$  be a sequence of real numbers such that  $\mu_n = -\mu_{-n}$ . Assume that for some  $d \geq 0$  and  $0 < \alpha < 1$*

$$\text{card}\{k \in \mathbb{Z} : 0 \leq \mu_k \leq t\} = dt + O(t^\alpha)$$

*Then we have the following properties*

(1) *For each  $T > 2\pi d$  and for each  $n \in \mathbb{Z}$  there exists a constant  $C_n > 0$  such that*

$$\int_0^T |f(t)|^2 dt \geq C_n |f_n|^2,$$

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for each function  $f$  of the form  $f = \sum_{k \in J} f_k e^{i\mu_k t}$ ,  $J$  being any finite subset of  $\mathbb{Z}$ .

(2) For each  $T < 2\pi d$  and for any finite sequence  $\{\alpha_n\}_{n \in F}$  of complex numbers having a non zero term, there exists no constant  $C > 0$  such that

$$\int_0^T |f(t)|^2 dt \geq C \left| \sum_{n \in F} \alpha_n f_n \right|^2,$$

holds for each  $f$  of the form  $f = \sum_{k \in J} f_k e^{i\mu_k t}$ , with  $F \subset J$ .

PROPOSITION 2. -Let  $\{\mu_n\}_{n \in \mathbb{Z}}$  be the sequence defined by (9). Then

$$\text{card}\{k \in \mathbb{Z} : 0 \leq \mu_k \leq t\} = \frac{T_0}{2\pi} t + O(1).$$

This latter proposition is essentially due to von-Below [4] and can be easily obtained with the aid of the max-min principle of Courant-Weyl.

From equality (8) it follows that

$$u_{i^*,x}(\mathbf{v}^*, t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \gamma_n e^{i\mu_n t} \varphi_{ni^*,x}(\mathbf{v}^*)$$

and then, from Proposition 1, if  $T > T_0$ ,

$$\int_0^T |u_{i^*,x}(\mathbf{v}^*, t)|^2 dt \geq C_n |\varphi_{ni^*,x}(\mathbf{v}^*)|^2 |\gamma_n|^2.$$

Consequently,

$$\int_0^T |u_{i^*,x}(\mathbf{v}^*, t)|^2 dt \geq \tilde{C}_n (|u_n^0|^2 + |u_n^1|^2) |\varphi_{ni^*,x}(\mathbf{v}^*)|^2.$$

This latter inequality implies that, if  $\varphi_{ni^*,x}(\mathbf{v}^*) \neq 0$ , inequality (10) is obtained.

On the other hand, it is easy to see that, if some eigenfunction  $\bar{\varphi}_n$  vanishes, together with  $\bar{\varphi}_{n,x}$  at the controlled node, then inequality (10) fails. Note that, in view of the fact that each eigenfunction satisfies a second order ordinary differential equation in each string, if  $\bar{\varphi}_n$  vanishes, together with  $\bar{\varphi}_{n,x}$  at some node, then it vanishes on the string that contains that node.

Assertion b) of the theorem is a consequence of Proposition 1 (2).  $\square$

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