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THE COST OF APPROXIMATE CONTROLLABILITY FOR HEAT EQUATIONS: THE LINEAR CASE

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Abstract. We consider linear heat equations in a bounded domain of $\mathbb{R}^d$ with Dirichlet boundary conditions. We analyze the problem of controllability when the control acts on a (small) open subset of the domain. It is by now well known that the system is approximately controllable, null-controllable and also finite-approximately controllable. This last property means that there exist controls by means of which we can simultaneously guarantee the approximate controllability and the exact controllability of a projection of the solution over a finite dimensional subspace. In this paper we obtain explicit bounds of the cost of approximate controllability, i.e., of the minimal norm of a control needed to control the system approximately. We also address the problem of simultaneous finite-approximate controllability. The methods we use combine global Carleman estimates, energy estimates for parabolic equations and the variational approach to approximate controllability. In the case of the constant coefficient heat equation, following a different approach, we are able to obtain better bounds. We also show that, in this particular case, the estimates are sharp. As a consequence of our estimates, we can determine the speed of convergence of the limiting process in which the approximate control is obtained through a sequence of penalized optimal control problems.

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1. Introduction and main results. Let Ω be a bounded domain of $\mathbb{R}^d$ with boundary of class C^2. Given $T > 0$, we consider linear parabolic equations of the form

$$\begin{cases} y_t - \Delta y + ay = v1_{\mathcal{O}} & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(x,0) = y_0(x) & \text{in} \quad \Omega\,. \end{cases} \tag{1.1}$$

In (1.1), $y = y(x,t)$ is the state and $v = v(x,t)$ is the control. This acts on the system through the non-empty open set $\mathcal{O} \subset \Omega$. $1_{\mathcal{O}}$ denotes the characteristic function of the set $\mathcal{O}$. The potential $a = a(x,t)$ is assumed to be in $L^\infty(\Omega \times (0,T))$, although this condition may be relaxed very often (the usual norm in $L^\infty(\Omega \times (0,T))$ will be denoted by $\|\cdot\|_\infty$).

We shall denote by Q the cylinder $\Omega \times (0,T)$ and by Σ the lateral boundary $\partial\Omega \times (0,T)$. We assume that $y_0 \in L^2(\Omega)$ and $v \in L^2(\mathcal{O} \times (0,T))$, so that (1.1) admits a unique solution y in the class

$$y \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))\,. \tag{1.2}$$

It is well known that system (1.1) is approximately controllable (see for instance [10] and [4]). In other words, given $y_0 \in L^2(\Omega)$, a final state $y_1 \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control $v \in L^2(\mathcal{O} \times (0,T))$ such that the solution of (1.1) satisfies

$$\|y(T) - y_1\|_{L^2} \le \varepsilon\,. \tag{1.3}$$

In particular, the *set of admissible controls* $\mathcal{U}_{ad}(y_0, y_1; \varepsilon)$ below is non-empty:

$$\mathcal{U}_{ad}(y_0, y_1; \varepsilon) = \left\{ v \in L^2(\mathcal{O} \times (0,T));\ \text{the solution } y \text{ of (1.1) satisfies (1.3)} \right\}. \tag{1.4}$$

Let us introduce the following quantity, which measures the *cost of approximate controllability* or, more precisely, the cost of achieving (1.3):

$$\mathcal{C}(y_0, y_1; \varepsilon) = \inf_{v \in \mathcal{U}_{ad}(y_0, y_1; \varepsilon)} \|v\|_{L^2(\mathcal{O} \times (0,T))}\,. \tag{1.5}$$

The first main goal of this paper is to obtain explicit bounds of $\mathcal{C}(y_0, y_1; \varepsilon)$. Taking into account that system (1.1) is linear, one can assume without loss of generality that $y_0 \equiv 0$. Indeed,

$$\mathcal{C}(y_0, y_1; \varepsilon) = \mathcal{C}(0, z_1; \varepsilon)\,, \tag{1.6}$$

where $z_1 = y_1 - z(T)$ and z is the solution of (1.1) with $v \equiv 0$.

The only case of interest is when $\mathcal{O} \neq \Omega$. Throughout this paper, this will be assumed. In the sequel, C stands for a generic positive constant only depending on Ω and $\mathcal{O}$, whose value can change from line to line.

The first main result of this paper is as follows:

Theorem 1.1. *For any* $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\varepsilon > 0$, $T > 0$ *and any potential* $a \in L^\infty(Q)$, *one has*

$$\mathcal{C}(0, y_1; \varepsilon) \le \exp\big[C\big[1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} + \frac{\|a\|_\infty \|y_1\|_{L^2} + \|\Delta y_1\|_{L^2}}{\varepsilon}\big]\big] \|y_1\|_{L^2} . \tag{1.7}$$

Remark 1.1.

1.- Note that (1.7) is only of interest when

$$\frac{\|\Delta y_1\|_{L^2}}{\lambda_1} > \varepsilon ,$$

with λ_1 being the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Otherwise, we would have $\|y_1\|_{L^2} \le \varepsilon$ and then, taking $v = 0$ in (1.1) for $y_0 = 0$, we would trivially obtain $y \equiv 0$ and

$$\|y(T) - y_1\|_{L^2} \le \varepsilon .$$

In other words,

$$\mathcal{C}(0, y_1; \varepsilon) = 0 \quad \text{if} \quad \frac{\|\Delta y_1\|_{L^2}}{\lambda_1} \le \varepsilon .$$

2.- If, instead of assuming $y_1 \in H^2(\Omega) \cap H_0^1(\Omega) = D(-\Delta)$, we assume $y_1 \in D((-\Delta)^{\gamma/2})$ with $0 < \gamma \le 2$ ($-\Delta$ is here the Dirichlet Laplacian), other estimates similar to (1.7) can be established. Namely, we have:

$$\mathcal{C}(0, y_1; \varepsilon) \le \exp\big[C\big[N(T, \|a\|_\infty) + M(\|a\|_\infty, y_1, \gamma, \varepsilon)\big]\big] \|y_1\|_{L^2} , \tag{1.8}$$

where

$$M(\|a\|_\infty, y_1, \gamma, \varepsilon) = \big(\frac{\|(-\Delta)^{\gamma/2} y_1\|_{L^2}}{\gamma\varepsilon}\big)^{2/\gamma} + \frac{\|a\|_\infty \|y_1\|_{L^2}}{\varepsilon} \tag{1.9}$$

and

$$N(T, \|a\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\,. \tag{1.10}$$

However, notice that, as far as the power of ε is concerned, the best estimate is obtained when $y_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ (in this case, the estimate is of the order of $\exp(C/\varepsilon)$).

3.- The proof of Theorem 1.1 shows the way the constant C in (1.7) depends on Ω and $\mathcal{O}$.

4.- The estimate in (1.7) depends on the H^2-norm of the target y_1. However, from (1.7) one can easily deduce estimates for any $y_1 \in L^2(\Omega)$. Given $y_1 \in L^2(\Omega)$ and $\varepsilon > 0$, we consider $y_{1,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\|y_{1,\varepsilon} - y_1\|_{L^2} \le \varepsilon/2\,. \tag{1.11}$$

Then

$$\mathcal{C}(0, y_1; \varepsilon) \le \mathcal{C}(0, y_{1,\varepsilon}; \varepsilon/2)\,. \tag{1.12}$$

Note however that $\|\Delta y_{1,\varepsilon}\|_{L^2}$ may tend to infinity as $\varepsilon \to 0$ arbitrarily fast.

5.- In the case of the constant coefficient heat equation, Theorem 1.1 may be improved. In particular, when $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, the estimate becomes of the order of $\exp(C/\sqrt{\varepsilon})$. This will be discussed in Section 6.

Approximate controllability can be viewed as a consequence of the null-controllability property of the heat equation (1.1). We recall that (1.1) is said to be *null-controllable* if, for any $y_0 \in L^2(\Omega)$, there exists $v \in L^2(\mathcal{O} \times (0,T))$ such that the solution of (1.1) satisfies

$$y(T) = 0 \ \text{ in } \ \Omega\,. \tag{1.13}$$

In our context, in order to obtain explicit bounds of the kind (1.7), we first have to obtain sharp bounds on the cost of controlling to zero. We refer to [15] for a survey on the first results on the null-controllability of the classical heat equation. More recently, G. Lebeau and L. Robbiano [9] proved the null-controllability in the setting we are working on provided the potential is independent of t, i.e., $a = a(x)$. More complete results covering all potentials $a \in L^\infty(Q)$ were obtained by A. Fursikov and O. Yu. Imanuvilov in [7], using global Carleman inequalities.

Let us introduce the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi_0(x) & \text{in} \quad \Omega\,. \end{cases} \tag{1.14}$$

The following holds:

Theorem 1.2. (Observability estimate). *For any solution of* (1.14) *and for all $a \in L^\infty(Q)$, one has*

$$\|\varphi(0)\|_{L^2}^2 \le \exp\big(C\big(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\big)\big) \iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt\,. \tag{1.15}$$

The proof of Theorem 1.2 relies on global Carleman inequalities as in [7], but paying special attention to the constants arising in the integrations by parts. Once (1.15) is known, Theorem 1.1 can be proved arguing as follows.

As proved in [10], the approximate control v of minimal norm in $L^2(\mathcal{O}\times(0,T))$ corresponding to $y_0 = 0$, $y_1 \in L^2(\Omega)$ and $\varepsilon > 0$ can be obtained by minimizing the convex functional

$$J_{y_1,\varepsilon}(\varphi_0) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|\varphi_0\|_{L^2} - \int_\Omega \varphi_0\,y_1\,dx \tag{1.16}$$

in $L^2(\Omega)$. More precisely, if the minimum of $J_{y_1,\varepsilon}$ in $L^2(\Omega)$ is attained at $\hat\varphi_0$, then the control $v = \hat\varphi 1_{\mathcal{O}}$, where $\hat\varphi$ is the solution of (1.14) with $\varphi_0 = \hat\varphi_0$, is such that the solution y of (1.14) with $y_0 = 0$ satisfies (1.3). Moreover, $\|v\|_{L^2(\mathcal{O}\times(0,T))} = \mathcal{C}(0, y_1; \varepsilon)$. We refer to [4] for a systematic analysis of this approach to approximate controllability.

As indicated in [4], one has

$$\liminf_{\|\varphi_0\|_{L^2}\to\infty} \frac{J_{y_1,\varepsilon}(\varphi_0)}{\|\varphi_0\|_{L^2}} \ge \varepsilon\,. \tag{1.17}$$

This coercivity estimate is a consequence of the following unique continuation property for the solutions of the adjoint system (1.14) (see for instance [16]): If φ solves (1.14) and $\varphi = 0$ in $\mathcal{O}\times(0,T)$, then necessarily $\varphi \equiv 0$.

The key point in the proof of Theorem 1.1 is to understand how observability inequalities of the form (1.15) may be used to obtain sharper results on the coercivity of $J_{y_1,\varepsilon}$. Of course, sharp coercivity estimates immediately yield sharp upper bounds on the norms of the minimizer.

It is reasonable to expect that inequalities like (1.15) yield sharper results on the coercivity in $J_{y_1,\varepsilon}$. Indeed, (1.15) and the backward uniqueness property for (1.14) (see [12] and [8]) lead to the unique-continuation result above with suitable estimates on the solution φ of (1.14). Thus, (1.15) is strictly stronger than the unique-continuation result above.

It is also well known that (1.15) implies null-controllability for (1.1). For the sake of completeness, we will explain now this. Let us fix $y_0 \in L^2(\Omega)$ and let us denote by $\mathcal{U}_{ad}(y_0, 0)$ the following set:

$$\mathcal{U}_{ad}(y_0,0) = \big\{ v \in L^2(\mathcal{O} \times (0,T));\ \text{the solution } y \text{ of (1.1) satisfies (1.13)} \big\}. \tag{1.18}$$

Let us put

$$\mathcal{C}(y_0,0) = \inf_{v \in \mathcal{U}_{ad}(y_0,0)} \|v\|_{L^2(\mathcal{O}\times(0,T))}. \tag{1.19}$$

Then we have the following result, that we state without proof:

Theorem 1.3. *For each $y_0 \in L^2(\Omega)$, the set $\mathcal{U}_{ad}(y_0,0)$ is non-empty. Furthermore, one has*

$$\mathcal{C}(y_0,0) \le \exp\big(C\big(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3}\big)\big)\|y_0\|_{L^2}. \tag{1.20}$$

The second main goal in this paper is to estimate the *cost of finite-approximate controllability.* Given a finite dimensional subspace E of $L^2(\Omega)$, we denote by Π_E the orthogonal projection from $L^2(\Omega)$ onto E. System (1.1) is said to be *finite-approximately controllable* if, for any y_0, $y_1 \in L^2(\Omega)$, $\varepsilon > 0$ and any finite dimensional subspace $E \subset L^2(\Omega)$, there exists a control $v \in L^2(\mathcal{O} \times (0,T))$ such that the solution y of (1.1) satisfies

$$\|y(T) - y_1\|_{L^2} \le \varepsilon, \qquad \Pi_E(y(T)) = \Pi_E(y_1). \tag{1.21}$$

Conditions (1.21) assert that the control v we are looking for is such that the solution y of (1.1), in addition to the approximate controllability condition (1.3), satisfies exactly a finite number of constraints at time $t = T$. A basic general result of Functional Analysis implies that approximate controllability implies finite-approximate controllability (see [13]). Moreover, as indicated in [17], the finite-approximate control may be found by minimizing, instead of $J_{y_1,\varepsilon}$, the following functional:

$$J_{y_1,\varepsilon,E}(\varphi_0) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|(I-\Pi_E)\varphi_0\|_{L^2} - \int_\Omega \varphi_0\,y_1\,dx\,. \tag{1.22}$$

Let us introduce the *set of admissible controls*

$$\mathcal{U}_{ad}(y_0,y_1;\varepsilon,E) = \Big\{v\in L^2(\mathcal{O}\times(0,T)); \tag{1.23}$$
$$\text{the solution } y \text{ of (1.1) satisfies (1.21)}\Big\}.$$

We then set

$$\mathcal{C}(y_0,y_1;\varepsilon,E) = \inf_{v\in\mathcal{U}_{ad}(y_0,y_1;\varepsilon,E)} \|v\|_{L^2(\mathcal{O}\times(0,T))}\,. \tag{1.24}$$

As in the case of approximate controllability we assume, without loss of generality, that $y_0\equiv 0$. Our second main result is as follows:

Theorem 1.4. *Assume that E is a finite dimensional subspace of $H_0^1(\Omega)$. Then*

$$\mathcal{C}(0,y_1;\varepsilon) \le \Big(\exp\big[C\big[N(T,\|a\|_\infty) + (1+\Lambda_E)\frac{M(\|a\|_\infty,y_1)}{\varepsilon}\big]\big] + \Lambda_E\Big)\|y_1\|_{L^2} \tag{1.25}$$

for any $y_1\in H^2(\Omega)\cap H_0^1(\Omega)$, $\varepsilon>0$ and $a\in L^\infty(Q)$. In (1.25), we have used the notation

$$\begin{aligned} M(\|a\|_\infty,y_1) &= \|a\|_\infty\|y_1\|_{L^2} + \|\Delta y_1\|_{L^2}\,,\\ N(T,\|a\|_\infty) &= 1+\frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} \end{aligned} \tag{1.26}$$

and Λ_E is given by

$$\Lambda_E = \|\Pi_E\|\exp\big[T[\mu(E)+\|a\|_\infty]e^{\frac{T}{4}\|a\|_\infty} + C\,N(T,\|a\|_\infty)\big], \tag{1.27}$$

where

$$\mu(E) = \max_{\varphi\in E\setminus\{0\}} \frac{\|\nabla\varphi_0\|_{L^2}^2}{\|\varphi_0\|_{L^2}^2}\,. \tag{1.28}$$

Remark 1.2

1.- The constant $\mu(E)$ in (1.28) is finite due to the fact that E is finite dimensional. Note that the maximum in (1.28) is achieved.

2.- In (1.27), $\|\Pi_E\|$ denotes the norm of the projection Π_E as a bounded linear operator from $L^2(\Omega)$ onto E. Note that (1.7) is a particular case of (1.25), with $E = \{0\}$. Furthermore, the proof of Theorem 1.4 shows the way the constants C in (1.25) and (1.27) depend on Ω and $\mathcal{O}$.

3.- The functional $J_{y_1,\varepsilon,E}$ satisfies the coercivity property (1.17). But further results on the coercivity of $J_{y_1,\varepsilon,E}$ seem much more difficult to obtain. Thus, in order to estimate the cost of the finite-approximate controllability, we will follow a different approach.

The proof of Theorem 1.4 relies on the construction of a control that effectively realizes (1.21), whose norm can be estimated. This control is built by superposing a control v_1 satisfying

$$\|y(T) - y_1\|_{L^2} \leq \varepsilon/R\,, \tag{1.29}$$

where $R \geq 1$ is sufficiently large and, then, a control v_2 that takes account of the finite dimensional projection in such a way that, even if (1.29) is lost, we keep (1.21). The construction of the latter requires finite dimensional observability inequalities that we are now going to explain.

Consider the adjoint system (1.14) with initial data φ_0 in a finite dimensional subspace E of $H_0^1(\Omega)$. In view of (1.15), by backward uniqueness and taking into account that all norms are equivalent in E, we conclude that

$$\|\varphi_0\|_{L^2}^2 \leq \text{Const.} \iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt \tag{1.30}$$

for all solutions φ of (1.14) with initial data $\varphi_0 \in E$.

Combining (1.15) and backward energy estimates in the spirit of [8], we can obtain the following estimate, which provides an explicit bound of the constant in (1.30):

Theorem 1.5. (Finite dimensional observability estimate). *Assume E is a finite dimensional subspace of $H_0^1(\Omega)$. For any solution φ of* (1.14) *with* $\varphi_0 \in E$ *and for all* $a \in L^\infty(Q)$*, one has*

$$\|\varphi_0\|_{L^2}^2 \le \exp\big(T[\mu(E) + \|a\|_\infty]e^{\frac{T}{4}\|a\|_\infty} + CN(T, \|a\|_\infty)\big) \iint_{\mathcal{O}\times(0,T)} |\varphi|^2\, dx\, dt, \tag{1.31}$$

where $N(T, \|a\|_\infty)$ *is given by the second equality in* (1.26)*.*

Remark 1.3. Observe that, in (1.31), we provide an estimate on the solution φ of (1.15) at $t = T$, which is the "initial" time for the adjoint system (1.15). In Theorem 1.2, we have only obtained estimates of the norm of φ at $t = 0$, which is a much weaker quantity due to the irreversibility of the system. Notice that we are only able to estimate φ at $t = T$ when φ_0 is known to belong to a finite dimensional subspace.

As in the case of (1.15), the finite dimensional observability estimate (1.31) leads to a controllability property for (1.1). More precisely, for given y_0, $y_1 \in L^2(\Omega)$, let us denote by $\mathcal{U}^*_{ad}(y_0, y_1; E)$ the set

$$\mathcal{U}^*_{ad}(y_0, y_1; E) = \Big\{ v \in L^2(\mathcal{O} \times (0,T)); \\ \text{the solution } y \text{ of (1.1) satisfies } \Pi_E(y(T)) = \Pi_E(y_1) \Big\}.$$

Let us also put

$$\mathcal{C}^*(y_0, y_1; E) = \inf_{v \in \mathcal{U}^*_{ad}(y_0, y_1; E)} \|v\|_{L^2(\mathcal{O}\times(0,T))} \,. \tag{1.32}$$

Then it can be deduced from Theorem 1.5 that $\mathcal{U}^*_{ad}(y_0, y_1; E)$ is never the empty set. We can also obtain from (1.31) an explicit bound for $\mathcal{C}^*(y_0, y_1; E)$. This is summarized in the following result, where we have taken $y_0 = 0$ for convenience:

Theorem 1.6. *For each* $y_1 \in L^2(\Omega)$*, the set* $\mathcal{U}^*_{ad}(0, y_1; E)$ *is non-empty. Furthermore, one has*

$$\mathcal{C}^*(0, y_1; E) \le \exp\big(T[\mu(E) + \|a\|_\infty]e^{\frac{T}{4}\|a\|_\infty} + C\, N(T, \|a\|_\infty)\big)\|\Pi_E y_1\|_{L^2} \,, \tag{1.33}$$

where $N(T, \|a\|_\infty)$ *is as in* (1.26)*.*

The results of this paper may be combined with the fixed point methods of [4] and [6] to deduce similar results for semilinear heat equations with globally Lipschitz nonlinearities or even when the nonlinearity grows slightly in a superlinear way. These problems will be addressed in a second part of this paper.

The rest of this paper is organized as follows. In Section 2 we prove in detail the observability estimate in Theorem 1.2. In Section 3 we use Theorem 1.2 and energy estimates for parabolic equations to prove Theorem 1.5. We then prove Theorem 1.6. In Section 4 we prove the first main result in this paper (Theorem 1.1). Section 5 is devoted to the proof of the second main result (Theorem 1.4). In Section 6 we consider the case of the constant coefficient heat equation. We will see that, in this case, the estimate in Theorem 1.1 can be improved and that the new estimate is sharp. In Section 7, using the above estimates of the cost of approximate controllability, we determine the speed of convergence of the limiting process in which the control is obtained by penalization. In Section 8 we discuss various possible extensions of the results of this paper. In an Appendix at the end of the paper, we give a detailed proof of a global Carleman inequality in the spirit of [7], for the sake of completeness.

2. Proof of the observability inequality. The observability estimate in Theorem 1.2 is a consequence of a suitable global Carleman estimate for the adjoint system (1.14).

Following [7], we introduce a function $\eta^0 = \eta^0(x)$ such that

$$\begin{cases} \eta^0 \in C^2(\overline{\Omega})\,, \\ \eta^0 > 0 \text{ in } \Omega\,, \quad \eta^0 = 0 \text{ on } \partial\Omega\,, \\ \nabla\eta^0 \neq 0 \text{ in } \overline{\Omega\setminus\mathcal{O}}\,. \end{cases} \tag{2.1}$$

We refer to [7] for the proof of the existence of a function satisfying (2.1).

Let $K_0 > 0$ be such that $K_0 \geq 5\max_{\overline{\Omega}}\eta^0 - 6\min_{\overline{\Omega}}\eta^0$ and set

$$\beta^0 = \eta^0 + K_0\,, \quad \bar\beta = \frac{5}{4}\max_{\overline{\Omega}}\beta^0\,, \quad \rho^1(x) = e^{\lambda\bar\beta} - e^{\lambda\beta^0} \tag{2.2}$$

where λ is a sufficiently large positive constant that only depends on Ω and $\mathcal{O}$ and will be fixed later on. Notice that $\rho^1 > 0$ in Ω. We also introduce

$$\phi(x,t) = \rho^1(x)/[t(T-t)]\,, \quad \rho(x,t) = \exp\big[\rho^1(x)/[t(T-t)]\big] = \exp(\phi(x,t)) \tag{2.3}$$

and the space

$$Z = \{q \in C^2(\overline{Q});\ q = 0 \text{ on } \Sigma\}. \tag{2.4}$$

The following holds:

Proposition 2.1. ([7], [6]). *There exist positive constants $C_*, s_1 > 0$ such that*

$$\begin{aligned} &\frac{1}{s}\iint_Q \rho^{-2s} t(T-t)\big(|q_t|^2 + |\Delta q|^2\big)\,dx\,dt \\ &+s\iint_Q \rho^{-2s}t^{-1}(T-t)^{-1}|\nabla q|^2\,dx\,dt + s^3\iint_Q \rho^{-2s}t^{-3}(T-t)^{-3}|q|^2\,dx\,dt \\ &\le C_*\big[\iint_Q \rho^{-2s}\,|\partial_t q + \Delta q|^2\ dx\,dt + s^3\iint_{\mathcal{O}\times(0,T)} \rho^{-2s}t^{-3}(T-t)^{-3}|q|^2\,dx\,dt\big] \end{aligned} \tag{2.5}$$

for all $q \in Z$ and $s \ge s_1$. Moreover, C_* depends only on Ω and $\mathcal{O}$ and s_1 is of the form

$$s_1 = \sigma_1(\Omega,\mathcal{O})\big(T + T^2\big) \tag{2.6}$$

where $\sigma_1(\Omega,\mathcal{O})$ is a positive constant that only depends on Ω and $\mathcal{O}$.

Remark 2.1. The proof of Proposition 2.1 is given in the Appendix at the end of the paper, for the sake of completeness. The choice of the constant λ needed for the definition of ρ^1 in (2.2) is made there. The dependence on the geometry of Ω and $\mathcal{O}$ of λ and the other constants found in the proof will not be analyzed. However, the dependence on T is essential in our analysis.

Let us now complete the proof of the observability inequality in Theorem 1.2 using Proposition 2.1. Let $\varphi_0 \in L^2(\Omega)$ be given. By density, we can write (2.5) for $p = \varphi$, with φ being the solution of the corresponding adjoint system (1.14). Taking into account that $\varphi_t + \Delta\varphi = a\varphi$, it follows that

$$\begin{aligned} &s\iint_Q \rho^{-2s}t^{-1}(T-t)^{-1}|\nabla\varphi|^2\,dx\,dt + s^3\iint_Q \rho^{-2s}t^{-3}(T-t)^{-3}|\varphi|^2\,dx\,dt \\ &\qquad\le C_*\big[\iint_Q \rho^{-2s}|a\varphi|^2\,dx\,dt + s^3\iint_{\mathcal{O}\times(0,T)} \rho^{-2s}t^{-3}(T-t)^{-3}|\varphi|^2\,dx\,dt\big]. \end{aligned} \tag{2.7}$$

In order to get rid of the first term in the right hand side of (2.7), we observe that

$$\iint_Q \rho^{-2s}|a\varphi|^2\,dx\,dt \le 2^{-6}T^6\,\|a\|_\infty^2 \iint_Q \rho^{-2s}t^{-3}(T-t)^{-3}|\varphi|^2\,dx\,dt\,. \quad (2.8)$$

Then, according to (2.7), we deduce that

$$\iint_Q \rho^{-2s}t^{-1}(T-t)^{-1}|\nabla\varphi|^2\,dx\,dt \le C_* s^2 \iint_{\mathcal{O}\times(0,T)} \rho^{-2s}t^{-3}(T-t)^{-3}|\varphi|^2 dx\,dt \quad (2.9)$$

provided

$$s \ge s_2 = \max\big(s_1, CT^2\|a\|_\infty^{2/3}\big). \quad (2.10)$$

Let us estimate the weights appearing in (2.9):

Lemma 2.1. *One has*

$$\|\rho^{-2s}t^{-3}(T-t)^{-3}\|_\infty \le 2^6T^{-6}\exp(-CsT^{-2}) \quad (2.11)$$

for all

$$s \ge s_3 = \max\big(s_2, 3T^2(8\min_{\overline{\Omega}}\rho^1(x))^{-1}\big). \quad (2.12)$$

Proof. We observe that

$$\rho(x,t)^{-2s}t^{-3}(T-t)^{-3} = 1/f_x(t) \quad (2.13)$$

for any $x\in\Omega$ and $t\in(0,T)$, with

$$f_x(t) = t^3(T-t)^3\exp\big(\frac{2s\rho^1(x)}{t(T-t)}\big) = \tau^3\exp\big(\frac{2s\rho^1(x)}{\tau}\big) = g_x(\tau)$$

and $\tau = t(T-t)\in[0,T^2/4]$. The minimum of g_x is achieved at $\hat\tau = \frac{2}{3}s\rho^1(x)$ and $g_x(\hat\tau) = (\frac{2}{3}s\rho^1(x))^3e^3$. On the other hand, $g_x(0)=\infty$ and g_x is decreasing for $\tau\in\big(0,\hat\tau\big)$ and increasing for $\tau>\hat\tau$. Thus,

$$\begin{aligned}\min_{0\le t\le T} f_x(t) &= \min_{0\le\tau\le T^2/4} g_x(\tau)\\ &= \begin{cases} g_x(\hat\tau) = (\frac{2}{3}s\rho^1(x))^3e^3 & \text{if } T^2/4 \ge \frac{2}{3}s\rho^1(x)\\ g_x(T^2/4) = 2^{-6}T^6\exp(8s\rho^1(x)T^{-2}) & \text{if } T^2/4 < \frac{2}{3}s\rho^1(x)\end{cases}\end{aligned}$$

Therefore, if $s \geq s_3$ with s_3 as in (2.12), we have

$$\min_{0 \leq t \leq T} f_x(t) \geq 2^{-6} T^6 \exp(CsT^{-2}) \quad \text{with} \quad C = 8 \min_{\overline{\Omega}} \rho^1(x)\,. \tag{2.14}$$

In view of (2.13) and (2.14), (2.11) holds.

Remark 2.2. Let us look at the structure of the constant s_3 appearing in (2.12). In view of (2.6), (2.10) and (2.12), we have

$$s_3 \leq s_4 = C\big(T + (1 + \|a\|_\infty^{2/3})T^2\big)\,. \tag{2.15}$$

Applying (2.11) with s as in the right hand side of (2.15), we deduce that

$$\|\rho^{-2s} t^{-3}(T-t)^{-3}\|_\infty \leq 2^6 T^{-6} \exp\big(-C\big(1 + \frac{1}{T} + \|a\|_\infty^{2/3}\big)\big). \tag{2.16}$$

Lemma 2.2. *One has*

$$\rho^{-2s} t^{-1}(T-t)^{-1} \geq \frac{16}{3} T^{-2} \exp(-CsT^{-2}) \quad \forall x \in \Omega\,,\ \forall t \in \big[T/4, 3T/4\big]\,, \tag{2.17}$$

whenever $s \geq s_4$.

Proof. We have

$$\rho(x,t)^{-2s} t^{-1}(T-t)^{-1} = 1/h_x(t)$$

with

$$h_x(t) = t(T-t)\exp\big(\frac{2s\rho^1(x)}{t(T-t)}\big) = \tau \exp\big(\frac{2s\rho^1(x)}{\tau}\big) = j_x(\tau)$$

and $\tau = t(T-t) \in [0, T^2/4]$. When $t \in [T/4, 3T/4]$, one has $\tau \in [3T^2/16, T^2/4]$. Proceeding as in the proof of Lemma 2.1, we deduce that

$$\max_{T/4 \leq t \leq 3T/4} h_x(t) \leq \frac{3}{16} T^2 \exp(CsT^{-2})\,,$$

provided $s \geq T^2(8 \min_{\overline{\Omega}} \rho^1(x))^{-1}$. In particular, this is the case if $s \geq s_4$. Therefore,

$$\rho^{-2s} t^{-1}(T-t)^{-1} \geq \frac{16}{3} T^{-2} \exp(-CsT^{-2}) \quad \text{in} \quad \Omega \times \big[T/4, 3T/4\big]$$

whenever $s \geq s_4$. This concludes the proof.

Remark 2.3. Arguing as in Remark 2.2, it follows that

$$\rho^{-2s}t^{-1}(T-t)^{-1} \geq \frac{16}{3}T^{-2}\exp\bigl(-C\bigl(1+\frac{1}{T}+\|a\|_\infty^{2/3}\bigr)\bigr) \quad \text{in} \quad \Omega\times\bigl[T/4,3T/4\bigr] \tag{2.18}$$

for $s = s_4$.

Coming back to (2.9) and using (2.16) and (2.18), we deduce that

$$\iint_{\Omega\times(T/4,3T/4)} |\nabla\varphi|^2\,dx\,dt \leq \exp\bigl(C\bigl(1+\frac{1}{T}+\|a\|_\infty^{2/3}\bigr)\bigr)\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt \tag{2.19}$$

for any solution φ of (1.14), for any potential $a\in L^\infty(Q)$ and for all $T>0$. On the other hand, multiplying in (1.14) by φ and integrating in Ω, we deduce that

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2\,dx + \int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega a|\varphi|^2\,dx = 0 \tag{2.20}$$

and

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |\varphi|^2\,dx \leq \|a\|_\infty \int_\Omega |\varphi|^2\,dx\,.$$

Thus,

$$\frac{d}{dt}\bigl(e^{2t\|a\|_\infty}\int_\Omega |\varphi|^2\,dx\bigr) \geq 0\,. \tag{2.21}$$

Integrating this inequality in $[T/4,t]$ for any $t\in[T/4,3T/4]$, we find

$$\begin{aligned}\int_\Omega |\varphi(x,t)|^2\,dx &\geq \exp\bigl(2\bigl(\frac{T}{4}-t\bigr)\|a\|_\infty\bigr)\int_\Omega |\varphi(x,T/4)|^2\,dx\\ &\geq \exp\bigl(-T\|a\|_\infty\bigr)\int_\Omega |\varphi(x,T/4)|^2\,dx \qquad \forall t\in\bigl[T/4,3T/4\bigr]\,.\end{aligned} \tag{2.22}$$

Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Combining (2.19) and (2.22), we deduce that

$$\begin{aligned}\frac{\lambda_1 T}{2}\exp\bigl(-T\|a\|_\infty\bigr)\int_\Omega |\varphi(x,T/4)|^2\,dx &\leq \lambda_1\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2\,dx\,dt\\ &\leq \exp\bigl(C\bigl(1+\frac{1}{T}+\|a\|_\infty^{2/3}\bigr)\bigr)\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt\,.\end{aligned}$$

On the other hand, integrating (2.21) in the time interval $[0, T/4]$, we also find that

$$\exp\big(\frac{T}{2}\|a\|_\infty\big)\int_\Omega |\varphi(x,T/4)|^2\,dx \geq \int_\Omega |\varphi(x,0)|^2\,dx\,,$$

whence

$$\|\varphi(0)\|_{L^2}^2 \leq \frac{2}{\lambda_1 T}\exp\big(C\big(1+\frac{1}{T}+\|a\|_\infty^{2/3}\big)+\frac{3T}{2}\|a\|_\infty\big)\iint_{\mathcal{O}\times(0,T)}|\varphi|^2\,dx\,dt\,. \tag{2.23}$$

Inequality (1.15) follows easily from (2.23). This completes the proof of Theorem 1.2.

3. The finite dimensional observability estimate and its consequences. This Section is devoted to prove Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.5. We proceed in two steps:
Step 1. We apply the observability estimate of Theorem 1.2 in the time interval $[T/2, T]$. It follows that

$$\|\varphi(T/2)\|_{L^2}^2 \leq \exp\big(C\big(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{2/3}\big)\big)\iint_{\mathcal{O}\times(0,T)}|\varphi|^2\,dx\,dt \tag{3.1}$$

for every solution of (1.14).
Step 2. We look for an estimate of the form

$$\|\varphi_0\|_{L^2}^2 \leq \text{Const.}\,\|\varphi(T/2)\|_{L^2}^2\,. \tag{3.2}$$

This is a backward estimate for the heat equation (1.14). Hence, it must be expected it holds only when φ_0 lies in a finite dimensional subspace E of $H_0^1(\Omega)$. We use again (2.20), which can be written in the form

$$-\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|_{L^2}^2+\Lambda(t)\|\varphi(t)\|_{L^2}^2+\int_\Omega a(t)\varphi(t)^2\,dx=0\,, \tag{3.3}$$

with

$$\Lambda(t)=\frac{\|\nabla\varphi(t)\|_{L^2}^2}{\|\varphi(t)\|_{L^2}^2}\,. \tag{3.4}$$

Let us set

$$\Lambda^* = \max_{T/2\le t\le T} \Lambda(t)\,. \tag{3.5}$$

We then have

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t)\|_{L^2}^2 \le \big(\Lambda^* + \|a\|_\infty\big)\|\varphi(t)\|_{L^2}^2\,.$$

Therefore,

$$\|\varphi(T)\|_{L^2}^2 = \|\varphi_0\|_{L^2}^2 \le e^{T(\Lambda^*+\|a\|_\infty)}\|\varphi(T/2)\|_{L^2}^2\,. \tag{3.6}$$

In view of (3.6), in order to obtain (3.2), it is sufficient to estimate the constant Λ^*. Taking into account that $\varphi_0 \in E$, we observe that

$$\Lambda(T) = \frac{\|\nabla\varphi_0\|_{L^2}^2}{\|\varphi_0\|_{L^2}^2} \le \mu(E)\,, \tag{3.7}$$

where $\mu(E)$ is given by (1.28). Consequently, it suffices to obtain an upper bound for $d\Lambda/dt$. We proceed following the ideas in [8]. We set $\widetilde\Lambda = \Lambda + \|a\|_\infty$. Then

$$\begin{aligned}
\frac{d\widetilde\Lambda}{dt} &= \frac{2\big(\int_\Omega \nabla\varphi\cdot\nabla\varphi_t\,dx\big)\big(\int_\Omega |\varphi|^2\,dx\big) - 2\big(\int_\Omega \varphi\varphi_t\,dx\big)\big(\int_\Omega |\nabla\varphi|^2\,dx\big)}{\big(\int_\Omega |\varphi|^2\,dx\big)^2}\\
&= -2\frac{\big(\int_\Omega \Delta\varphi\varphi_t\,dx\big)\big(\int_\Omega |\varphi|^2\,dx\big) + \big(\int_\Omega \varphi\varphi_t\,dx\big)\big(\int_\Omega |\nabla\varphi|^2\,dx\big)}{\big(\int_\Omega |\varphi|^2\,dx\big)^2}\\
&= -\frac{2}{\int_\Omega |\varphi|^2\,dx}\int_\Omega (\varphi_t - \Lambda\varphi)(\Delta\varphi + \Lambda\varphi)\,dx\,.
\end{aligned} \tag{3.8}$$

In the last step we have used the fact that

$$\int_\Omega \varphi(\Delta\varphi + \Lambda\varphi)\,dx = 0\,.$$

Going back to (3.8), we see that

$$\begin{aligned}
\frac{d\widetilde{\Lambda}}{dt} &= -\frac{2}{\displaystyle\int_\Omega |\varphi|^2 dx}\int_\Omega (\varphi_t+\Delta\varphi)(\Delta\varphi+\Lambda\varphi)dx + \frac{2}{\displaystyle\int_\Omega |\varphi|^2 dx}\int_\Omega |\Delta\varphi+\Lambda\varphi|^2 dx\\
&= -\frac{2}{\displaystyle\int_\Omega |\varphi|^2\, dx}\int_\Omega a\varphi(\Delta\varphi+\Lambda\varphi)\, dx + \frac{2}{\displaystyle\int_\Omega |\varphi|^2\, dx}\int_\Omega |\Delta\varphi+\Lambda\varphi|^2\, dx\\
&= \frac{2}{\displaystyle\int_\Omega |\varphi|^2\, dx}\Big(\int_\Omega |\Delta\varphi+\Lambda\varphi-\frac{1}{2}a\varphi|^2\, dx - \frac{1}{4}\int_\Omega a^2|\varphi|^2\, dx\Big)\\
&\geq -\frac{1}{2}\frac{\displaystyle\int_\Omega a^2|\varphi|^2\, dx}{\displaystyle\int_\Omega |\varphi|^2\, dx} \geq -\frac{1}{2}\|a\|_\infty^2 \geq -\frac{1}{2}\|a\|_\infty\widetilde{\Lambda}\,.
\end{aligned}$$

Thus,

$$\frac{d}{dt}\Big(\exp\big(\frac{t}{2}\|a\|_\infty\big)\widetilde{\Lambda}(t)\Big) \geq 0\,. \tag{3.9}$$

From (3.9) we deduce that

$$\widetilde{\Lambda}(t) \leq \exp\big(\frac{T-t}{2}\|a\|_\infty\big)\widetilde{\Lambda}(T) \quad \forall t\in[T/2,T]\,.$$

In other words,

$$\begin{aligned}
\Lambda(t)+\|a\|_\infty &\leq [\Lambda(T)+\|a\|_\infty]\exp\big(\tfrac{T-t}{2}\|a\|_\infty\big)\\
&\leq [\mu(E)+\|a\|_\infty]\exp\big(\tfrac{T}{4}\|a\|_\infty\big)
\end{aligned} \tag{3.10}$$

for all $t\in[T/2,T]$. From (3.5) and (3.10), we get

$$\Lambda^*+\|a\|_\infty \leq \big[\mu(E)+\|a\|_\infty\big]\exp\big(\frac{T}{4}\|a\|_\infty\big). \tag{3.11}$$

Finally, (3.6) and (3.11) yield

$$\|\varphi_0\|_{L^2}^2 \leq \exp\Big(T\big[\mu(E)+\|a\|_\infty\big]\exp\big(\frac{T}{4}\|a\|_\infty\big)\Big)\|\varphi(T/2)\|_{L^2}^2 \tag{3.12}$$

for all $\varphi_0\in E$. Combining (3.1) and (3.12), we immediately deduce the estimate (1.31) of Theorem 1.5.

Proof of Theorem 1.6. Given $y_1 \in H_0^1(\Omega)$, there exist controls $v \in L^2(\mathcal{O} \times (0,T))$ such that the corresponding solution y of (1.1) satisfies

$$\Pi_E[y(T)] = \Pi_E(y_1)\,. \tag{3.13}$$

This can be seen minimizing over E the functional

$$J_{y_1,E}(\varphi_0) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt - \int_\Omega y_1\,\varphi_0\,dx\,. \tag{3.14}$$

In the definition (3.14) of $J_{y_1,E}$, φ is of course the solution of the adjoint system (1.14).

Indeed, (1.31) implies that the minimum of $J_{y_1,E}$ is achieved at a unique $\hat\varphi_0 \in E$. The control $v = \hat\varphi 1_{\mathcal{O}}$, where $\hat\varphi$ is the solution of (1.14) associated to the minimizer, is such that the solution of (1.1) with $y_0 = 0$ satisfies (3.13). This shows that $\mathcal{U}^*_{ad}(0, y_1; E)$ is not the empty set.

On the other hand, also according to the observability inequality (1.31), it follows immediately that

$$\|v\|_{L^2(\mathcal{O}\times(0,T))} \le \exp\bigl(T\bigl[\mu(E) + \|a\|_\infty\bigr]e^{\frac{T}{4}\|a\|_\infty} + C\,N(T,\|a\|_\infty)\bigr)\|\Pi_E y_1\|_{L^2}\,,$$

where $N(T,\|a\|_\infty)$ is given by the second equality in (1.26). This leads to (1.33) and concludes the proof of Theorem 1.6.

4. On the cost of approximate controllability. This Section is devoted to prove Theorem 1.1.

As indicated in the Introduction, without loss of generality we can assume that $y_0 \equiv 0$. Let us fix $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\varepsilon > 0$. Then, we consider the functional

$$J_{y_1,\varepsilon}(\varphi_0) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt + \varepsilon\|\varphi_0\|_{L^2} - \int_\Omega y_1\,\varphi_0\,dx$$

defined for all $\varphi_0 \in L^2(\Omega)$, where φ denotes the solution of the adjoint system (1.14).

It is easy to see that $J_{y_1,\varepsilon} : L^2(\Omega) \mapsto \mathbb{R}$ is continuous and strictly convex. On the other hand, as proved in [4], one has (1.17). Thus, $J_{y_1,\varepsilon}$ achieves its minimum at some $\hat\varphi_0 \in L^2(\Omega)$. It is shown in [10] and [4] that the control

$$v = \hat\varphi 1_{\mathcal{O}} \tag{4.1}$$

(where $\hat{\varphi}$ solves (1.14) with $\varphi_0 = \hat{\varphi}_0$) is such that the solution of (1.1) satisfies (1.3). Moreover, v is of minimal L^2-norm in the set of admissible controls $\mathcal{U}_{ad}(0, y_1; \varepsilon)$.

In order to prove the estimate (1.7) on the cost of controlling, we have to analyze more in detail the coercivity of the functional $J_{y_1,\varepsilon}$.

Let us rewrite $J_{y_1,\varepsilon}(\varphi_0)$ in the form

$$J_{y_1,\varepsilon}(\varphi_0) = \widetilde{J}_{y_1,\delta}(\varphi_0) + \varepsilon\|\varphi_0\|_{L^2} - \int_\Omega y_1\big(\varphi_0 - \varphi(T-\delta)\big)\,dx\,, \tag{4.2}$$

where $\widetilde{J}_{y_1,\delta}(\varphi_0)$ is given by

$$\widetilde{J}_{y_1,\delta}\big(\varphi_0\big) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt - \int_\Omega y_1\,\varphi(T-\delta)\,dx \quad \forall \varphi_0 \in L^2(\Omega)\,. \tag{4.3}$$

The positive number $\delta > 0$, small enough, will be fixed later on in such a way that

$$\varepsilon\|\varphi_0\|_{L^2} - \int_\Omega y_1\big(\varphi_0 - \varphi(T-\delta)\big)dx \geq 0 \quad \forall \varphi_0 \in L^2(\Omega)\,. \tag{4.4}$$

With this choice of δ, we have

$$I_1 = \min_{\varphi_0 \in L^2(\Omega)} J_{y_1,\varepsilon}\big(\varphi_0\big) \geq \inf_{\varphi_0 \in L^2(\Omega)} \widetilde{J}_{y_1,\delta}\big(\varphi_0\big) = I_2\,. \tag{4.5}$$

Recall that $\hat{\varphi}_0$ is the minimizer of $J_{y_1,\varepsilon}$. It is easy to check that

$$I_1 = -\frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |\hat{\varphi}|^2\,dx\,dt\,, \tag{4.6}$$

where $\hat{\varphi}$ is the solution of (1.14) for $\varphi_0 = \hat{\varphi}_0$. Combining (4.5) and (4.6), we deduce that

$$\mathcal{C}(0, y_1; \varepsilon)^2 = \iint_{\mathcal{O}\times(0,T)} |\hat{\varphi}|^2\,dx\,dt \leq -2\,I_2\,. \tag{4.7}$$

Thus, our task is reduced to estimate I_2 in terms of δ and to choose δ such that (4.4) holds.

Applying the observability inequality (1.15) given in Theorem 1.2, we have

$$\|\varphi(T-\delta)\|_{L^2}^2 \le \exp\big(C\big(1+\frac{1}{\delta}+\delta\|a\|_\infty+\|a\|_\infty^{2/3}\big)\big)\iint_{\mathcal{O}\times(0,T)}|\varphi|^2dx\,dt \tag{4.8}$$

for any solution of (1.14). From (4.8) and the definition of I_2, it is easy to see that

$$I_2 \ge -\frac{1}{2}\exp\big(C\big(1+\frac{1}{\delta}+\delta\|a\|_\infty+\|a\|_\infty^{2/3}\big)\big)\|y_1\|_{L^2}^2\,.$$

Hence, we obtain the following estimate:

$$\mathcal{C}(0,y_1;\varepsilon)^2 \le -2\,I_2 \le \exp\big(C\big(1+\frac{1}{\delta}+\delta\|a\|_\infty+\|a\|_\infty^{2/3}\big)\big)\|y_1\|_{L^2}^2\,. \tag{4.9}$$

Now, going back to (4.4), we observe that it suffices to find $\delta>0$ such that

$$\Big|\int_\Omega y_1\big(\varphi_0-\varphi(T-\delta)\big)\,dx\Big| \le \varepsilon\|\varphi_0\|_{L^2}\,. \tag{4.10}$$

Let us prove that

$$\Big|\int_\Omega y_1\big(\varphi_0-\varphi(T-\delta)\big)\,dx\Big| \le \|\varphi_0\|_{L^2}\big[C\delta\|\Delta y_1\|_{L^2}+\big(e^{\delta\|a\|_\infty}-1\big)\|y_1\|_{L^2}\big]. \tag{4.11}$$

We start from the identity

$$\varphi(T-\delta)=S(\delta)\varphi_0-\int_0^\delta S(\delta-s)\big(a(T-s)\varphi(T-s)\big)\,ds\,, \tag{4.12}$$

where $S(\cdot)$ is the semigroup generated by the classical heat equation in Ω with Dirichlet boundary conditions, i.e., $u(t)=S(t)\varphi_0$ if and only if

$$\begin{cases} u_t-\Delta u=0 & \text{in} \quad \Omega\times(0,\infty)\\ u=0 & \text{on} \quad \partial\Omega\times(0,\infty)\\ u(x,0)=\varphi_0(x) & \text{in} \quad \Omega\,. \end{cases} \tag{4.13}$$

From (4.12), we have that

$$\int_\Omega y_1\big(\varphi_0-\varphi(T-\delta)\big)\,dx=\int_\Omega y_1\big(\varphi_0-S(\delta)\varphi_0\big)\,dx \\ +\int_\Omega y_1\Big[\int_0^\delta S(\delta-s)\big(a(T-s)\varphi(T-s)\big)\,ds\Big]dx\,.$$

Thus,

$$\begin{aligned}\Big|\int_\Omega y_1\big(\varphi_0-\varphi(T-\delta)\big)dx\Big| &\le \Big|\int_\Omega y_1\big(\varphi_0-S(\delta)\varphi_0\big)dx\Big| \\ &+\int_\Omega |y_1|\,\Big|\int_0^\delta S(\delta-s)\big(a(T-s)\varphi(T-s)\big)\,ds\Big|\,dx \\ &\le \|\Delta y_1\|_{L^2}\,\|\varphi_0-S(\delta)\varphi_0\|_{H^{-2}} \\ &+\|y_1\|_{L^2}\,\|\int_0^\delta S(\delta-s)\big(a(T-s)\varphi(T-s)\big)\,ds\,\|_{L^2}\,.\end{aligned} \tag{4.14}$$

We also have

$$\begin{aligned}\|\varphi_0-S(\delta)\varphi_0\|_{H^{-2}} &= \|\int_0^\delta u_t\,ds\,\|_{H^{-2}}=\|\int_0^\delta \Delta u\,ds\,\|_{H^{-2}} \\ &\le C\int_0^\delta \|u\|_{L^2}\,ds\le C\delta\max_{0\le s\le\delta}\|u(s)\|_{L^2}\,,\end{aligned}$$

with u being the solution of (4.13). Multiplying in (4.13) by u and integrating in the time interval $[0,\delta]$, we deduce that

$$\max_{0\le s\le\delta}\|u(s)\|_{L^2}\le\|\varphi_0\|_{L^2}\,.$$

Hence,

$$\|\varphi_0-S(\delta)\varphi_0\|_{H^{-2}}\le C\delta\|\varphi_0\|_{L^2}\,. \tag{4.15}$$

On the other hand, taking into account that S is a contraction semigroup in $L^2(\Omega)$, we see that

$$\|\int_0^\delta S(\delta-s)\big(a(T-s)\varphi(T-s)\big)\,ds\,\|_{L^2}\le\|a\|_\infty\int_0^\delta\|\varphi(T-s)\|_{L^2}\,ds\,. \tag{4.16}$$

Recall that, according to (2.21),

$$\|\varphi(t)\|_{L^2}\le e^{(T-t)\|a\|_\infty}\|\varphi_0\|_{L^2}\quad\forall t\in[0,T]\,. \tag{4.17}$$

Combining (4.16) and (4.17), we deduce that

$$\begin{aligned}&\|\int_0^\delta S(\delta-s)\big(a(T-s)\varphi(T-s)\big)\,ds\,\|_{L^2} \\ &\le\|a\|_\infty\|\varphi_0\|_{L^2}\int_0^\delta e^{s\|a\|_\infty}\,ds=\|\varphi_0\|_{L^2}\big(e^{\delta\|a\|_\infty}-1\big).\end{aligned} \tag{4.18}$$

Using (4.14), (4.15) and (4.18), we obtain (4.11). This inequality shows that (4.10) holds as soon as

$$C\delta\|\Delta y_1\|_{L^2} + \big(e^{\delta\|a\|_\infty} - 1\big)\|y_1\|_{L^2} \le \varepsilon.$$

Thus, it suffices to take

$$\delta = \min\big(T\,,\ \frac{\varepsilon}{2C\|\Delta y_1\|_{L^2}}\,,\ \frac{1}{\|a\|_\infty}\log\big(1 + \frac{\varepsilon}{2\|y_1\|_{L^2}}\big)\big). \tag{4.19}$$

Note that the unique case of interest is when

$$\frac{\|\Delta y_1\|_{L^2}}{\lambda_1} > \varepsilon \tag{4.20}$$

(recall that λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$). Otherwise, $\|y_1\|_{L^2} \le \varepsilon$ and the control $v = 0$ is such that the solution of (1.1) with $y_0 = 0$, i.e., $y \equiv 0$, satisfies

$$\|y(T) - y_1\|_{L^2} = \|y_1\|_{L^2} \le \varepsilon\,.$$

Thus, $\mathcal{C}(0, y_1; \varepsilon) = 0$ when (4.20) is violated.

For δ as in (4.19), it is not difficult to see that

$$\frac{1}{\delta} + \delta\|a\|_\infty \le C\big(\frac{1}{T} + T\|a\|_\infty + \frac{\|\Delta y_1\|_{L^2} + \|a\|_\infty\|y_1\|_{L^2}}{\varepsilon}\big) \tag{4.21}$$

In view of (4.9) written for δ as in (4.19) and (4.21), we find (1.7). This concludes the proof of Theorem 1.1.

Remark 4.1. As indicated in Remark 1.1, one can get a whole family of estimates of the form (1.7) assuming that $y_1 \in D((-\Delta)^{\gamma/2})$, for $0 < \gamma \le 2$. By $(-\Delta)^s$ we denote the fractional power of the Dirichlet Laplacian and by $D(\cdot)$ its domain. Indeed, going back to the proof of Theorem 1.1 above and more precisely to (4.14), we observe that

$$\begin{aligned} \Big|\int_\Omega y_1\big(\varphi_0 - \varphi(T-\delta)\big)\,dx\Big| &\le \|(-\Delta)^{\gamma/2} y_1\|_{L^2}\|\int_0^\delta u_t\,ds\,\|_{H^{-\gamma}} \\ &+ \|y_1\|_{L^2}\|\int_0^\delta S(\delta - s)\big(a(T-s)\varphi(T-s)\big)\,ds\,\|_{L^2}\,. \end{aligned} \tag{4.22}$$

We can estimate the last term in (4.22) as in the proof of Theorem 1.1. Let us now estimate the first term in the right hand side. We have

$$\| \int_0^\delta u_t \, ds \, \|_{H^{-\gamma}} \leq \int_0^\delta \|\Delta u\|_{H^{-\gamma}} \, ds \leq C \int_0^\delta \|u\|_{H^{2-\gamma}} \, ds$$
$$\leq C \big(\int_0^\delta s^{-(1-\gamma/2)} \, ds \big) \|\varphi_0\|_{L^2} = \frac{2C}{\gamma} \delta^{\gamma/2} \|\varphi_0\|_{L^2} \, .$$

This shows that, when $y_1 \in D((-\Delta)^{\gamma/2})$, the choice of δ in (4.19) must be modified to take

$$\delta = \min\big(T \, , \, \big(\frac{\gamma\varepsilon}{4C\|(-\Delta)^{\gamma/2} y_1\|_{L^2}}\big)^{2/\gamma} \, , \, \frac{1}{\|a\|_\infty} \log\big(1 + \frac{\varepsilon}{2\|y_1\|_{L^2}}\big)\big). \tag{4.23}$$

From (4.9) written for δ as in (4.23), we deduce that

$$\mathcal{C}(0, y_1; \varepsilon) \leq \exp\big[C\big[N(T, \|a\|_\infty) + M(\|a\|_\infty, y_1, \gamma, \varepsilon)\big]\big] \|y_1\|_{L^2}$$

for any $\gamma \in (0, 2]$, where $M(\|a\|_\infty, y_1, \gamma, \varepsilon)$ is given by (1.9) and $N(T, \|a\|_\infty)$ is given by (1.10). Obviously, when $\gamma = 2$ we recover the inequality (1.7) in Theorem 1.1.

Note that, as far as the power of ε is concerned, the best estimate is obtained when $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. In this case, the estimate is of the order of $\exp(C/\varepsilon)$.

5. On the cost of finite-approximate controllability. The goal of this Section is to prove Theorem 1.4. Let us first explain how we can construct a control v satisfying (1.21). Without loss of generality we may assume that $y_0 = 0$. We fix $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\varepsilon > 0$ and a finite dimensional subspace E of $H_0^1(\Omega)$. For some $R > 0$ large enough (that will be fixed later on), we choose first a control $v_1 \in L^2(\mathcal{O} \times (0, T))$ such that the solution $\widehat{y}$ of (1.1) with $v = v_1$ satisfies

$$\|\widehat{y}(T) - y_1\|_{L^2} \leq \frac{\varepsilon}{R} \tag{5.1}$$

and also

$$\|v_1\|_{L^2(\mathcal{O}\times(0,T))} \leq \exp\big[C\big[N(T, \|a\|_\infty) + R\frac{M(\|a\|_\infty, y_1)}{\varepsilon}\big]\big] \|y_1\|_{L^2} \, . \tag{5.2}$$

Here, $M(\|a\|_\infty, y_1)$ and $N(T, \|a\|_\infty)$ are given by (1.10). This can be made, according to Theorem 1.1. In particular, when $\|y_1\|_{L^2} \leq \varepsilon/R$, we take $v_1 = 0$.

On the other hand, there exists $v_2 \in L^2(\mathcal{O}\times(0,T))$ such that the solution $\widetilde{y}$ of (1.1) with control $v = v_2$ satisfies

$$\Pi_E\big(\widetilde{y}(T)\big) = \Pi_E\big(y_1 - \widehat{y}(T)\big) \tag{5.3}$$

and

$$\|v_2\|_{L^2(\mathcal{O}\times(0,T))} \leq C_E \|\Pi_E\big(y_1 - \widehat{y}(T)\big)\|_{L^2} \leq C_E\|\Pi_E\|\frac{\varepsilon}{R}\,, \tag{5.4}$$

with a constant C_E that can be bounded as follows:

$$C_E \leq \exp\big(T[\mu(E) + \|a\|_\infty]e^{\frac{T}{4}\|a\|_\infty} + C\,N(T, \|a\|_\infty)\big). \tag{5.5}$$

Here, $\|\Pi_E\|$ stands for the norm of the orthogonal projection from $L^2(\Omega)$ onto E. The existence of this control v_2 follows immediately from Theorem 1.6. Notice that

$$\|v_2\|_{L^2(\mathcal{O}\times(0,T))} \leq C_E\|\Pi_E\|\,\|y_1\|_{L^2} \quad \text{when} \quad \|y_1\|_{L^2} \leq \varepsilon/R \tag{5.6}$$

Let us put $v = v_1 + v_2$. Then the corresponding solution $y = \widehat{y} + \widetilde{y}$ of (1.1) satisfies, according to (5.3),

$$\Pi_E(y(T)) = \Pi_E(y_1)\,. \tag{5.7}$$

We will show that, with an appropriate choice of R, one has $\|y(T) - y_1\|_{L^2} \leq \varepsilon$. Indeed,

$$\|y(T) - y_1\|_{L^2} \leq \|\widehat{y}(T) - y_1\|_{L^2} + \|\widetilde{y}(T)\|_{L^2} \leq \frac{\varepsilon}{R} + \|\widetilde{y}(T)\|_{L^2}\,. \tag{5.8}$$

Multiplying by $\widetilde{y}$ in the heat equation satisfied by $\widetilde{y}$, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\widetilde{y}|^2\,dx + \int_\Omega |\nabla\widetilde{y}|^2\,dx &= \int_{\mathcal{O}} v_2\,\widetilde{y}\,dx - \int_\Omega a|\widetilde{y}|^2\,dx \\ &\leq \int_{\mathcal{O}} |v_2|\,|\widetilde{y}|\,dx + \|a\|_\infty \int_\Omega |\widetilde{y}|^2\,dx\,.\end{aligned}$$

By Gronwall's inequality we deduce that

$$\|\widetilde{y}\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{e^{T\|a\|_\infty}}{\sqrt{2\lambda_1}}\|v_2\|_{L^2(\mathcal{O}\times(0,T))} \tag{5.9}$$

and, combining (5.8), (5.9) and (5.4), we also deduce that

$$\|y(T) - y_1\|_{L^2} \leq \frac{\varepsilon}{R}\big(1 + C_E\|\Pi_E\|\frac{e^{T\|a\|_\infty}}{\sqrt{2\lambda_1}}\big). \tag{5.10}$$

It immediately follows that, in addition to (5.7), the inequality

$$\|y(T) - y_1\|_{L^2} \leq \varepsilon \tag{5.11}$$

holds, provided

$$R = 1 + \|\Pi_E\| C_E \frac{e^{T\|a\|_\infty}}{\sqrt{2\lambda_1}}\,. \tag{5.12}$$

Let us take R as in (5.12). With this choice of R, we always have

$$\|v_2\|_{L^2(\mathcal{O}\times(0,T))} \leq C_E\|\Pi_E\|\,\|y_1\|_{L^2}\,. \tag{5.13}$$

From (5.2) and (5.13), we also find

$$\begin{aligned}\mathcal{C}(0, y_1; \varepsilon, E) &\leq \|v_1\|_{L^2(\mathcal{O}\times(0,T))} + \|v_2\|_{L^2(\mathcal{O}\times(0,T))} \\ &\leq \exp\big[C\big[N(T, \|a\|_\infty) + R\frac{M(\|a\|_\infty, y_1)}{\varepsilon}\big]\big]\|y_1\|_{L^2} + C_E\|\Pi_E\|\,\|y_1\|_{L^2}\,.\end{aligned} \tag{5.14}$$

It is clear that the constant R in (5.12) is of the order of

$$R \sim 1 + \|\Pi_E\| \exp\big[T[\mu(E) + \|a\|_\infty]e^{\frac{T}{4}\|a\|_\infty} + N(T, \|a\|_\infty)\big]. \tag{5.15}$$

Consequently, we have (1.25) with Λ_E is as in (1.27). This ends the proof of Theorem 1.4.

Remark 5.1. Proceeding as in Remark 4.1, one can also estimate $\mathcal{C}(0, y_1; \varepsilon, E)$ when $y_1 \in D((-\Delta)^{\gamma/2})$ for $\gamma \in (0, 2]$. When $E = \{0\}$, we have $\|\Pi_E\| = 0$ and we recover the estimates (1.7) and (1.8). Notice that, again, the best behavior in ε is obtained when $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, with a bound for $\mathcal{C}\big(0, y_1; \varepsilon\big)$ of the order of $\exp(C/\varepsilon)$.

6. The particular case of the constant coefficient heat equation. This Section is concerned with the particular case in which $a \equiv$ Const. We will see that, in this case, the estimate in Theorem 1.1 can be improved. More precisely, we will obtain a bound of the cost of approximate controllability of the order of $\exp(C/\sqrt{\varepsilon})$. Also, it will be shown that the estimates in this Section are optimal in an appropriate sense.

Without any loss of generality, we can assume that $a \equiv 0$ and (again) $y_0 = 0$. The state equation is then

$$\begin{cases} y_t - \Delta y = v 1_{\mathcal{O}} & \text{in} \quad \Omega \times (0,T) \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ y(x,0) = 0 & \text{in} \quad \Omega\,. \end{cases} \tag{6.1}$$

We have the following result:

Theorem 6.1. *For any $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, $\varepsilon > 0$ and $T > 0$, one has*

$$\mathcal{C}(0, y_1; \varepsilon) \le \exp\big[C\big[1 + \frac{1}{T} + \big(\frac{(1+T)\|\Delta y_1\|_{L^2}}{\varepsilon}\big)^{1/2}\big]\big]\, \|y_1\|_{L^2}\,. \tag{6.2}$$

Proof. Let us consider the adjoint system

$$\begin{cases} -\varphi_t - \Delta \varphi = 0 & \text{in} \quad \Omega \times (0,T) \\ \ \ \varphi = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ \ \ \varphi(x,T) = \varphi_0(x) & \text{in} \quad \Omega\,, \end{cases} \tag{6.3}$$

where $\varphi_0 \in L^2(\Omega)$. We will need the following:

Proposition 6.1. *For any solution φ of* (6.3)*, one has*

$$\iint_Q \exp\big(\frac{-2C(1+T)}{T-t}\big)|\varphi|^2 dx\,dt \le \exp\big(C\big(1 + \frac{1}{T}\big)\big) \iint_{\mathcal{O}\times(0,T)} |\varphi|^2 dx\,dt \tag{6.4}$$

Assuming for the moment that Proposition 6.1 holds, let us prove Theorem 6.1. Let $\lambda_1, \lambda_2, \dots$ and $w_1, w_2, \dots$ be the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. For any $\varphi_0 = \sum_{k\ge1} a_k w_k$ in $L^2(\Omega)$, the solution of (6.3) is given by $\varphi = \sum_{k\ge1} a_k e^{-\lambda_k(T-t)} w_k$. Using (6.4), let us prove that

$$\sum_{k\ge1} \exp\big(-C\sqrt{(1+T)\lambda_k}\big) a_k^2 \le \exp\big(C\big(1+\frac{1}{T}\big)\big) \iint_{\mathcal{O}\times(0,T)} |\varphi|^2\, dx\, dt\,. \tag{6.5}$$

Indeed, we have

$$\iint_Q \exp\big(\frac{-2C(1+T)}{T-t}\big)|\varphi|^2\,dx\,dt = \sum_{k\geq 1}\big(\int_0^T e^{\frac{-2C(1+T)}{T-t}-2\lambda_k(T-t)}\,dt\big)a_k^2$$

and (6.4) can also be written in the form

$$\sum_{k\geq 1}\Big(\int_0^T e^{\frac{-2C(1+T)}{T-t}-2\lambda_k(T-t)}dt\Big)a_k^2 \leq \exp\big(C\big(1+\frac{1}{T}\big)\big)\iint_{\mathcal{O}\times(0,T)}|\varphi|^2 dx\,dt\,. \quad (6.6)$$

It is not difficult to see that, as $\lambda \to +\infty$,

$$\int_0^T e^{\frac{-2C(1+T)}{T-t}-2\lambda(T-t)}\,dt \sim \big(\frac{\pi^2 C(1+T)}{4\lambda^3}\big)^{1/4}\exp\big(-4\sqrt{C(1+T)\lambda}\big) \quad (6.7)$$

(for instance, see [1] or [2]). Hence, we can write

$$\int_0^T e^{\frac{-2C(1+T)}{T-t}-2\lambda_k(T-t)}\,dt \geq C\exp\big(-C\sqrt{(1+T)\lambda_k}\big) \quad (6.8)$$

for all k. From (6.6) and (6.8), we obtain (6.5).

Remark 6.1. From these inequalities, we deduce there exist positive constants C_1 and C_2, only depending on Ω, $\mathcal{O}$ and T, such that the following holds for the solutions of (6.3):

$$\sum_{k\geq 1} e^{-C_1\sqrt{\lambda_k}}a_k^2 \leq C_2\iint_{\mathcal{O}\times(0,T)}|\varphi|^2\,dx\,dt$$

(again, $a_1\,, a_2\,, \ldots$ are the Fourier coefficients of φ_0). This implies exact controllability for the linear, constant coefficient, heat equation (with L^2-controls) in the space of final states $y_1 = \sum_{k\geq 1} b_k w_k$, where $\sum_{k\geq 1} e^{C_1\sqrt{\lambda_k}}b_k^2 < +\infty$. This result is better than those guaranteeing exact controllability in the range of the associate semigroup. Indeed, in the definition of this space, the summability condition above is changed by: $\sum_{k\geq 1} e^{C_1\lambda_k}b_k^2 < +\infty$. A similar result to the one we have derived here was obtained by D.L. Russell in [14], when he deduced the null controllability of the heat equation as a consequence of the exact controllability of the wave equation. Notice however that our argument is valid without any geometric restriction on the control set $\mathcal{O}$.

A first consequence of (6.5) is that any z_1 spanned by a finite number of eigenfunctions w_k can be controlled exactly. A second consequence is that we can estimate the corresponding cost of controllability. More precisely, assume that

$$z_1 = \sum_{k=1}^{m} b_k w_k \,. \tag{6.9}$$

For each $\alpha > 0$, let us consider the function $J_{z_1,\alpha}$, given by

$$J_{z_1,\alpha}(\varphi_0) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |\varphi|^2 \, dx \, dt + \alpha \|\varphi_0\|_{L^2} - \int_\Omega \varphi_0 \, z_1 \, dx \tag{6.10}$$

for all $\varphi_0 \in L^2(\Omega)$. Let $\hat\varphi_{0,\alpha}$ be the unique minimizer of $J_{z_1,\alpha}$ in $L^2(\Omega)$. Then the control $v_\alpha = \hat\varphi_\alpha 1_{\mathcal{O}}$, where $\hat\varphi_\alpha$ is the solution of (6.3) with $\varphi_0 = \hat\varphi_{0,\alpha}$, is such that the associate solution y_α of (6.1) satisfies

$$\|y_\alpha(T) - z_1\|_{L^2} \le \alpha \tag{6.11}$$

and, moreover,

$$\mathcal{C}(0, z; \alpha) = \|v_\alpha\|_{L^2(\mathcal{O}\times(0,T))} \,. \tag{6.12}$$

The key point is that (6.5) implies a uniform estimate of $\|v_\alpha\|_{L^2(\mathcal{O}\times(0,T))}$. Indeed, since $J_{z_1,\alpha}$ attains its minimum at $\hat\varphi_{0,\alpha}$, we must have

$$\iint_{\mathcal{O}\times(0,T)} |\hat\varphi^\alpha|^2 \, dx \, dt + \alpha \|\hat\varphi_0^\alpha\|_{L^2} = \int_\Omega \hat\varphi_0^\alpha \, z_1 \, dx \,.$$

By putting $\hat\varphi_{0,\alpha} = \sum_{k\ge1} \hat{a_k} w_k$ and using (6.9) and (6.5), we find

$$\begin{aligned}
\iint_{\mathcal{O}\times(0,T)} |\hat\varphi_\alpha|^2 \, dx \, dt &\le \sum_{k=1}^{m} \hat{a_k} b_k \\
&\le \Big(\sum_{k\ge1} e^{-C\sqrt{(1+T)\lambda_k}} \, \hat{a_k}^2\Big)^{1/2} \Big(\sum_{k=1}^{m} e^{C\sqrt{(1+T)\lambda_k}} \, b_k^2\Big)^{1/2} \\
&\le e^{C\left(1+\frac{1}{T}\right)} \Big(\iint_{\mathcal{O}\times(0,T)} |\hat\varphi_\alpha|^2 \, dx \, dt\Big)^{1/2} \Big(\sum_{k=1}^{m} e^{C\sqrt{(1+T)\lambda_k}} \, b_k^2\Big)^{1/2} \\
&\le e^{C\left(1+\frac{1}{T}\right)} e^{C\sqrt{(1+T)\lambda_m}} \Big(\iint_{\mathcal{O}\times(0,T)} |\hat\varphi_\alpha|^2 \, dx \, dt\Big)^{1/2} \|z_1\|_{L^2}.
\end{aligned}$$

Hence,

$$\|v_\alpha\|_{L^2(\mathcal{O}\times(0,T))} = \Big(\iint_{\mathcal{O}\times(0,T)} |\hat\varphi_\alpha|^2 dx\,dt\Big)^{\frac{1}{2}} \leq e^{C(1+\frac{1}{T}+\sqrt{(1+T)\lambda_m})}\|z_1\|_{L^2}. \quad (6.13)$$

Taking (for instance) $\alpha = 1/n$ for each $n \geq 1$ and letting $n \to \infty$, we can obtain a bounded sequence of controls $v_n \in L^2(\mathcal{O} \times (0,T))$ such that the corresponding states y_n satisfy

$$\|y_n(T) - z_1\|_{L^2} \leq 1/n\,. \quad (6.14)$$

Let v be the weak limit in $L^2(\mathcal{O}\times(0,T))$ of a subsequence of $\{v_n\}$. The corresponding solution of (6.1) is such that $y(T)=z_1$. Furthermore, $\|v\|_{L^2(\mathcal{O}\times(0,T))}$ is bounded as in (6.13). Thus, we have proved that (6.1) can be controlled exactly to any final state z_1 as in (6.9), with cost

$$\mathcal{C}(0,z_1;0) \leq \exp\big(C\big(1+\frac{1}{T}+\sqrt{(1+T)\lambda_m}\big)\big)\|z_1\|_{L^2}. \quad (6.15)$$

Now, assume that $y_1 \in H^2(\Omega)\cap H^1_0(\Omega)$ and $\varepsilon > 0$ are given. Let us put

$$y_1 = \sum_{k\geq 1} b_k w_k\,, \text{ with } \sum_{k\geq 1} \lambda_k^2 b_k^2 < +\infty\,. \quad (6.16)$$

Let us introduce

$$y_{1,\varepsilon} = \sum_{k=1}^{m(\varepsilon)} b_k w_k\,, \quad (6.17)$$

where $m(\varepsilon)$ is such that $\sum_{k\geq m(\varepsilon)+1} b_k^2 \leq \varepsilon^2$. It is then clear that

$$\|y_1 - y_{1,\varepsilon}\| \leq \varepsilon \quad (6.18)$$

and, consequently,

$$\mathcal{C}(0,y_1;\varepsilon) \leq \mathcal{C}(0,y_{1,\varepsilon};0)\,. \quad (6.19)$$

From (6.15) written for $z_1 = y_{1,\varepsilon}$, (6.17) and (6.19), we obtain the estimate

$$\mathcal{C}(0,y_1;\varepsilon) \leq \exp\big(C\big(1+\frac{1}{T}+\sqrt{(1+T)\lambda_{m(\varepsilon)}}\big)\big)\|y_1\|_{L^2}. \quad (6.20)$$

Notice that (6.20) must hold whenever $m(\varepsilon)$ is such that $\sum_{k\geq m(\varepsilon)+1} b_k^2 \leq \varepsilon^2$. We are now going to make a particular choice of $m(\varepsilon)$ which leads to (6.2). First, observe that, as in the proof of Theorem 1.1 in Section 4, the unique case of interest is when

$$\frac{\|\Delta y_1\|_{L^2}}{\lambda_1} > \varepsilon\,. \tag{6.21}$$

Let $m(\varepsilon)$ be the first integer m satisfying

$$\frac{\|\Delta y_1\|_{L^2}}{\lambda_{m+1}} \leq \varepsilon\,. \tag{6.22}$$

Because of (6.21), this is well defined. For this choice of $m(\varepsilon)$, we have

$$\sum_{k\geq m(\varepsilon)+1} b_k^2 \leq \frac{1}{\lambda_{m(\varepsilon)+1}^2} \sum_{k\geq 1} \lambda_k^2 b_k^2 \leq \frac{\|\Delta y_1\|_{L^2}^2}{\lambda_{m(\varepsilon)+1}^2}$$

and, consequently, (6.20) has to be satisfied. We also have

$$\sqrt{\lambda_{m(\varepsilon)}} < \Big(\frac{\|\Delta y_1\|_{L^2}}{\varepsilon}\Big)^{1/2}. \tag{6.23}$$

From (6.20) and (6.23), we obtain (6.2). This proves Theorem 6.1.

Let us now prove Proposition 6.1.

Proof of Proposition 6.1. We start from the Carleman estimate (2.5) written for an arbitrary solution φ of (6.3) and $s \geq s_1$ (recall (2.6)). Obviously, we have:

$$\iint_Q \rho^{-2s} t^{-3}(T-t)^{-3}|\varphi|^2\,dx\,dt \leq C_* \iint_{\mathcal{O}\times(0,T)} \rho^{-2s} t^{-3}(T-t)^{-3}|\varphi|^2\,dx\,dt\,.$$

By virtue of Lemma 2.1, we know that

$$\|\rho^{-2s} t^{-3}(T-t)^{-3}\|_\infty \leq 2^6 T^{-6} \exp(-CsT^{-2})\,,$$

whenever s is as in (2.12). In particular, this gives:

$$\iint_Q \rho^{-2s} t^{-3}(T-t)^{-3}|\varphi|^2\,dx\,dt \leq CT^{-6} \iint_{\mathcal{O}\times(0,T)} |\varphi|^2\,dx\,dt\,. \tag{6.24}$$

From the definition of ρ, it is not difficult to deduce that

$$\rho^{-2s}t^{-3}(T-t)^{-3} \geq CT^{-6}\exp\bigl(-CsT^{-1}(T-t)^{-1}\bigr) \tag{6.25}$$

$\forall x \in \Omega$, $\forall t \in \bigl[T/2, T\bigr]$, provided $s \geq s_4$, where s_4 is given by (2.15) (with $a \equiv 0$). Let us fix $s = s_4$. From (6.25), it is clear that

$$\rho^{-2s}t^{-3}(T-t)^{-3} \geq CT^{-6}\exp\bigl(-\frac{2C(1+T)}{T-t}\bigr) \tag{6.26}$$

$\forall x \in \Omega$, $\forall t \in \bigl[T/2, T\bigr]$. Using (6.24) and (6.26), we deduce the following:

$$\iint_{\Omega\times(T/2,T)} e^{\frac{-2C(1+T)}{T-t}}|\varphi|^2\,dx\,dt \leq C\iint_{\mathcal{O}\times(0,T)}|\varphi|^2\,dx\,dt\,. \tag{6.27}$$

On the other hand, we also have

$$\int_\Omega |\varphi(x,t)|^2\,dx \leq \int_\Omega |\varphi(x,T/2)|^2\,dx \quad \forall t \in \bigl[0,T/2\bigr]. \tag{6.28}$$

This implies

$$\iint_{\Omega\times(0,T/2)} e^{\frac{-2C(1+T)}{T-t}}|\varphi|^2\,dx\,dt \leq \bigl(\int_0^{T/2} e^{\frac{-2C(1+T)}{T-t}}\,dt\bigr)\|\varphi(T/2)\|^2\,. \tag{6.29}$$

Finally, notice that we have an observability estimate, similar to (1.15), in the time interval $\bigl[T/2, T\bigr]$. It is as follows:

$$\|\varphi(T/2)\|^2_{L^2} \leq \exp\bigl(C\bigl(1+\frac{1}{T}\bigr)\bigr)\iint_{\mathcal{O}\times(T/2,T)}|\varphi|^2\,dx\,dt\,. \tag{6.30}$$

Combining (6.27), (6.29) and (6.30), we find (6.4). This ends the proof of Proposition 6.1.

Remark 6.2. 1) There are several important differences between the approach in Section 4 and the one in this Section. In Section 4, the argument was equivalent to controlling exactly a target close to y_1 on a trajectory starting from y_1. In this Section, we have preferred to control exactly an approximation of y_1 spanned by a finite number of eigenfunctions. We have seen that, this way, a better estimate can be obtained for the corresponding cost. Unfortunately, we do not know whether this can be made in the general case of system (1.1), with $a = a(x,t)$ (see however some related comments in subsection 8.2).

2) From the arguments above, it is clear again that, assuming that $y_1 \in D((-\Delta)^{\gamma/2})$ with $0 < \gamma \le 2$, other estimates similar to (6.2) hold. As far as the power of ε is concerned, the best estimate is obtained when $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ and, in this case, the estimate is of the order of $\exp(C/\sqrt{\varepsilon})$.

At present, we are going to prove the optimality of the estimate in Theorem 6.1.

Theorem 6.2. *Assume that $\mathcal{O} \subset \Omega$ is a non-empty open set with $\mathcal{O} \neq \Omega$ and $T > 0$. Then, there exist a sequence $\{y_1^\varepsilon\}_{\varepsilon>0}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ and a constant $K > 0$ depending on Ω, $\mathcal{O}$ and T such that*

$$\|\Delta y_1^\varepsilon\|_{L^2} = 1 \quad \forall \varepsilon > 0 \tag{6.31}$$

and

$$\mathcal{C}(0, y_1^\varepsilon; \varepsilon) \ge \exp\big(K/\sqrt{\varepsilon}\big)\|y_1^\varepsilon\|_{L^2} \quad \text{as} \quad \varepsilon \to 0\,. \tag{6.32}$$

Proof. Let us consider again the adjoint system (6.3). The following holds:

Proposition 6.2. *There exists a sequence $\{\psi_0^\varepsilon\}_{\varepsilon>0}$ of $H^2(\Omega) \cap H_0^1(\Omega)$ data such that*

$$\|\psi_0^\varepsilon\|_{L^2}\Big[\frac{\|\psi_0^\varepsilon\|_{L^2}}{\|\Delta\psi_0^\varepsilon\|_{L^2}} - \varepsilon\Big] \ge \exp\big(K/\sqrt{\varepsilon}\big)\frac{\|\psi_0^\varepsilon\|_{L^2}}{\|\Delta\psi_0^\varepsilon\|_{L^2}} \quad \text{as} \quad \varepsilon \to 0 \tag{6.33}$$

for a suitable $K > 0$ depending only on Ω, $\mathcal{O}$ and T and, moreover,

$$\iint_{\mathcal{O}\times(0,T)} |\psi^\varepsilon|^2 \, dx\, dt = 1 \quad \forall \varepsilon > 0\,. \tag{6.34}$$

Here, ψ^ε is the solution of (6.3) with $\varphi_0 = \psi_0^\varepsilon$.

Assuming for the moment that Proposition 6.2 holds, let us prove Theorem 6.2. We set

$$y_1^\varepsilon = \frac{1}{\|\Delta\psi_0^\varepsilon\|_{L^2}}\,\psi_0^\varepsilon\,, \tag{6.35}$$

where the functions ψ_0^ε are furnished by Proposition 6.2. Clearly, (6.31) holds. Let $v^\varepsilon \in L^2(\mathcal{O} \times (0,T))$ be a control such that the corresponding solution y^ε of (6.1) satisfies

$$\|y^\varepsilon(T) - y_1^\varepsilon\|_{L^2} \le \varepsilon. \tag{6.36}$$

Then, multiplying by ψ^ε in the equation satisfied by y^ε, we obtain

$$\iint_{\mathcal{O}\times(0,T)} v^\varepsilon \psi^\varepsilon dx\,dt = \int_\Omega y^\varepsilon(T)\psi_0^\varepsilon dx = \int_\Omega \big(y^\varepsilon(T) - y_1^\varepsilon\big)\psi_0^\varepsilon dx + \int_\Omega y_1^\varepsilon \psi_0^\varepsilon dx. \quad (6.37)$$

According to (6.34), (6.36) and (6.37), we deduce that

$$\|v^\varepsilon\|_{L^2\left(\mathcal{O}\times(0,T)\right)} \geq \int_\Omega y_1^\varepsilon\, \psi_0^\varepsilon\, dx - \varepsilon\, \|\psi_0^\varepsilon\|_{L^2}\ . \quad (6.38)$$

By the definition (6.35) of y_1^ε, the right hand side in (6.38) can be rewritten in the form

$$\int_\Omega y_1^\varepsilon\, \psi_0^\varepsilon\, dx - \varepsilon\|\psi_0^\varepsilon\|_{L^2} = \|\psi_0^\varepsilon\|_{L^2}\Big[\frac{\|\psi_0^\varepsilon\|_{L^2}}{\|\Delta\psi_0^\varepsilon\|_{L^2}} - \varepsilon\Big]$$

and, in view of (6.33) and (6.38), we may conclude that

$$\|v^\varepsilon\|_{L^2(\mathcal{O}\times(0,T))} \geq \exp\big(K/\sqrt{\varepsilon}\big)\|y_1^\varepsilon\|_{L^2} \quad \text{as} \quad \varepsilon \to 0\,. \quad (6.39)$$

This proves Theorem 6.2, since we have shown that (6.39) must be true for any control satisfying (6.36).

Let us now proceed to prove Proposition 6.2.

Proof of Proposition 6.2. We can assume without loss of generality that $0 \in \Omega\setminus\overline{\mathcal{O}}$ and that $|x| > A$ in $\overline{\mathcal{O}} \cup \big(\mathbb{R}^d \setminus \Omega\big)$ for some $A > 0$. Let us introduce the function

$$u(x,t) = \cos\Big(\frac{Ax_1}{2t}\Big)\exp(A^2/4t)\,G(x,t)\,, \quad (6.40)$$

where G is the fundamental solution of the heat equation in $\mathbb{R}^d$, i.e.,

$$G(x,t) = (4\pi t)^{-d/2}\exp(-|x|^2/4t) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+. \quad (6.41)$$

It is easy to see that u solves the heat equation:

$$u_t - \Delta u = 0 \ \text{ in } \ \mathbb{R}^d \times \mathbb{R}^+\,. \quad (6.42)$$

Note that u is the real part of the inverse Fourier transform of $e^{A\xi_1}\widehat{G}$. Accordingly, u can be viewed as a derivative of infinite order of the fundamental solution G. For any $\varepsilon > 0$, we denote by u^ε the function

$$u^\varepsilon(x,t) = u(x,t+\delta(\varepsilon)) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad (6.43)$$

where the parameter $\delta(\varepsilon) \in (0,1)$ will be fixed below. The choice of $\delta(\varepsilon)$ will be such that, in particular, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Observe that

$$0 < C_1 \le \iint_{\mathcal{O}\times(0,T)} |u^\varepsilon|^2 \, dx \, dt \le C_2 < \infty \tag{6.44}$$

uniformly in $0 < \delta(\varepsilon) < 1$, for suitable positive constants. This is due to the fact that $|x| > A$ in $\overline{\mathcal{O}}$. We also have

$$|u^\varepsilon| \le C_3 \quad \text{on} \quad \partial\Omega \times (0,T) \tag{6.45}$$

for a suitable C_3, again independent of ε. Let us introduce the solutions h^ε of the following auxiliary problems

$$\begin{cases} h^\varepsilon_t - \Delta h^\varepsilon = 0 & \text{in} \quad \Omega \times (0,T) \\ h^\varepsilon = u^\varepsilon & \text{on} \quad \partial\Omega \times (0,T) \\ h^\varepsilon(x,0) = 0 & \text{in} \quad \Omega\,. \end{cases} \tag{6.46}$$

In view of (6.45) and the maximum principle, we deduce that

$$|h^\varepsilon| \le C_3 \quad \text{in} \quad \Omega \times (0,T) \tag{6.47}$$

for all $\varepsilon > 0$. Let us put $w^\varepsilon = u^\varepsilon - h^\varepsilon$. Then

$$\begin{cases} w^\varepsilon_t - \Delta w^\varepsilon = 0 & \text{in} \quad \Omega \times (0,T) \\ w^\varepsilon = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ w^\varepsilon(x,0) = u^\varepsilon(x,0) & \text{in} \quad \Omega\,. \end{cases} \tag{6.48}$$

In view of (6.44) and (6.47), we also have that

$$\iint_{\mathcal{O}\times(0,T)} |w^\varepsilon|^2 \, dx \, dt \le C_4 \tag{6.49}$$

uniformly on ε.

Let us introduce the normalized functions

$$\tilde{w}^\varepsilon = \Big(\iint_{\mathcal{O}\times(0,T)} |w^\varepsilon|^2 \, dx \, dt\Big)^{-1/2} w^\varepsilon\,. \tag{6.50}$$

These are well defined, since the integrals in (6.50) never vanish. Otherwise, by the unique-continuation property, we should have $w^\varepsilon \equiv 0$ for some ε, which is not obviously the case. In view of the definition (6.50), we have

$$\iint_{\mathcal{O}\times(0,T)} |\tilde{w}^\varepsilon|^2 \, dx \, dt = 1\,. \tag{6.51}$$

Let us analyze the behavior of $\tilde{w}^\varepsilon(0)$ as $\varepsilon \to 0$. We have

$$\tilde{w}^\varepsilon(0) = \Big(\iint_{\mathcal{O}\times(0,T)} |w^\varepsilon|^2\,dx\,dt\Big)^{\frac{-1}{2}} w^\varepsilon(0) = \Big(\iint_{\mathcal{O}\times(0,T)} |w^\varepsilon|^2\,dx\,dt\Big)^{\frac{-1}{2}} u(\delta(\varepsilon))\,. \tag{6.52}$$

Thus,

$$\frac{\|\tilde{w}^\varepsilon(0)\|_{L^2}}{\|\Delta\tilde{w}^\varepsilon(0)\|_{L^2}} = \frac{\|u(\delta(\varepsilon))\|_{L^2}}{\|\Delta u(\delta(\varepsilon))\|_{L^2}}\,.$$

Taking into account that $|x| > A$ in $\mathbb{R}^d \setminus \Omega$, it is easy to see that

$$\lim_{\delta(\varepsilon)\to 0} \frac{\|u(\delta(\varepsilon))\|_{L^2}}{\|u(\delta(\varepsilon))\|_{L^2(\mathbb{R}^d)}} = 1 \quad\text{and}\quad \lim_{\delta(\varepsilon)\to 0} \frac{\|\Delta u(\delta(\varepsilon))\|_{L^2}}{\|\Delta u(\delta(\varepsilon))\|_{L^2(\mathbb{R}^d)}} = 1\,.$$

Therefore, it will be sufficient to analyze the behavior of $\|u(\delta(\varepsilon))\|_{L^2(\mathbb{R}^d)}$ and $\|\Delta u(\delta(\varepsilon))\|_{L^2(\mathbb{R}^d)}$ as $\varepsilon \to 0$. Using Plancherel's identity, we find that

$$\|u(\delta)\|^2_{L^2(\mathbb{R}^d)} = \|e^{A\xi_1}e^{-\delta|\xi|^2}\|^2_{L^2(\mathbb{R}^d)} \sim C_5\delta^{-d/2}\exp(A^2/2\delta)$$

for some $C_5 > 0$ independent of δ. Also,

$$\begin{aligned}
&\|\Delta u(\delta)\|^2_{L^2(\mathbb{R}^d)} = \|\,|\xi|^2 e^{A\xi_1}e^{-\delta|\xi|^2}\|^2_{L^2(\mathbb{R}^d)}\\
&= \int_{\mathbb{R}^d} |\xi|^4 e^{2A\xi_1}e^{-2\delta|\xi|^2}\,d\xi = \int_{\mathbb{R}^d} |\xi|^4 \exp\Big(-2\delta\Big|\xi - \frac{Ae_1}{2\delta}\Big|^2 + \frac{A^2}{2\delta}\Big)\,d\xi\\
&= e^{A^2/2\delta}\int_{\mathbb{R}^d} \Big|\eta + \frac{Ae_1}{2\delta}\Big|^4 \exp(-2\delta|\eta|^2)\,d\eta\\
&= e^{A^2/2\delta}\int_{\mathbb{R}^d} \Big[\frac{A^4}{16\delta^4} + \frac{A^3}{2\delta^3}\eta_1 + \cdots\Big] \exp(-2\delta|\eta|^2)\,d\eta\\
&\sim C_6\delta^{-d/2-4}\exp(A^2/2\delta) \quad\text{as}\quad \delta\to 0\,.
\end{aligned}$$

In the computations above, we have used the notation $e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$. Of course, C_6 is independent of δ. Therefore,

$$\frac{\|u(\delta(\varepsilon))\|_{L^2}}{\|\Delta u(\delta(\varepsilon))\|_{L^2}} \sim C_7\delta(\varepsilon)^2\,. \tag{6.53}$$

On the other hand,

$$\|u(\delta(\varepsilon))\|_{L^2} \sim C_5\delta(\varepsilon)^{-d/4}\exp(A^2/4\delta(\varepsilon))\,. \tag{6.54}$$

Combining (6.53) and (6.54) and choosing $\delta(\varepsilon) = C_8\sqrt{\varepsilon}$, where $C_8 > 0$ is large enough, it follows that

$$\|u(\delta(\varepsilon))\|_{L^2}\Big[\frac{\|u(\delta(\varepsilon))\|_{L^2}}{\|\Delta u(\delta(\varepsilon))\|_{L^2}} - \varepsilon\Big] \geq C_9\varepsilon^{7d/8}\exp(A^2/4C_8\sqrt{\varepsilon})\,.$$

Reversing the sense of time, i.e., setting $\psi^\varepsilon(x,t) = u^\varepsilon(x, T-t)$, $\psi_0^\varepsilon(x) = u^\varepsilon(x,0) = u(x,\delta(\varepsilon))$, we obtain a sequence in $H^2(\Omega)\cap H_0^1(\Omega)$ satisfying the conditions of Proposition 6.2.

7. The speed of convergence of a penalization approach. In this section, we recall an algorithm that can be used to construct a sequence of controls whose associated final states converge to a prescribed y_1. The algorithm is obtained via penalization and was introduced in [11] (see also [5] and [18]). Let us consider again system (1.1) with $y_0 = 0$. For every $v \in L^2(\mathcal{O}\times(0,T))$, let us denote by Lv the corresponding final state $y(T)$. The mapping $L : L^2(\mathcal{O}\times(0,T)) \mapsto L^2(\Omega)$ is linear and bounded. Assume y_1 is given in $H^2(\Omega)\cap H_0^1(\Omega)$ and set

$$J_n(v) = \frac{1}{2}\iint_{\mathcal{O}\times(0,T)}|v|^2dx\,dt + \frac{n}{2}\|Lv - y_1\|_{L^2}^2 \quad \forall v \in L^2(\mathcal{O}\times(0,T)) \quad (7.1)$$

for each $n \geq 1$. We are going to minimize each J_n over $L^2(\mathcal{O}\times(0,T))$. In this way, we will obtain a sequence $\{v_n\}$ such that the corresponding Lv_n converge to y_1 as $n \to +\infty$. Notice that J_n is C^1 and strictly convex on $L^2(\mathcal{O}\times(0,T))$ and satisfies

$$\lim_{\|v\|_{L^2(\mathcal{O}\times(0,T))}\to+\infty} J_n(v) = +\infty\,.$$

Consequently, for each n there exists exactly one $v_n \in L^2(\mathcal{O}\times(0,T))$ such that $J_n(v_n) \leq J_n(v)\ \forall v \in L^2(\mathcal{O}\times(0,T))$. Furthermore, $J_n'(v_n) = 0$, that is to say

$$v_n + nL^*(Lv_n - y_1) = 0 \quad \forall n \geq 1\,, \quad (7.2)$$

where $L^* : L^2(\Omega) \mapsto L^2(\mathcal{O}\times(0,T))$ is the adjoint of L.

Let us denote by y_n the solution of (1.1) with $v = v_n$ (and $y_0 = 0$). Notice that $L^*\varphi_0$ is, for each $\varphi_0 \in L^2(\Omega)$, the restriction to $\mathcal{O}\times(0,T)$ of φ, where φ solves the corresponding adjoint system (1.14). Consequently, we have from (7.2) that v_n is the restriction to $\mathcal{O} \times (0,T)$ of $-n\varphi_n$, where

$$\begin{cases} -\partial_t\varphi_n - \Delta\varphi_n + a\varphi_n = 0 & \text{in} \quad Q \\ \varphi_n = 0 & \text{on} \quad \Sigma \\ \varphi_n(x,T) = y_n(x,T) - y_1(x) & \text{in} \quad \Omega\,. \end{cases} \tag{7.3}$$

Of course, we also have

$$\begin{cases} \partial_t y_n - \Delta y_n + a y_n = -n\varphi_n 1_{\mathcal{O}} & \text{in} \quad Q \\ y_n = 0 & \text{on} \quad \Sigma \\ y_n(x,0) = 0 & \text{in} \quad \Omega\,. \end{cases} \tag{7.4}$$

Here and in the sequel, we denote by ∂_t partial differentiation with respect to time.

In this Section, the main result is the following:

Theorem 7.1. *For all $n \geq 1$, one has*

$$\|y_n(T) - y_1\|_{L^2} \leq \big(CM(\|a\|_\infty, y_1) + \exp\big(CN(T, \|a\|_\infty)\big)\|y_1\|_{L^2}\big)\frac{1}{\log n} \tag{7.5}$$

and

$$\|v_n\|_{L^2(\mathcal{O}\times(0,T))} \leq \big(CM(\|a\|_\infty, y_1) + \exp\big(CN(T, \|a\|_\infty)\big)\|y_1\|_{L^2}\big)\frac{\sqrt{n}}{\log n}, \tag{7.6}$$

where $M(\|a\|_\infty, y_1)$ and $N(T, \|a\|_\infty)$ are given by (1.26).

Proof. Let us put $M = M(\|a\|_\infty, y_1)$ and $N = N(T, \|a\|_\infty)$. We will first see that

$$\|y_n(T) - y_1\|_{L^2}^2 \leq \frac{1}{n}\exp\big(C\big(N + \frac{M}{\varepsilon}\big)\big)\|y_1\|_{L^2}^2 + \varepsilon^2 \tag{7.7}$$

for all $\varepsilon > 0$ and for all $n \geq 1$.

Thus, let us fix $n \geq 1$. For each $\varepsilon > 0$, there exists $v_\varepsilon \in L^2(\mathcal{O} \times (0,T))$ such that the corresponding solution to (1.1) satisfies $\|y(T) - y_1\|_{L^2} \leq \varepsilon$. This means that

$$\|Lv_\varepsilon - y_1\|_{L^2} \leq \varepsilon\,. \tag{7.8}$$

We also know from Theorem 1.1 that v_ε can be chosen such that

$$\|v_\varepsilon\|_{L^2(\mathcal{O}\times(0,T))} \le \exp\big(C\big(N + \frac{M}{\varepsilon}\big)\big)\|y_1\|_{L^2}\,. \tag{7.9}$$

We then have

$$J_n(v_n) \le J_n(v_\varepsilon) \le \frac{1}{2}\exp\big(C\big(N + \frac{M}{\varepsilon}\big)\big)\|y_1\|^2_{L^2} + \frac{n}{2}\varepsilon^2\,. \tag{7.10}$$

For each $n \ge 1$, let us set

$$f_n(\varepsilon) = \frac{1}{n}\exp\big(C\big(N + \frac{M}{\varepsilon}\big)\big)\|y_1\|^2_{L^2} + \varepsilon^2 \quad \forall \varepsilon > 0\,.$$

Using that $J_n(v_n) \ge \frac{n}{2}\|Lv_n - y_1\|^2_{L^2}$ in (7.10), after multiplication by $\frac{2}{n}$, we obtain that

$$\|y_n(T) - y_1\|^2_{L^2} \le f_n(\varepsilon)\,, \tag{7.11}$$

which is (7.7). For instance, let us put $\varepsilon = 2CM/\log n$ in the inequality (7.7). We find $f_n(\varepsilon) = \big[4C^2M^2 + \frac{(\log n)^2}{n^{1/2}}e^{CN}\|y_1\|^2_{L^2}\big]\frac{1}{(\log n)^2}$ and this can be bounded above by $\big[4C^2M^2 + C_0e^{CN}\|y_1\|^2_{L^2}\big]\frac{1}{(\log n)^2}$, with C_0 independent of n. This, together with (7.11), yields (7.5). Then, (7.6) is an immediate consequence of (7.1) and (7.10) written for $\varepsilon = 2CM/\log n$.

Remark 7.1. 1) We have used the regularity of y_1 in (7.9). Of course, it is possible to generalize (7.5) for y_1 in $D((-\Delta)^{\gamma/2})(0 < \gamma \le 2)$. Namely, one has

$$\|y_n(T) - y_1\|_{L^2} \le \frac{C_\gamma}{(\log n)^{\gamma/2}} \quad \forall n \ge 1, \tag{7.12}$$

where C_γ depends on $\Omega, \mathcal{O}, T, \gamma, \|y_1\|_{H^\gamma}$ and $\|a\|_{L^\infty(Q)}$.

2) When $a \equiv 0$, an argument similar to the one in the proof but using Theorem 6.1 instead of Theorem 1.1, leads to the following estimates:

$$\|y_n(T) - y_1\|_{L^2} \le \big(C(1+T)\|\Delta y_1\|_{L^2} + e^{C\left(1+\frac{1}{T}\right)}\|y_1\|_{L^2}\big)\frac{1}{(\log n)^2} \tag{7.13}$$

and

$$\|v_n\|_{L^2(\mathcal{O}\times(0,T))} \le \big(C(1+T)\|\Delta y_1\|_{L^2} + e^{C\left(1+\frac{1}{T}\right)}\|y_1\|_{L^2}\big)\frac{\sqrt{n}}{(\log n)^2}\,. \tag{7.14}$$

3) In view of (7.5) and (7.13), we see that the coupled systems (7.3)–(7.4) provide a sequence of controls v_n such that the corresponding final states converge "slowly" to y_1. For each n, the control v_n is roughly of size $\sqrt{n}$ and the associate error $y_n(T) - y_1$ is of the order of $(\log n)^{-1}$ and $(\log n)^{-2}$, respectively.

4) It is possible to prove that the sequence $\{y_n(T)\}$ generated by (7.3)–(7.4) converges to y_1 without using directly the approximate controllability of (1.1). The argument is given in [11] and relies mainly on the identity

$$\int_\Omega y_n(T)\big(y_n(T) - y_1\big)dx + n \iint_{\mathcal{O}\times(0,T)} |\varphi_n|^2\, dx\, dt = 0\,. \tag{7.15}$$

Nevertheless, one has to use in an essential way the unique-continuation property of solutions of the adjoint system: If φ solves

$$\begin{cases} -\varphi_t - \Delta\varphi + a\varphi = 0 & \text{in} \quad Q \\ \quad \varphi = 0 & \text{on} \quad \Sigma \end{cases}$$

and $\varphi = 0$ in $\mathcal{O}\times(0,T)$ then, necessarily, $\varphi \equiv 0$. Furthermore, this approach does not provide any estimate on the speed of convergence of $y_n(T)$ and only leads to a bound of $\|v_n\|_{L^2(\mathcal{O}\times(0,T))}$ of the order of $\sqrt{n}$.

8. Further comments and results.

8.1. Equations with convective terms. The results of this paper can be extended to more general equations of the form

$$\begin{cases} y_t - \Delta y + \operatorname{div}(By) + ay = v1_{\mathcal{O}} & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0(x) & \text{in} \quad \Omega\,, \end{cases} \tag{8.1}$$

where $a \in L^\infty(Q)$ and $B \in L^\infty(Q;\mathbb{R}^d)$.

To do that, it is sufficient to obtain appropriate observability estimates for the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi - B\cdot\nabla\varphi + a\varphi = 0 & \text{in} \quad Q \\ \quad \varphi = 0 & \text{on} \quad \Sigma\,. \end{cases} \tag{8.2}$$

This can be done following the methods developed in this paper with minor changes. More precisely, arguing as in Section 2, we find that

$$\|\varphi(0)\|^2_{L^2} \le \exp\Big(C\big(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|^{2/3}_\infty + T^2\|B\|^2_\infty\big)\Big) \iint_{\mathcal{O}\times(0,T)} |\varphi|^2 dx\, dt \tag{8.3}$$

for any solution of (8.2) and for all $a \in L^\infty(Q)$, $B \in L^\infty(Q;\mathbb{R}^d)$. Then, an argument similar to the one in Section 4, leads to a family of estimates of the cost of approximate controllability in the case of system (8.2). For instance, when $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, one obtains the following:

$$\mathcal{C}(0, y_1; \varepsilon) \leq \exp\big[C\big[N(T, \|a\|_\infty, \|B\|_\infty) + \frac{M(\|a\|_\infty, \|B\|_\infty, y_1)}{\varepsilon}\big]\big] \|y_1\|_{L^2} \,,$$

where $M(\|a\|_\infty, \|B\|_\infty, y_1) = \|\Delta y_1\|_{L^2} + \|a\|_\infty \|y_1\|_{L^2} + \|B\|_\infty \|y_1\|_{H_0^1}$ and $N(T, \|a\|_\infty, \|B\|_\infty) = 1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{2/3} + (1+T^2)\|B\|_\infty^2 \,.$

The situation is different when the state equation is of the form

$$\begin{cases} y_t - \Delta y + B \cdot \nabla y = 0 & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y_0(x) & \text{in} \quad \Omega \,. \end{cases} \tag{8.4}$$

Obviously, when $B \in W^{1,\infty}(Q;\mathbb{R}^d)$, system (8.4) can be written in the form (8.1) and the methods of this paper apply. However, when B is only assumed to be in $L^\infty(Q;\mathbb{R}^d)$, the situation is much more delicate. Indeed, the adjoint systems take the form

$$\begin{cases} -\varphi_t - \Delta\varphi - \operatorname{div}(B\varphi) = 0 & \text{in} \quad Q \\ \quad \varphi = 0 & \text{on} \quad \Sigma \,, \end{cases} \tag{8.5}$$

and, therefore, the global Carleman inequality of Proposition 2.1 cannot be applied without further regularity assumptions on B. Roughly speaking, in order to address system (8.5) we would need the H^{-1}-version of the L^2-Carleman inequality in Proposition 2.1. The unique-continuation problem for system (8.5) with $B \in L^\infty(Q;\mathbb{R}^d)$ has recently been solved by C. Fabre in [3]. Unfortunately, the methods in [3] do not seem to provide explicit observability estimates.

8.2. Potentials of a particular kind. Theorem 6.1 remains essentially true, with a very similar proof, for nonvanishing potentials of the form $a(x,t) = a_1(x) + a_2(t)$, where $a_1 \in L^\infty(\Omega)$ and $a_1 \in L^\infty(0,T)$. The unique change is that, in (6.2), the constant C also depends on a. We do not know whether the same result holds for general potentials $a = a(x,t)$.

8.3. L^p-potentials. All along this paper we have assumed the potential a to be in $L^\infty(Q)$. The results of this paper may be extended to the case where $a \in L^p(Q)$, with p large enough (depending on the spatial dimension d). Results of this kind become important when dealing with semilinear control problems in which the nonlinearity grows at infinity in a superlinear way, as in [6]. This will be discussed in a future paper.

8.4. L^p-estimates of the cost. From the Carleman inequality in Proposition 2.1, it is also possible to deduce observability estimates similar to (1.15) but with an integral in the right hand side of the form

$$\iint_{\mathcal{O}\times(0,T)} |\varphi|^{p'}\, dx\, dt\,, \quad p' < 2\,.$$

This leads to estimates of the cost of approximate controllability in $L^p(\mathcal{O}\times(0,T))$ for suitable $p > 2$. Again, this plays a crucial role in the analysis of some superlinear parabolic systems.

8.5. Semilinear state equations. Let us consider the semilinear system:

$$\begin{cases} y_t - \Delta y + f(y) = v 1_{\mathcal{O}} & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(0) = y_0 & \text{in} \quad \Omega\,. \end{cases} \tag{8.6}$$

When f is globally Lipschitz, the approximate controllability of (8.6) was proved in [4]. The methods of this paper, combined with the fixed point scheme of [4], can be applied to obtain estimates on the cost of approximate and finite-approximate controllability for system (8.6).

As mentioned in the previous subsections, one can also deal with nonlinearities that grow in a slight superlinear way, see [6]. Namely, with nonlinearities satisfying

$$|f(s)| \le C|s| \log |s| \quad \text{for} \quad |s| \ge s_0 > 0, \tag{8.7}$$

but without any sign condition of f.

8.6. Boundary control. The methods of this paper can also be applied to obtain estimates on the cost of controllability when the control acts on an open subset of the boundary of Ω.

8.7. Sharper upper bounds. In Theorem 1.1 we have obtained an upper bound on the cost of approximate controllability of the order of $\exp(C/\varepsilon)$ when $y_0 = 0$ and $y_1 \in H^2(\Omega) \cap H_0^1(\Omega)$. We know that every y_1 belonging to the range of the uncontrolled semigroup is exactly reachable. More precisely, let

$$R(T) = \left\{ y(T); y \text{ solves (1.1) with } y_0 \in L^2(\Omega) \text{ and } v \equiv 0 \right\}. \tag{8.8}$$

Then, every $y_1 \in R(T)$ is exactly reachable. In other words, for any $y_1 \in R(T)$ there exists $v \in L^2(\mathcal{O} \times (0,T))$ such that the solution of (1.1) with $y_0 \equiv 0$ satisfies $y(T) = y_1$. Consequently, the associate cost of approximate controllability $\mathcal{C}(0, y_1; \varepsilon)$ remains bounded as $\varepsilon \to 0$.

This can also be seen by analyzing the behavior of the functional $J_{y_1,\varepsilon}$ in (1.16). Indeed, assume $y_1 = z(T)$, with z being the solution of

$$\begin{cases} z_t - \Delta z + az = 0 & \text{in} \quad Q \\ z = 0 & \text{on} \quad \Sigma \\ z(x,0) = z_0(x) & \text{in} \quad \Omega\,. \end{cases} \tag{8.9}$$

It is then easy to see that

$$\int_\Omega y_1 \, \varphi_0 \, dx = - \int_\Omega z_0 \, \varphi(0) \, dx \tag{8.10}$$

for any solution φ of the adjoint system. Thus, $J_{y_1,\varepsilon}$ can be rewritten in the form

$$J_{y_1,\varepsilon}(\varphi_0) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |\varphi|^2 \, dx \, dt + \varepsilon \|\varphi_0\|_{L^2} + \int_\Omega z_0 \, \varphi(0) \, dx \tag{8.11}$$

and

$$J_{y_1,\varepsilon}(\varphi_0) \geq J_{y_1}(\varphi_0) \quad \forall \varphi_0 \in L^2(\Omega) \quad \forall \varepsilon > 0\,, \tag{8.12}$$

where

$$J_{y_1}(\varphi_0) = \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} |\varphi|^2 \, dx \, dt + \int_\Omega z_0 \varphi(0) \, dx\,. \tag{8.13}$$

It is not difficult to see that the functional J_{y_1} is coercive. This is a consequence of the observability estimate (1.15). Thus, the functionals $J_{y_1,\varepsilon}$ are uniformly coercive with respect to $\varepsilon > 0$. As a consequence, if $\hat{\varphi}_{0,\varepsilon}$ minimizes $J_{y_1,\varepsilon}$ and $\hat{\varphi}_\varepsilon$ is the solution of the corresponding adjoint system (1.14), it follows that

$$\iint_{\mathcal{O}\times(0,T)} |\hat{\varphi}_\varepsilon|^2\,dx\,dt$$

is uniformly bounded. Recalling that $v_\varepsilon = \hat{\varphi}_\varepsilon 1_{\mathcal{O}}$ is the control of minimal norm, we deduce that $\mathcal{C}(0, y_1; \varepsilon)$ is bounded as $\varepsilon \to 0$.

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Appendix. *Proof of the global Carleman inequality.* With the notation of Section 2, assume $q \in Z$ and $s > 0$. Let us set $\psi = \rho^{-s}q$, $f = \partial_t q + \Delta q$ and $g = \rho^{-s} f$. Recall that $\rho = e^\phi$ where $\phi = \rho^1(x)/[t(T-t)], \rho^1 = e^{\lambda\bar{\beta}} - e^{\lambda\beta^0}$ with β^0 given by (2.2), $\bar{\beta} = \frac{5}{4}\max_{\overline{\Omega}}\beta^0$ and $\lambda > 0$. The constant λ will be fixed later on.

We will denote by $\sigma_0(\Omega, \mathcal{O})$, $\lambda_0(\Omega, \mathcal{O})$, ... various constants (large enough) only depending on Ω and $\mathcal{O}$. The equality

$$\rho^{-s}\big(\partial_t(\rho^s\psi) + \Delta(\rho^s\psi)\big) = g \tag{A.1}$$

can be written in the form

$$M_1\psi + M_2\psi = g - s(\Delta\phi)\psi\,, \tag{A.2}$$

where

$$M_1\psi = \partial_t\psi + 2s\nabla\phi\cdot\nabla\psi, \quad M_2\psi = \Delta\psi + s^2|\nabla\phi|^2\psi + s\big(\partial_t\phi\big)\psi\,. \tag{A.3}$$

From (A.2), we obtain

$$\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + 2\big(M_1\psi, M_2\psi\big)_{L^2(Q)} = \|g - s(\Delta\phi)\psi\|^2_{L^2(Q)} \tag{A.4}$$

Let us compute the scalar product in the left side of (A.4). We have

$$\big(M_1\psi, M_2\psi\big)_{L^2(Q)} = I_1 + 2sI_2 + s^2 I_3 + 2s^3 I_4 + sI_5 + 2s^2 I_6\,,$$

where

$$I_1 = \iint_Q (\partial_t\psi)\Delta\psi = -\frac{1}{2}\iint_Q \partial_t|\nabla\psi|^2 = 0\,,$$
$$I_2 = \iint_Q (\nabla\phi\cdot\nabla\psi)\Delta\psi = \iint_Q \big(-\partial_i\partial_j\phi + \frac{1}{2}(\Delta\phi)\delta_{ij}\big)\partial_i\psi\partial_j\psi + \frac{1}{2}\iint_\Sigma \big(\partial_n\phi\big)|\partial_n\psi|^2 d\Gamma\,dt$$

(here, the usual convention of repeated indices has been used; $\partial_n\phi$ and $\partial_n\psi$ denote outward normal derivatives; unless otherwise specified, we are integrating with respect to the

Lebesgue measure in Q),

$$
\begin{aligned}
I_3 &= \iint_Q (\partial_t\psi)|\nabla\phi|^2\psi = -\frac{1}{2}\iint_Q \partial_t(|\nabla\phi|^2)|\psi|^2\,,\\
I_4 &= \iint_Q (\nabla\phi\cdot\nabla\psi)|\nabla\phi|^2\psi = \iint_Q \big(-\partial_i\partial_j\phi - \frac{1}{2}(\Delta\phi)\delta_{ij}\big)\partial_i\phi\partial_j\phi|\psi|^2\,,\\
I_5 &= \iint_Q (\partial_t\psi)\big(\partial_t\phi\big)\psi = -\frac{1}{2}\iint_Q \big(\partial_t^2\phi\big)|\psi|^2,\\
I_6 &= \iint_Q (\nabla\phi\cdot\nabla\psi)\big(\partial_t\phi\big)\psi = -\frac{1}{2}\iint_Q \big(\partial_i\big(\partial_t\phi\big)\partial_i\phi + \big(\partial_t\phi\big)\Delta\phi\big)|\psi|^2\,.
\end{aligned}
$$

Thus, we have from (A.4) the following identity:

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + 4s^3\iint_Q \big(-\partial_i\partial_j\phi\big)\partial_i\phi\partial_j\phi|\psi|^2 + 2s\iint_\Sigma \big(\partial_n\phi\big)|\partial_n\psi|^2\,d\Gamma\,dt\\
&= \|g - s(\Delta\phi)\psi\|^2_{L^2(Q)} + 2s^2\iint_Q \big(2\partial_i\big(\partial_t\phi\big)\big(\partial_i\phi\big) + \big(\partial_t\phi\big)\Delta\phi\big)|\psi|^2 \qquad \text{(A.5)}\\
&+ 4s\iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi + s\iint_Q \big(\partial_t^2\phi\big)|\psi|^2 - 2\iint_Q \big(s(\Delta\phi)|\nabla\psi|^2 - s^3(\Delta\phi)|\nabla\phi|^2|\psi|^2\big).
\end{aligned}
$$

Since $\partial_n\beta^0 \le 0$ on $\partial\Omega$, one has $\partial_n\phi \ge 0$ on Σ and the boundary integral in the left side of (A.6) is nonnegative. Let us put

$$
\alpha(x,t) = \frac{e^{\lambda\beta^0(x)}}{t(T-t)}\,. \qquad \text{(A.6)}
$$

Then, for some $\lambda_0(\Omega,\mathcal{O}) \ge 1$, we have

$$
-\big(\partial_i\partial_j\phi\big)\partial_i\phi\partial_j\phi \ge C(\Omega,\mathcal{O})\lambda|\nabla\alpha|^3 \quad \text{in } Q_0 = (\Omega\setminus\mathcal{O})\times(0,T) \quad \forall\lambda\ge\lambda_0(\Omega,\mathcal{O}). \qquad \text{(A.7)}
$$

On the other hand, it is not difficult to check that

$$
|\Delta\phi|^2 = \frac{e^{2\lambda\beta^0}}{t^2(T-t)^2}\big[\lambda\Delta\beta^0 + \lambda^2|\nabla\beta^0|^2\big]^2 \le \frac{C(\Omega,\mathcal{O})\lambda^4 e^{2\lambda\beta^0}}{t^2(T-t)^2} \le C(\Omega,\mathcal{O})T^2\lambda^4\alpha^3, \qquad \text{(A.8)}
$$

$$
\big|\partial_i\big(\partial_t\phi\big)\partial_i\phi\big| = \frac{|T-2t|\lambda^2 e^{2\lambda\beta^0}}{t^3(T-t)^3}|\nabla\beta^0|^2 \le C(\Omega,\mathcal{O})T\lambda^2\alpha^3\,, \qquad \text{(A.9)}
$$

$$
|(\partial_t\phi)\Delta\phi| = \frac{|T-2t|e^{\lambda\beta^0}\big(e^{\lambda\bar\beta} - e^{\lambda\beta^0}\big)}{t^3(T-t)^3}\,\big|\lambda\Delta\beta^0 + \lambda^2|\nabla\beta^0|^2\big| \le C(\Omega,\mathcal{O})T\lambda^2\alpha^3, \qquad \text{(A.10)}
$$

$$
\big|\partial_t^2\phi\big| = \frac{2\,\big|T^2 - 3Tt + 3t^2\big|}{t^3(T-t)^3}\big(e^{\lambda\bar\beta} - e^{\lambda\beta^0}\big) \le \frac{C(\Omega,\mathcal{O})T^2 e^{\frac{3}{2}\lambda\beta^0}}{t^3(T-t)^3} \le C(\Omega,\mathcal{O})T^2|\alpha|^3. \qquad \text{(A.11)}
$$

In (A.10) and (A.11), we have used that $e^{2\lambda\bar\beta} \le e^{3\lambda\beta^0}$. This is implied by the fact that

$$
2\bar\beta = \tfrac{5}{2}\max_{\overline\Omega}\beta^0 \le 3\min_{\overline\Omega}\beta^0\,, \qquad \text{(A.12)}
$$

which is a consequence of the choice of K_0 in (2.2). Using (A.7)–(A.11) in (A.5), we obtain:

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + Cs^3\lambda \iint_{Q_0} |\nabla\alpha|^3|\psi|^2 \\
&\le 2\|g\|^2_{L^2(Q)} + Cs^2 \iint_Q \big[|\Delta\phi|^2 + |\partial_i(\partial_t\phi)\partial_i\phi| + |(\partial_t\phi)\Delta\phi|\big]|\psi|^2 + Cs\iint_Q \big|\partial_t^2\phi\big|\,|\psi|^2 \\
&- 2\iint_Q \big[s\Delta\phi|\nabla\psi|^2 - s^3\Delta\phi|\nabla\phi|^2\big]|\psi|^2 + 4s\iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi \\
&\le 2\|g\|^2_{L^2(Q)} + Cs^2(T^2\lambda^4 + T\lambda^2)\iint_Q |\alpha|^3|\psi|^2 + CsT^2\iint_Q |\alpha|^3|\psi|^2 \\
&- 2\iint_Q \big[s\Delta\phi|\nabla\psi|^2 - s^3\Delta\phi|\nabla\phi|^2|\psi|^2\big]|\psi|^2 + 4s\iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi\,.
\end{aligned}$$

Notice that $|\nabla\alpha| \ge C\lambda\alpha$ in Q_0. Consequently, we also have

$$\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + Cs^3\lambda\iint_{Q_0} |\nabla\alpha|^3|\psi|^2 \le 2\|g\|^2_{L^2(Q)} \qquad \text{(A.13)}\\
&+ Cs^3\lambda^4\iint_{\mathcal{O}\times(0,T)} \alpha^3|\psi|^2 - 2s\iint \big[s\Delta\phi|\nabla\psi|^2 - s^3\Delta\phi|\nabla\phi|^2|\psi|^2\big] + 4s\iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi\,,
\end{aligned}$$

provided we have taken $s \ge \sigma_0(\Omega,\mathcal{O})(T+T^2)$ and $\lambda \ge \lambda_0(\Omega,\mathcal{O})$. Let us set

$$S = \iint_Q \big(s(\Delta\phi)|\nabla\psi|^2 - s^3(\Delta\phi)|\nabla\phi|^2|\psi|^2\big). \qquad \text{(A.14)}$$

Then, we have from (A.2) that

$$\begin{aligned}
S &= s\iint_Q (\Delta\phi)|\nabla\psi|^2 + s\iint_Q \big(M_1\psi + \Delta\psi + s\big(\partial_t\phi\big)\psi - g + s(\Delta\phi)\psi\big)(\Delta\phi)\psi \\
&= \frac{s}{2}\iint_Q \big(\Delta^2\phi\big)|\psi|^2 + s^2\iint_Q \big(\partial_t\phi\big)(\Delta\phi)|\psi|^2 + s\iint_Q \big(M_1\psi - g + s(\Delta\phi)\psi\big)(\Delta\phi)\psi\,.
\end{aligned}$$

Hence,

$$|S| \le \frac{1}{4}\|M_1\psi\|^2_{L^2(Q)} + \|g\|^2_{L^2(Q)} + Cs^2\iint_Q \big[|\Delta\phi|^2 + |(\partial_t\phi)\Delta\phi|\big]|\psi|^2 + Cs\iint_Q \big(\Delta^2\phi\big)|\psi|^2\,. \qquad \text{(A.15)}$$

On the other hand,

$$\big|\Delta^2\phi\big| = \frac{|\Delta^2(e^{\lambda\beta^0})|}{t(T-t)} \le \frac{C\lambda^4 e^{\lambda\beta^0}}{t(T-t)} \le \frac{C\lambda^4T^4e^{\lambda\beta^0}}{t^3(T-t)^3} \le C\lambda^4T^4\alpha^3\,. \qquad \text{(A.16)}$$

Using (A.8), (A.10) and (A.16) in (A.15), we deduce that

$$|S| \le \frac{1}{4}\|M_1\psi\|^2_{L^2(Q)} + \|g\|^2_{L^2(Q)} + Cs^2(T^2\lambda^4 + T\lambda^2)\iint_Q \alpha^3|\psi|^2 + CsT^4\lambda^4\iint_Q \alpha^3|\psi|^2\,. \qquad \text{(A.17)}$$

From (A.13), (A.14) and (A.17), it is immediate that

$$\begin{aligned}&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + Cs^3\lambda \iint_{Q_0} |\nabla\alpha|^3|\psi|^2 \\ &\le C\|g\|^2_{L^2(Q)} + Cs^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} \alpha^3|\psi|^2 + Cs\iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi\,,\end{aligned} \tag{A.18}$$

for all $s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2)$ and $\lambda \ge \lambda_0(\Omega,\mathcal{O})$. Recalling (A.7), we also obtain

$$\begin{aligned}&\|M_1\psi\|^2_{L^2(Q)} + \|M_2\psi\|^2_{L^2(Q)} + Cs^3\lambda^4 \iint_Q e^{3\lambda\beta^0}t^{-3}(T-t)^{-3}|\psi|^2 \\ &\le C\|g\|^2_{L^2(Q)} + Cs^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{3\lambda\beta^0}t^{-3}(T-t)^{-3}|\psi|^2 + Cs\iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi\,.\end{aligned} \tag{A.19}$$

Notice that

$$\|M_2\psi\|^2_{L^2(Q)} + Cs^3\lambda^4 \iint_Q e^{3\lambda\beta^0}t^{-3}(T-t)^{-3}|\psi|^2 \ge \frac{C}{s}\iint_Q e^{-\lambda\beta^0}t(T-t)|\Delta\psi|^2 \tag{A.20}$$

for all $s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2)$ and $\lambda \ge \lambda_0(\Omega,\mathcal{O})$. Indeed,

$$\begin{aligned}&\frac{1}{s}\iint_Q e^{-\lambda\beta^0}t(T-t)|\Delta\psi|^2 = \frac{1}{s}\iint_Q e^{-\lambda\beta^0}t(T-t)|M_2\psi - s^2|\nabla\phi|^2\psi - s\big(\partial_t\phi\big)\psi|^2 \\ &\le \frac{1}{s}\iint_Q e^{-\lambda\beta^0}t(T-t)|M_2\psi|^2 + Cs^3\iint_Q e^{-\lambda\beta^0}t(T-t)|\nabla\phi|^4|\psi|^2 \\ &\quad + Cs\iint_Q e^{-\lambda\beta^0}t(T-t)|\partial_t\phi|^2\psi^2.\end{aligned}$$

Using that $s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2)$ and the inequalities

$$|\nabla\phi| \le \frac{C\lambda e^{\lambda\beta^0}}{t(T-t)}\,, \quad |\partial_t\phi| \le \frac{CTe^{2\lambda\beta^0}}{t^2(T-t)^2}\,, \tag{A.21}$$

we are led to:

$$\begin{aligned}&\frac{1}{s}\iint_Q e^{-\lambda\beta^0}t(T-t)|\Delta\psi|^2 \\ &\le \frac{CT^2}{s}\iint_Q e^{-\lambda\beta^0}|M_2\psi|^2 + C\big(s^3T^4 + sT^2\big)\iint_Q e^{3\lambda\beta^0}t^{-3}(T-t)^{-3}|\psi|^2 \\ &\le C\|M_2\psi\|^2_{L^2(Q)} + Cs^3\lambda^4\iint_Q e^{3\lambda\beta^0}t^{-3}(T-t)^{-3}|\psi|^2,\end{aligned}$$

which gives (A.20). We also have

$$\begin{aligned}&\frac{1}{s}\iint_Q e^{-\lambda\beta^0}t(T-t)|\Delta\psi|^2 + s^3\lambda^4\iint_Q e^{3\lambda\beta^0}t^{-3}(T-t)^{-3}|\psi|^2 \\ &\ge Cs\lambda^2\iint_Q e^{\lambda\beta^0}t^{-1}(T-t)^{-1}|\nabla\psi|^2\end{aligned} \tag{A.22}$$

for all $s \geq \sigma_1(\Omega, \mathcal{O})(T+T^2)$ and $\lambda \geq \lambda_0(\Omega, \mathcal{O})$. This can be seen as follows. Integrating by parts, one has

$$\begin{aligned} 2s\lambda^2 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2 &= 2s\lambda^2 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}(-\Delta\psi)\psi \\ &-2s\lambda^3 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}(\nabla\beta^0\cdot\nabla\psi)\psi\,. \end{aligned} \tag{A.23}$$

But the last term coincides with

$$s\lambda^3 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}(\Delta\beta^0)|\psi|^2 + s\lambda^4 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\beta^0|^2|\psi|^2.$$

Thus,

$$\begin{aligned} &2s\lambda^2 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2 = 2s\lambda^2 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}(-\Delta\psi)\psi \\ &\quad+s\lambda^3 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}(\Delta\beta^0)|\psi|^2 + s\lambda^4 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\beta^0|^2|\psi|^2 \\ &\leq \frac{1}{s} \iint_Q e^{-\lambda\beta^0} t(T-t)|\Delta\psi|^2 + s^3\lambda^4 \iint_Q e^{3\lambda\beta^0} t^{-3}(T-t)^{-3}|\psi|^2 \\ &\quad+CsT^4\lambda^4 \iint_Q e^{\lambda\beta^0} t^{-3}(T-t)^{-3}|\psi|^2 \\ &\leq \frac{1}{s} \iint_Q e^{-\lambda\beta^0} t(T-t)|\Delta\psi|^2 + Cs^3\lambda^4 \iint_Q e^{3\lambda\beta^0} t^{-3}(T-t)^{-3}|\psi|^2 \end{aligned}$$

and this leads to (A.22). Combining (A.20) and (A.22), we obtain

$$\begin{aligned} &\|M_2\psi\|^2_{L^2(Q)} + s^3\lambda^4 \iint_Q e^{3\lambda\beta^0} t^{-3}(T-t)^{-3}|\psi|^2 \\ &\geq \frac{C}{s} \iint_Q e^{-\lambda\beta^0} t(T-t)|\Delta\psi|^2 + Cs\lambda^2 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2 \end{aligned} \tag{A.24}$$

for all $s \geq \sigma_1(\Omega, \mathcal{O})(T+T^2)$ and $\lambda \geq \lambda_0(\Omega, \mathcal{O})$ This will be used in (A.19) in order to bound the left hand side from below. Notice that, on the other hand, the last integral in the right side of (A.19) satisfies

$$Cs \iint_Q \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi \leq Cs\lambda \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2. \tag{A.25}$$

This is a consequence of the particular form of ϕ. Indeed,

$$\begin{aligned} \big(\partial_i\partial_j\phi\big)\partial_i\psi\partial_j\psi &= -e^{\lambda\beta^0} t^{-1}(T-t)^{-1}\big(\lambda^2\partial_i\beta^0\partial_j\beta^0 + \lambda\partial_i\partial_j\beta^0\big)\partial_i\psi\partial_j\psi \\ &= -\lambda^2 e^{\lambda\beta^0} t^{-1}(T-t)^{-1}\big(\nabla\beta^0\cdot\nabla\psi\big)^2 - \lambda e^{\lambda\beta^0} t^{-1}(T-t)^{-1}\big(\partial_i\partial_j\beta^0\big)\partial_i\psi\partial_j\psi \\ &\leq C\lambda e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2. \end{aligned} \tag{A.26}$$

From (A.19), (A.24) and (A.25), we have

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \frac{C}{s}\iint_Q e^{-\lambda\beta^0} t(T-t)|\Delta\psi|^2 + Cs\lambda^2 \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2 \\
&\quad + Cs^3\lambda^4 \iint_Q e^{3\lambda\beta^0} t^{-3}(T-t)^{-3}|\psi|^2 \\
&\le C\|g\|^2_{L^2(Q)} + Cs^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{3\lambda\beta^0} t^{-3}(T-t)^{-3}|\psi|^2 \qquad \text{(A.27)} \\
&\quad + Cs\lambda \iint_Q e^{\lambda\beta^0} t^{-1}(T-t)^{-1}|\nabla\psi|^2 \quad \forall s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2),\ \forall\lambda \ge \lambda_0(\Omega,\mathcal{O}).
\end{aligned}
$$

It is now clear that, whenever $\lambda \ge \lambda_1(\Omega,\mathcal{O})$, the third term in the left side in (A.27) is greater than twice the last term in the right. In the sequel, we fix $\lambda = \lambda_1(\Omega,\mathcal{O})$. Obviously, we can put

$$
\begin{aligned}
&\|M_1\psi\|^2_{L^2(Q)} + \frac{C}{s}\iint_Q t(T-t)|\Delta\psi|^2 + Cs\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2 + Cs^3\iint_Q t^{-3}(T-t)^{-3}|\psi|^2 \\
&\le C\big(\|g\|^2_{L^2(Q)} + s^3 \iint_{\mathcal{O}\times(0,T)} t^{-3}(T-t)^{-3}|\psi|^2\big) \qquad \forall s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2). \qquad \text{(A.28)}
\end{aligned}
$$

Observe that

$$
\|M_1\psi\|^2_{L^2(Q)} + s\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2 \ge \frac{C}{s}\iint_Q t(T-t)\,|\partial_t\psi|^2\,. \qquad \text{(A.29)}
$$

Indeed,

$$
\begin{aligned}
\frac{1}{s}\iint_Q t(T-t)\,|\partial_t\psi|^2 &= \frac{1}{s}\iint_Q t(T-t)\,|M_1\psi - 2s\nabla\phi\cdot\nabla\psi|^2 \\
&\le \frac{1}{s}\iint_Q t(T-t)\,|M_1\psi|^2 + Cs\iint_Q t(T-t)|\nabla\phi|^2|\nabla\psi|^2 \\
&\le \frac{CT^2}{s}\|M_1\psi\|^2_{L^2(Q)} + Cs\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2
\end{aligned}
$$

and this is bounded by $C\|M_1\psi\|^2_{L^2(Q)} + Cs\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2$ provided $s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2)$. Using (A.29) in (A.28), we obtain

$$
\begin{aligned}
&\frac{1}{s}\iint_Q t(T-t)\big(|\partial_t\psi|^2 + |\Delta\psi|^2\big) + s\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2 + s^3\iint_Q t^{-3}(T-t)^{-3}|\psi|^2 \\
&\le C\big(\|g\|^2_{L^2(Q)} + s^3\iint_{\mathcal{O}\times(0,T)} t^{-3}(T-t)^{-3}|\psi|^2\big) \quad \forall s \ge \sigma_1(\Omega,\mathcal{O})(T+T^2) \qquad \text{(A.30)}
\end{aligned}
$$

Let us deduce from (A.30) that (2.5) holds for all $s \ge s_1$ where $s_1 = \sigma_1(\Omega,\mathcal{O})(T+T^2)$. Recall that $\psi = e^{-s\phi}q$. Then,

$$
\partial_t\psi = e^{-s\phi}\big(\partial_t q - s\big(\partial_t\phi\big)q\big), \quad \partial_i\psi = e^{s\phi}\big(\partial_i q - s\big(\partial_i\phi\big)q\big) \qquad \text{(A.31)}
$$

$$
\Delta\psi = e^{-s\phi}\big(\Delta q - 2s\nabla\phi\cdot\nabla q + s^2|\nabla\phi|^2 q - s(\Delta\phi)q\big). \qquad \text{(A.32)}
$$

Consequently, we find the following:

$$\begin{aligned}
&\frac{1}{s}\iint_Q t(T-t)\,|\partial_t\psi|^2 = \frac{1}{s}\iint_Q e^{-2s\phi}t(T-t)\,\big|\partial_t q - s\big(\partial_t\phi\big)q\big|^2\\
&\geq \frac{1}{2s}\iint_Q e^{-2s\phi}t(T-t)\,|\partial_t q|^2 - s\iint_Q e^{-2s\phi}t(T-t)\,|\partial_t\phi|^2\,|q|^2 \qquad \text{(A.33)}\\
&\geq \frac{1}{2s}\iint_Q e^{-2s\phi}t(T-t)\,|\partial_t q|^2 - C_1 s^3\iint_Q e^{-2s\phi}t^{-3}(T-t)^{-3}|q|^2
\end{aligned}$$

where C_1 only depends on Ω and $\mathcal{O}$. In a similar way, we obtain

$$\begin{aligned}
&s\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2 = s\iint_Q e^{-2s\phi}t^{-1}(T-t)^{-1}|\nabla q - sq\nabla\phi|^2\\
&\geq \frac{s}{2}\iint_Q e^{-2s\phi}t^{-1}(T-t)^{-1}|\nabla q|^2 - s^3\iint_Q e^{-2s\phi}t^{-1}(T-t)^{-1}|\nabla\phi|^2|q|^2 \qquad \text{(A.34)}\\
&\geq \frac{s}{2}\iint_Q e^{-2s\phi}t^{-1}(T-t)^{-1}|\nabla q|^2 - C_2 s^3\iint_Q e^{-2s\phi}t^{-3}(T-t)^{-3}|q|^2\,.
\end{aligned}$$

Also,

$$\begin{aligned}
&\frac{1}{s}\iint_Q t(T-t)|\Delta\psi|^2 = \frac{1}{s}\iint_Q e^{-2s\phi}t(T-t)\,\big|\Delta q - 2s\nabla\phi\cdot\nabla q + s^2|\nabla\phi|^2 q - s(\Delta\phi)q\big|^2\\
&\geq \frac{1}{2s}\iint_Q e^{-2s\phi}t(T-t)|\Delta q|^2 - C_3 s\iint_Q e^{-2s\phi}t^{-1}(T-t)^{-1}|\nabla q|^2 \qquad \text{(A.35)}\\
&- C_4 s^3\iint_Q e^{-2s\phi}t^{-3}(T-t)^{-3}|q|^2\,.
\end{aligned}$$

Let us choose a and b such that (for instance)

$$0 < b < \min\big(1, \frac{1}{2C_2}\big), \qquad 0 < a < \min\big(1, \frac{1}{2(C_1+C_4)}, \frac{b}{2C_3}\big).$$

Then

$$\begin{aligned}
&\frac{a}{s}\iint_Q t(T-t)\big(|\partial_t\psi|^2 + |\Delta\psi|^2\big) + bs\iint_Q t^{-1}(T-t)^{-1}|\nabla\psi|^2 + s^3\iint_Q t^{-3}(T-t)^{-3}|\psi|^2\\
&\geq \frac{a}{2s}\iint_Q e^{-2s\phi}t(T-t)\big(|\partial_t q|^2 + |\Delta q|^2\big) + \big(\frac{b}{2} - aC_3\big)s\iint_Q e^{-2s\phi}t^{-1}(T-t)^{-1}|\nabla q|^2\\
&+ \big(1 - a(C_1+C_4) - bC_2\big)s^3\iint_Q e^{-2s\phi}t^{-3}(T-t)^{-3}|q|^2\,.
\end{aligned}$$

This, together with (A.30), gives (2.5) for all $s \geq \sigma_1(\Omega,\mathcal{O})(T+T^2)$. This proves Proposition 2.1.

REFERENCES

[1] C. Bender and S. Orzag, "Advanced Mathematical Methods for Scientists and Engineers," McGraw-Hill, New York 1978.

[2] J. Dieudonné, "Calcul Infinitéssimal," Hermann, Paris 1971.

[3] C. Fabre, *Uniqueness results for Stokes equations and their consequences in linear and nonlinear control problems,* ESAIM: COCV, 1 (1996), 267–302 (web site `http://www.emath.fr/cocv/`).

[4] C. Fabre, J.P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation,* Proc. Roy. Soc. Edinburgh, 125A (1995), 31–61.

[5] L. A. Fernández and E. Zuazua, *Approximate controllability for the semilinear heat equation involving gradient terms,* preprint, 1997.

[6] E. Fernández-Cara, *Null controllability of the semilinear heat equation,* ESAIM: COCV, 2 (1997), 87–107.

[7] A. Fursikov and O. Yu. Imanuvilov, "Controllability of Evolution Equations," Lecture Notes Series # 34, Seoul National University, Korea, 1996.

[8] J.M. Ghidaglia, *Some backward uniqueness results,* Nonlinear Anal. TMA, 10 (1986), 777–790.

[9] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur,* Comm. P.D.E., 20 (1995), 335–356.

[10] J.-L. Lions, *Remarks on approximate controllability,* J. Anal. Math., 59 (1992), 103–116.

[11] J.-L. Lions, *Remarques sur la contrôlabilité approchée,* in "Jornadas Hispano-Francesas sobre Control de Sistemas Distribuidos," University of Malaga, Spain, 1991, p.77–87.

[12] J.-L. Lions and B. Malgrange, *Sur l'unicité rétrograde des équations paraboliques,* Math. Scand., 8 (1960), 277–286.

[13] J.-L. Lions and E. Zuazua, *The cost of controlling unstable systems: Time irreversible systems,* Rev. Mat. Complutense, 10 (1997).

[14] D.L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations,* Studies in Appl. Math., LII (3) (1973), 189–211.

[15] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations,* Recent progress and open questions, SIAM Rev., 20 (1978), 639–739.

[16] J.C. Saut and B. Scheurer, *Unique continuation for some evolution equations,* J. Diff. Equations, 66 (1) (1987).

[17] E. Zuazua, *Finite dimensional null-controllability of the semilinear heat equation,* J. Math. pures et appl., 76 (1997), 237–264.

[18] E. Zuazua, *Approximate controllability of the semilinear heat equation: boundary control,* in "Computational Sciences for the 21st Century," M.O. Bristeau et al. eds., John Wiley & Sons, 1997, 738–747.