

# NULL CONTROLLABILITY OF A SYSTEM OF VISCOELASTICITY WITH A MOVING CONTROL

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ABSTRACT. In this paper, we consider the wave equation with both a viscous Kelvin-Voigt and frictional damping as a model of viscoelasticity in which we incorporate an internal control with a moving support. We prove the null controllability when the control region, driven by the flow of an ODE, covers all the domain. The proof is based upon the interpretation of the system as, roughly, the coupling of a heat equation with an ordinary differential equation (ODE). The presence of the ODE for which there is no propagation along the space variable makes the controllability of the system impossible when the control is confined into a subset in space that does not move. The null controllability of the system with a moving control is established in using the observability of the adjoint system and some Carleman estimates for a coupled system of a parabolic equation and an ODE with the same singular weight, adapted to the geometry of the moving support of the control. This extends to the multi-dimensional case the results by P. Martin *et al.* on the one-dimensional case, employing  $1 - d$  Fourier analysis techniques.

RÉSUMÉ. Dans cet article, on considère comme modèle de la viscoélasticité l'équation des ondes avec un amortissement de Kelvin-Voigt et un amortissement frictionnel, dans laquelle on incorpore un contrôle interne à support mobile. On prouve la contrôlabilité à zéro de l'équation lorsque le support du contrôle, qui est transporté par le flot d'une équation différentielle ordinaire (EDO), parcourt tout le domaine. La preuve est basée sur l'interprétation du système de la viscoélasticité comme un système couplant une équation de la chaleur et une EDO. La présence de l'EDO, pour laquelle il n'y a pas de propagation suivant la variable d'espace, rend la contrôlabilité du système impossible lorsque le contrôle est confiné à une région qui ne bouge pas. La contrôlabilité à zéro du système avec un contrôle mobile est établie en utilisant l'observabilité du système adjoint et des inégalités de Carleman pour l'équation de la chaleur et l'ODE avec le même poids singulier, qui est adapté au support mobile du contrôle. Ceci permet d'étendre à une dimension quelconque les résultats de P. Martin *et al.* établis en dimension un à l'aide de techniques basées sur l'analyse de Fourier.

## 1. INTRODUCTION

We are concerned with the controllability of the following model of viscoelasticity consisting of a wave equation with both viscous Kelvin-Voigt and frictional damping:

$$y_{tt} - \Delta y - \Delta y_t + b(x)y_t = 1_{\omega(t)}h, \quad x \in \Omega, t \in (0, T), \quad (1.1)$$

$$y = 0, \quad x \in \partial\Omega, t \in (0, T), \quad (1.2)$$

$$y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \quad x \in \Omega. \quad (1.3)$$

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Here  $\Omega$  is a smooth, bounded open set in  $\mathbb{R}^N$ ,  $b \in L^\infty(\Omega)$  is a given function determining the frictional damping and  $h = h(x, t)$  denotes the control. To simplify the presentation and notation, and without loss of generality, the viscous constant has been taken to be the unit one  $\nu = 1$ . The same system could be considered with an arbitrary viscosity constant  $\nu > 0$  leading to the more general system

$$y_{tt} - \Delta y - \nu \Delta y_t + b(x)y_t = 1_{\omega(t)}h, \quad (1.4)$$

but the analysis would be the same.

The control  $h$  acting on the right hand side term as an external force is, for all  $0 < t < T$ , localized in a subset of  $\Omega$ . This fact is modeled by the multiplicative factor  $1_{\omega(t)}$  which stands for the characteristic function of the set  $\omega(t)$  that, for any  $0 < t < T$ , constitutes the support of the control, localized in a moving subset  $\omega(t)$  of  $\Omega$ .

Typically we shall consider control sets  $\omega(t)$  determined by the evolution of a given reference subset  $\omega$  of  $\Omega$  through a smooth flow  $X(x, t, 0)$ .

We consider the problem of null controllability. In other words, given a final time  $T$  and initial data for the system  $(y_0, y_1)$  in a suitable functional setting, we analyze the existence of a control  $h = h(x, t)$  such that the corresponding solution satisfies the rest condition at the final time  $t = T$ :

$$y(x, T) \equiv y_t(x, T) \equiv 0, \quad \text{in } \Omega.$$

One of the distinguished features of the system under consideration is that, for this null controllability condition to be fulfilled, the control needs to move in time. Indeed, if  $\omega(t) \equiv \omega$  for all  $0 < t < T$ , i.e. if the support of the control does not move in time as it is often considered, the system under consideration is not controllable. This can be easily seen at the level of the dual observability problem. In fact, the structure of the underlying PDE operator and, in particular, the existence of time-like characteristic hyperplanes, makes impossible the propagation of information in the space-like directions, thus making the observability inequality also impossible. This was already observed in the work by P. Martin *et al.* in [25] in the  $1 - d$  setting. There, for the  $1 - d$  model, it was shown that this obstruction could be removed by making the control move so that its support covers the whole domain where the equation evolves.

More precisely, in [25], the  $1 - d$  version of the problem above was considered in the torus, with periodic boundary conditions,  $b \equiv 0$  and  $\omega(t) = \{x - t; x \in \omega\}$ , i.e.

$$y_{tt} - y_{xx} - y_{xxt} = 1_{\omega(t)}h(x, t), \quad x \in \mathbb{T}. \quad (1.5)$$

Recall that this system with boundary control, i.e.  $h \equiv 0$  and the boundary conditions

$$y(0, t) = 0, \quad y(1, t) = g(t),$$

$g = g(t)$  being the boundary control, fails to be spectrally controllable, because of the existence of a limit point in the spectrum of the adjoint system [28]. In the moving frame  $x' = x + t$ , (1.5) may be written as

$$z_{tt} - 2z_{xt} - z_{txx} + z_{xxx} = a(x)h(x + t, t) \quad (1.6)$$

where  $z(x, t) = y(x + t, t)$ . In [25] the spectrum of the adjoint system to (1.6) was shown to be split into a hyperbolic part and a parabolic one. As a consequence, equation (1.6) was proved

to be null controllable in large time. A similar result was proved in [31] for the Benjamin-Bona-Mahony equation

$$y_t - y_{txx} + y_x + yy_x = a(x - ct)h(x, t), \quad x \in \mathbb{T}.$$

Once again this system turns out to be globally controllable and exponentially stabilizable in  $H^1(\mathbb{T})$  for any  $c \neq 0$ . But, as noticed in [26], the linearized equation fails to be spectrally controllable with a control supported in a fixed domain.

As mentioned above, in both cases, the lack of controllability of these systems with immobile controls is due to the fact that the underlying PDE operators exhibit the presence of time-like characteristic lines thus making propagation in the space-like directions impossible. By the contrary, when analyzing the problem in a moving frame, the characteristic lines are oblique ones in  $(x, t)$ , thus facilitating propagation properties.

The main goal of this paper is to extend the  $1 - d$  analysis in [25] to the multi-dimensional case. This can not be done with the techniques in [25] based on Fourier analysis. Our approach is rather inspired on the fact that system (1.1)-(1.2) can be rewritten as a system coupling a parabolic equation with an ordinary differential equation (ODE). The presence of this ODE, in the case of a fixed support of the control, independent of  $t$ , is responsible for the lack of controllability of the system, due to the absence of propagation in the space-like direction. Letting the control move introduces an effect similar to adding a transport term in the ODE but keeping the control immobile, thus changing the structure of the system into a parabolic-transport coupled one. This new system turns out to be controllable under the condition that all characteristics of the transport equation enter within the control set in the given control time, a condition that is reminiscent of the so-called Geometric Control Condition in the context of the wave equation (see [2]).

The approach in [25] would suggest to do the following splitting of (1.1):

$$v_t - \Delta v = 1_{\omega(t)}h + (1 - b)(v - y) \tag{1.7}$$

$$y_t + y = v. \tag{1.8}$$

However, the splitting can be performed in an alternative manner as follows:

$$y_t - \Delta y + (b - 1)y = z \tag{1.9}$$

$$z_t + z = 1_{\omega(t)}h + (b - 1)y. \tag{1.10}$$

It can easily be seen that  $y$  solves equation (1.1) if, and only if, it is the first component of the solution of system (1.9)-(1.10).

Our analysis of the Carleman inequalities for these systems is analog to that in [1] for a system of thermoelasticity coupling the heat and the wave equation. The key in [1] and in our own analysis is to use the same weight function both for the Carleman inequality of the heat and the hyperbolic model. In [1], since dealing with the wave equation, rather strong geometric conditions were needed on the subset where the control or the observation mechanism acts. In our case, since we are considering the simpler transport equation, the geometric assumptions will be milder, consisting mainly on a characteristic condition ensuring that all characteristic lines of the transport equation intersect the control/observation set. This suffices for the Carleman inequality to hold for the transport equation and is also sufficient for the heat equation that it

is well-known to be controllable/observable from any open non-empty subset of the space-time cylinder where the equation is formulated.

It is important to mention that, as far as we know, all the Carleman inequalities for the heat equation available in the literature are done for the case where the control region is fixed. In the case we are dealing, the control region is moving in time. Therefore, a new Carleman inequality must be proved in this framework. The proof of a Carleman inequality for the heat equation when the control region is moving is one of the novelties of this paper.

In order to state the main result of this paper we first describe precisely the class of moving trajectories for the control for which our null controllability result will hold.

**Admissible trajectories:** In practice the trajectory of the control can be taken to be determined by the flow  $X(x, t, t_0)$  generated by some vector field  $f \in C([0, T]; W^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N))$ , i.e.  $X$  solves

$$\begin{cases} \frac{\partial X}{\partial t}(x, t, t_0) = f(X(x, t, t_0), t), \\ X(x, t_0, t_0) = x. \end{cases} \quad (1.11)$$

For instance, any translation of the form:

$$X(x, t, t_0) = x + \gamma(t) - \gamma(t_0), \quad (1.12)$$

where  $\gamma \in C^1([0, T]; \mathbb{R}^N)$ , is admissible. (Pick  $f(x, t) = \dot{\gamma}(t)$ ).

We assume that there exist a bounded, smooth, open set  $\omega_0 \subset \mathbb{R}^N$ , a curve  $\Gamma \in C^\infty([0, T]; \mathbb{R}^N)$ , and two times  $t_1, t_2$  with  $0 \leq t_1 < t_2 \leq T$  such that:

$$\Gamma(t) \in X(\omega_0, t, 0) \cap \Omega, \quad \forall t \in [0, T]; \quad (1.13)$$

$$\bar{\Omega} \subset \cup_{t \in [0, T]} X(\omega_0, t, 0) = \{X(x, t, 0); \ x \in \omega_0, \ t \in [0, T]\}; \quad (1.14)$$

$$\Omega \setminus \overline{X(\omega_0, t, 0)} \text{ is nonempty and connected for } t \in [0, t_1] \cup [t_2, T]; \quad (1.15)$$

$$\Omega \setminus \overline{X(\omega_0, t, 0)} \text{ has two (nonempty) connected components for } t \in (t_1, t_2); \quad (1.16)$$

$$\forall \gamma \in C([0, T]; \Omega), \ \exists t \in [0, T], \quad \gamma(t) \in X(\omega_0, t, 0). \quad (1.17)$$

The main result in this paper is as follows.

**Theorem 1.1.** *Let  $T > 0$ ,  $X(x, t, t_0)$  and  $\omega_0$  be as in (1.13)-(1.17), and let  $\omega$  be any open set in  $\Omega$  such that  $\bar{\omega}_0 \subset \omega$ . Then for all  $(y_0, y_1) \in L^2(\Omega)^2$  with  $y_1 - \Delta y_0 \in L^2(\Omega)$ , there exists a function  $h \in L^2(0, T; L^2(\Omega))$  for which the solution of*

$$y_{tt} - \Delta y - \Delta y_t + b(x)y_t = \mathbf{1}_{X(\omega, t, 0)}(x)h, \quad (x, t) \in \Omega \times (0, T), \quad (1.18)$$

$$y(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \quad (1.19)$$

$$y(\cdot, 0) = y_0, \ y_t(\cdot, 0) = y_1, \quad (1.20)$$

fulfills  $y(\cdot, T) = y_t(\cdot, T) = 0$ .

A few remarks are in order in what concerns the functional setting of this model:

- Viewing the system of viscoelasticity under consideration as a damped wave equation, a natural functional setting would be the following: For data in  $H_0^1(\Omega) \times L^2(\Omega)$  and, say, right hand side term of (1.1) in  $L^2(0, T; H^{-1}(\Omega))$ , there exists a unique solution  $y \in C([0, T]; H_0^1(\Omega) \cap C^1([0, T]; L^2(\Omega)))$ . Furthermore  $y_t \in L^2(0, T; H_0^1(\Omega))$ . The latter is an added integrability/regularity property of the solution that is due to the strong damping effect of the system. This can be seen naturally by considering the energy of the system

$$E(t) = \frac{1}{2} \int_{\Omega} [|y_t|^2 + |\nabla y|^2] dx,$$

that fulfills the energy dissipation law

$$\frac{d}{dt} E(t) = - \int_{\Omega} [|\nabla y_t|^2 + b(x)|y_t|^2] dx + \int_{\omega(t)} h y_t dx.$$

- We can also solve (1.9)-(1.10) so that  $y$ , solution of the heat equation, lies in the space  $y \in C([0, T]; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$  and  $z$ , solution of the ODE, in  $C([0, T]; L^2(\Omega))$ . This can be done provided  $(y_0, y_1 - \Delta y_0) \in H_0^1(\Omega) \times L^2(\Omega)$ . The functional setting is not exactly the same but this is due to the fact that, in some sense, in one case we see the system as a perturbation of the wave equation, while, in the other one, as a variant of the heat equation.
- From a control theoretical point of view it is much more efficient to analyze the system in the second setting, as a perturbation of the heat equation, through the coupling with the ODE or, after changing variables, with a transport equation. If we view the system of viscoelasticity as a perturbation of the wave equation, standard hyperbolic techniques such as multiplier, Carleman inequalities or microlocal tools do not apply since, actually, the viscoelastic term determines the principal part of the underlying PDE operator and cannot be viewed as a perturbation of the wave dynamics.

The analysis is particularly simple in the special case where  $b \equiv 1$ . Indeed, in that case, the system (1.7)-(1.8) (with a second control incorporated in the ODE) takes the following cascade form

$$v_t - \Delta v = 1_{\omega(t)} \tilde{h}, \quad (1.21)$$

$$y_t + y = 1_{\omega(t)} \tilde{k} + v, \quad (1.22)$$

where the parabolic equation (1.21) is *uncoupled*. This system will be investigated in a separate section (Section 2) since some of the basic ideas allowing to handle the general case emerge already in its analysis. Note that, in this particular case, roughly, one can first control the heat equation by a suitable control  $\tilde{h}$  and then, once this is done, viewing  $v$  as a given source term, control the transport equation by a convenient  $\tilde{k}$ . This case is also important because the only assumption needed to prove Theorem 1.1 in this case is (1.14) (i.e. we don't assume that (1.13) and (1.15)-(1.17) are satisfied).

In this particular case  $b \equiv 1$  a similar argument can be used with the second decomposition.

The paper is organized as follows. Section 2 is devoted to address the particular case  $b \equiv 1$ . In Section 3 we give some examples showing the importance of the taken assumptions on the trajectories. In Section 4, we go back to the general system (1.9)-(1.10). We prove that this

system is null controllable in  $L^2(\Omega)^2$  (see Theorem 4.1) by deriving the observability of the adjoint system from two Carleman estimates with the same singular weight, adapted to the flow determining the moving control. The details of the construction of the weight function based on (1.13)-(1.17) are given in Lemma 4.3. Theorem 1.1 is then a direct consequence of Theorem 4.1. We finish this paper with two further sections devoted to comment some closely related issues and open problems

## 2. ANALYSIS OF THE DECOUPLED CASCADE SYSTEM

In this section, we give a proof of Theorem 1.1 in the special situation when  $b \equiv 1$ , and  $\omega(t) = X(\omega_0, t, 0)$ , where  $X$  is given by (1.11) for some  $f \in C(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N))$ .

As we said before, we will prove Theorem 1.1 in the case  $b \equiv 1$  by proving a null controllability result for the decomposition (1.21)-(1.22). The idea of the proof is as follows. We take some appropriate  $0 < \epsilon < T$  and then drive the solution of the heat equation (1.21) to zero in time  $\epsilon$  by means of a control  $\tilde{h}$ . Next, we let equation (1.22) evolves freely in  $[0, \epsilon]$ , i.e.,  $\tilde{k} \equiv 0$ , and then we control this equation by means of a smooth control  $\tilde{k}$  in the time interval  $[\epsilon, T]$ . This gives the null controllability of the system (1.21)-(1.22) in the whole time interval  $[0, T]$ .

### Proof of Theorem 1.1 in the case $b \equiv 1$ .

Suppose (1.14) is satisfied and let

$$T_0 = \inf\{T > 0; \bar{\Omega} \subset \cup_{0 \leq t \leq T} X(\omega_0, t, 0)\}. \quad (2.1)$$

Pick any  $T > T_0$ , and pick some  $\epsilon \in (0, T - T_0)$  and some nonempty open set  $\omega_{-1} \subset \omega_0$  such that

$$\bar{\Omega} \subset \cup_{\epsilon \leq t \leq T} X(\omega_0, t, 0), \quad (2.2)$$

$$\omega_{-1} \subset X(\omega_0, t, 0) \quad \forall t \in (0, \epsilon). \quad (2.3)$$

Let  $T' = T - \epsilon$ , and pick any  $(v_0, y_0) \in L^2(\Omega)^2$ . Then, it is well known (see [16]) that there exists some control input  $h \in L^2(0, \epsilon; L^2(\Omega))$  such that the solution  $v = v(x, t)$  of

$$v_t - \Delta v = \mathbf{1}_{\omega_{-1}} h, \quad x \in \Omega, \quad t \in (0, \epsilon), \quad (2.4)$$

$$v(x, 0) = v_0(x), \quad x \in \Omega, \quad (2.5)$$

satisfies

$$v(x, \epsilon) = 0, \quad x \in \Omega.$$

Set

$$\begin{aligned} \tilde{h}(x, t) &= \mathbf{1}_{\omega_{-1}}(x)h(x, t), & x \in \Omega, \quad t \in (0, \epsilon), \\ \tilde{h}(x, t) &= 0, & x \in \Omega, \quad t \in (\epsilon, T), \\ \tilde{k}(x, t) &= 0, & x \in \Omega, \quad t \in (0, \epsilon) \end{aligned}$$

Then the solution  $v$  of

$$v_t - \Delta v = \mathbf{1}_{X(\omega_0, t, 0)}(x) \tilde{h} \quad x \in \Omega, \quad t \in (0, T), \quad (2.6)$$

$$v(x, 0) = v_0(x), \quad x \in \Omega, \quad (2.7)$$

satisfies  $v(x, t) = 0$  for  $t \in [\varepsilon, T]$ . We claim that the system

$$y_t + y = \mathbf{1}_{X(\omega_0, t, 0)}(x) k(x, t), \quad x \in \Omega, \quad t \in (\varepsilon, T), \quad (2.8)$$

$$y(x, \varepsilon) = y_0(x), \quad (2.9)$$

is exactly controllable in  $L^2(\Omega)$  on the time interval  $(\varepsilon, T)$ . By duality, this is equivalent to proving that the corresponding observability inequality

$$\int_{\Omega} |q_0(x)|^2 dx \leq C \int_{\varepsilon}^T \int_{\Omega} \mathbf{1}_{X(\omega_0, t, 0)}(x) |q(x, t)|^2 dx dt \quad (2.10)$$

is fulfilled with a uniform constant  $C > 0$  for all solution  $q$  of the adjoint system

$$-q_t + q = 0, \quad x \in \Omega, \quad t \in (\varepsilon, T), \quad (2.11)$$

$$q(x, T) = q_0(x). \quad (2.12)$$

Since the solution of (2.11)-(2.12) is given by  $q(x, t) = e^{t-T} q_0(x)$ , we have that

$$\int_{\varepsilon}^T \int_{\Omega} \mathbf{1}_{X(\omega_0, t, 0)}(x) |q(x, t)|^2 dx dt \geq e^{2(\varepsilon-T)} \int_{\Omega} |q_0(x)|^2 \left( \int_{\varepsilon}^T \mathbf{1}_{X(\omega_0, t, 0)}(x) dt \right) dx. \quad (2.13)$$

From (2.1) and the smoothness of  $X$ , we see that for all  $x \in \bar{\Omega}$ , there is some  $t_0 \in (\varepsilon, T)$ , and some  $\delta > 0$  such that for any  $y \in B(x, \delta)$  and any  $t \in (\varepsilon, T) \cap (t_0 - \delta, t_0 + \delta)$  we have  $y \in X(\omega_0, t, 0)$ . From the compactness of  $\bar{\Omega}$ , we see that there exists a number  $\delta_0 > 0$  such that

$$\int_{\varepsilon}^T \mathbf{1}_{X(\omega_0, t, 0)}(x) dt > \delta_0, \quad \forall x \in \bar{\Omega}.$$

Combined with (2.13), this yields (2.10). Thus, (2.8)-(2.9) is exactly controllable in  $L^2(\Omega)$  on  $(\varepsilon, T)$  with some controls  $k \in C([\varepsilon, T]; L^2(\Omega))$ . Let  $y_1(x) = e^{-\varepsilon} y_0(x) + \int_0^{\varepsilon} e^{s-\varepsilon} v(x, s) ds$ . Extend  $\tilde{k}$  to  $(0, T)$  so that  $\tilde{k} \in L^2(0, T; L^2(\Omega))$  and the solution of

$$y_t + y = \mathbf{1}_{X(\omega_0, t, 0)}(x) \tilde{k}, \quad x \in \Omega, \quad t \in (\varepsilon, T), \quad (2.14)$$

$$y(x, \varepsilon) = y_1(x), \quad (2.15)$$

satisfies  $y(\cdot, T) = 0$ . Thus the control  $(\tilde{h}, \tilde{k})$  steers the solution of (1.21)-(1.22) from  $(v_0, y_0)$  at  $t = 0$  to  $(0, 0)$  at  $t = T$ . Applying the operator  $\partial_t - \Delta$  in each side of (1.22) results in

$$y_{tt} - \Delta y - \Delta y_t + y_t = \mathbf{1}_{X(\omega_0, t, 0)}(x) \tilde{h} + (\partial_t - \Delta)[\mathbf{1}_{X(\omega_0, t, 0)}(x) \tilde{k}], \quad (2.16)$$

This proves Theorem 1.1, except the fact that the control does not live in  $L^2(0, T; L^2(\Omega))$ . Assume now that  $(v_0, y_0) \in L^2(\Omega) \times [H^2(\Omega) \cap H_0^1(\Omega)]$ . To get a control  $\tilde{k} \in L^2(0, T; H^2(\Omega))$ , it is sufficient to replace  $\mathbf{1}_{\omega(t)}$  by  $a(X(x, 0, t))$  in (1.22), where  $a$  is a function satisfying

$$a \in C_0^\infty(\omega),$$

$$a(x) = 1 \quad \forall x \in \omega_0.$$

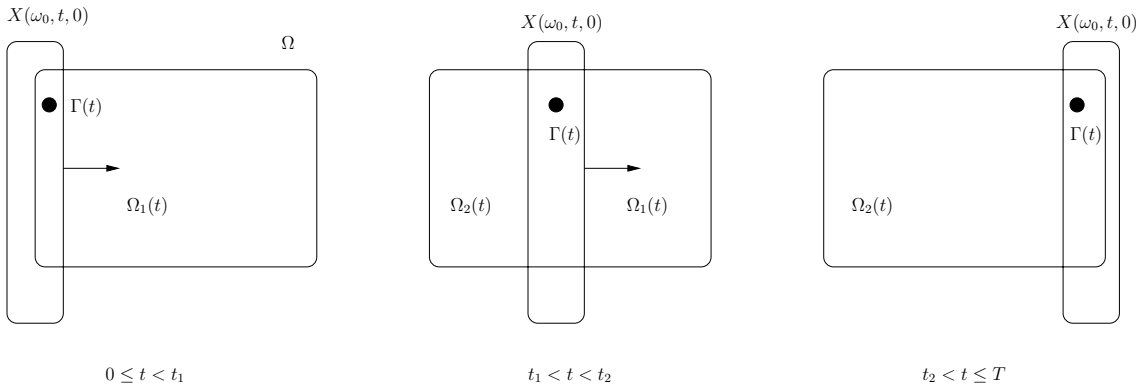


FIGURE 1. Example for which conditions (1.13)-(1.17) are satisfied.

The proof is completed by showing the observability inequality

$$\|q_0\|_{X'}^2 \leq C \int_{\varepsilon}^T \|(t - \varepsilon)a(X(\cdot, 0, t))q(\cdot, t)\|_{X'}^2 dt$$

for the solution  $q$  of system (2.11)-(2.12), where  $X = H^2(\Omega) \cap H_0^1(\Omega)$  and  $X'$  stands for its dual space. This can be done as in [30, Proposition 2.1]. Next, using the HUM operator, we notice that  $\tilde{k} \in C^1([\varepsilon, T]; X)$  with  $\tilde{k}(\cdot, \varepsilon) = 0$ , since  $q \in C^1([\varepsilon, T]; X')$  for any  $q_0 \in X'$ . Thus, with this small change, the right hand side term in (2.16) can be written  $\mathbf{1}_{X(\omega, t, 0)}u(x, t)$ , where  $u \in L^2(0, T; L^2(\Omega))$ .

**Remark 2.1.** Observe that the situation when  $X(\omega_0, t, 0)$  moves as in Figure 2, Figure 4 or in Figure 5 (see below) is admissible in the case when  $b \equiv 1$ .

### 3. EXAMPLES

In this section, we provide some geometric examples to illustrate the assumptions (1.13)-(1.17). We use simple shapes (like rectangles) just for convenience.

- Figure 1 shows how a control region should move in order to satisfy conditions (1.13)-(1.17).
- Figure 2 depicts a situation for which Theorem 1.1 cannot be applied, except in the case when  $b \equiv 1$ , as condition (1.16) fails.
- In Figure 3, we modify the example given in Figure 1 by shifting the time. Theorem 1.1 cannot be applied as it is, since (1.15) fails. However, the conclusion of Theorem 1.1 remains valid. Indeed, assume that  $\Omega \setminus \overline{\omega(t)}$  has two connected components (resp. one) for  $t \in [0, \tau_1) \cup (\tau_2, T]$  (resp. for  $t \in [\tau_1, \tau_2]$ ). Assume that the “jump” of  $\omega(t)$  occurs at  $t = \tau_3$ , with  $\tau_1 < \tau_3 < \tau_2$ . Let

$$\mathcal{O}_1 := \cup_{0 \leq t \leq \tau_3} \omega(t), \quad (3.1)$$

$$\mathcal{O}_2 := \cup_{\tau_3 \leq t \leq T} \omega(t) \quad (3.2)$$



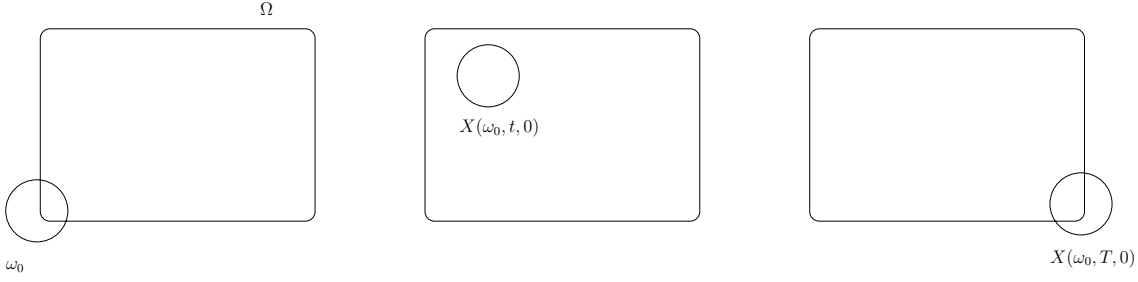


FIGURE 2. Example for which condition (1.16) fails.

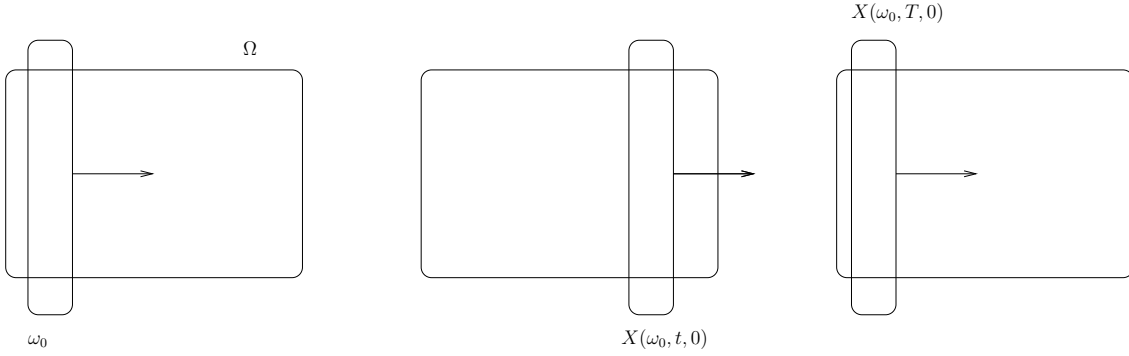


FIGURE 3. Example for which condition (1.15) fails.

and let  $\eta \in C^\infty(\Omega; [0, 1])$  be such that

$$\text{supp}(\eta) \subset \mathcal{O}_1, \quad (3.3)$$

$$\text{supp}(1 - \eta) \subset \mathcal{O}_2, \quad (3.4)$$

$$\text{supp}(\nabla \eta) \subset \omega_0. \quad (3.5)$$

Then, applying the Carleman estimate in Lemma 4.6 to  $(p_1, q_1) = \eta(X(x, 0, t))(p, q)$  in  $\Omega \cap \{\eta > 0\}$  on the time interval  $[0, \tau_3]$ , and to  $(p_2, q_2) = (1 - \eta(X(x, 0, t)))(p, q)$  in  $\Omega \cap \{\eta < 1\}$  on the time interval  $[\tau_3, T]$ , we can easily prove the observability inequality (4.11).

- Figure 4 shows that the assumption (1.13), which is needed to construct the weight function  $\psi$  in Lemma 4.3 cannot be replaced by the simpler condition

$$X(\omega_0, t, 0) \cap \Omega \neq \emptyset, \quad \forall t \in [0, T].$$

- Figure 5 shows that the assumption (1.17), which is also needed to construct the weight function  $\psi$  in Lemma 4.3, does not result from the other assumptions (1.13)-(1.16).

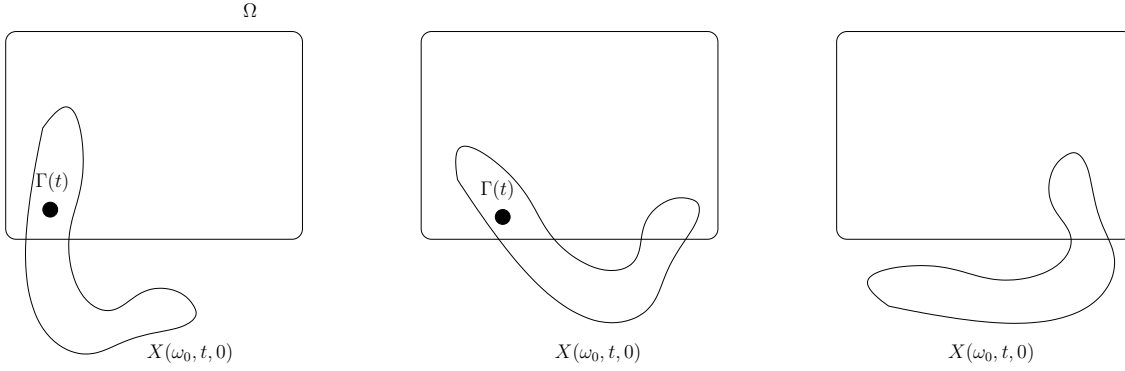


FIGURE 4. Example showing that  $X(\omega_0, t, 0) \cap \Omega \neq \emptyset \forall t \in [0, T]$  does not imply (1.13).

#### 4. NULL CONTROLLABILITY OF SYSTEM (1.21)-(1.22).

In this section we proof Theorem 1.1. Using decomposition (1.9)-(1.10), it is easy to see that the null controllability of (1.1)-(1.3) turns out to be equivalent to the null controllability of the system

$$y_t - \Delta y + (b(x) - 1)y = z, \quad (x, t) \in \Omega \times (0, T), \quad (4.1)$$

$$z_t + z = \mathbf{1}_{X(\omega, t, 0)}(x)h + (b(x) - 1)y, \quad (x, t) \in \Omega \times (0, T), \quad (4.2)$$

$$y(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \quad (4.3)$$

$$z(x, 0) = z_0(x), \quad x \in \Omega, \quad (4.4)$$

$$y(x, 0) = y_0(x), \quad x \in \Omega. \quad (4.5)$$

More precisely, Theorem 1.1 is a direct consequence of the following result.

**Theorem 4.1.** *Let  $T$ ,  $X(x, t, t_0)$  and  $\omega_0$  be as in (1.13)-(1.17), and let  $\omega$  be as in Theorem 1.1. Then for all  $(y_0, z_0) \in L^2(\Omega)^2$ , there exists a control function  $h \in L^2(0, T; L^2(\Omega))$  for which the solution  $(y, z)$  of (4.1)-(4.5) satisfies  $y(\cdot, T) = z(\cdot, T) = 0$ .*

From now on we concentrate in the proof of Theorem 4.1.

It is well-known (see [12]) that Theorem 4.1 is equivalent to prove an observability inequality for the adjoint system of (4.1)-(4.5), namely

$$-p_t - \Delta p + (b(x) - 1)p = (b(x) - 1)q, \quad (x, t) \in \Omega \times (0, T), \quad (4.6)$$

$$-q_t + q = p, \quad (x, t) \in \Omega \times (0, T), \quad (4.7)$$

$$p(x, t) = 0, \quad (x, t) \in \partial\Omega \times (0, T), \quad (4.8)$$

$$p(x, T) = p_0(x), \quad x \in \Omega, \quad (4.9)$$

$$q(x, T) = q_0(x), \quad x \in \Omega. \quad (4.10)$$

In fact, one can show that Theorem 4.1 is equivalent to the following:

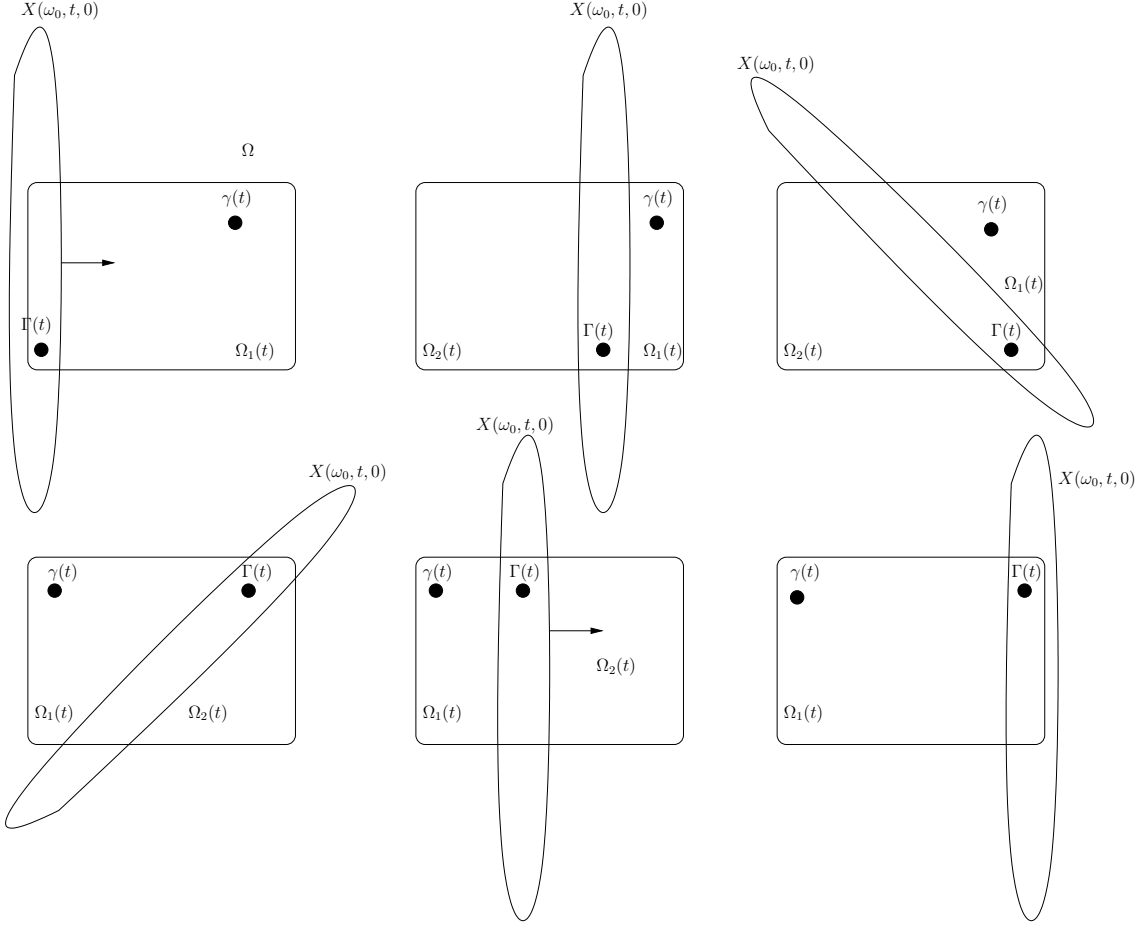


FIGURE 5. Example showing that (1.13)-(1.16) does not imply (1.17).

**Proposition 4.2.** *Let  $T$ ,  $X$ ,  $\omega_0$  and  $\omega$  be as in Theorem 1.1. Then there exists a constant  $C > 0$  such that for all  $(p_0, q_0) \in L^2(\Omega)^2$ , the solution  $(p, q)$  of (4.6)-(4.10) satisfies*

$$\int_{\Omega} [ |p(x, 0)|^2 + |q(x, 0)|^2 ] dx \leq C \int_0^T \int_{X(\omega, t, 0)} |q(x, t)|^2 dx dt. \quad (4.11)$$

*Proof of Proposition 4.2.* Inspired in part by [1] (which was concerned with a heat-wave system<sup>1</sup>), we shall establish some Carleman estimates for the (backward) parabolic equation (4.6) and the ODE (4.7) with the same singular weight.

For a better comprehension, the proof will be divided into two steps as follows:

<sup>1</sup>See also [9] for some Carleman estimates for a coupled system of parabolic-hyperbolic equations.

*Step 1.* We apply suitable Carleman estimates for the parabolic equation (4.6) and the ODE (4.7), with the same weights and with a moving control region.

*Step 2.* We estimate a local integral of  $p$  in terms of a local integral of  $q$  and some small order terms. Finally, we combine all the estimates obtained in the first step and derive the desired Carleman inequality.

The basic weight function we need in order to prove such inequalities is given by the following Lemma.

**Lemma 4.3.** *Let  $X$ ,  $\omega_0$  and  $\omega$  be as in Theorem 1.1, and let  $\omega_1$  be a nonempty open set in  $\mathbb{R}^N$  such that*

$$\overline{\omega_0} \subset \omega_1, \quad \overline{\omega_1} \subset \omega. \quad (4.12)$$

*Then there exist a number  $\delta \in (0, T/2)$  and a function  $\psi \in C^\infty(\overline{\Omega} \times [0, T])$  such that*

$$\nabla \psi(x, t) \neq 0, \quad t \in [0, T], \quad x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \quad (4.13)$$

$$\psi_t(x, t) \neq 0, \quad t \in [0, T], \quad x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \quad (4.14)$$

$$\psi_t(x, t) > 0, \quad t \in [0, \delta], \quad x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \quad (4.15)$$

$$\psi_t(x, t) < 0, \quad t \in [T - \delta, T], \quad x \in \overline{\Omega} \setminus X(\omega_1, t, 0), \quad (4.16)$$

$$\frac{\partial \psi}{\partial n}(x, t) \leq 0, \quad t \in [0, T], \quad x \in \partial \Omega, \quad (4.17)$$

$$\psi(x, t) > \frac{3}{4} \|\psi\|_{L^\infty(\Omega \times (0, T))}, \quad t \in [0, T], \quad x \in \overline{\Omega}. \quad (4.18)$$

The proof of Lemma 4.3 will be given in Appendix A.

Next, we pick a function  $g \in C^\infty(0, T)$  such that

$$g(t) = \begin{cases} \frac{1}{t} & \text{for } 0 < t < \delta/2, \\ \text{strictly decreasing} & \text{for } 0 < t \leq \delta, \\ 1 & \text{for } \delta \leq t \leq \frac{T}{2}, \\ g(T-t) & \text{for } \frac{T}{2} \leq t < T \end{cases}$$

and define the weights

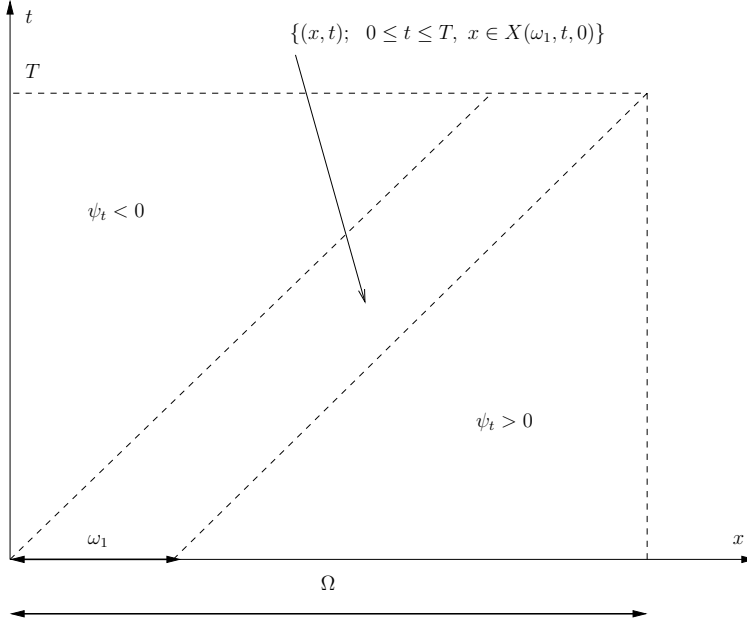
$$\begin{aligned} \varphi(x, t) &= g(t)(e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty}} - e^{\lambda\psi(x,t)}), & (x, t) \in \Omega \times (0, T), \\ \theta(x, t) &= g(t)e^{\lambda\psi(x,t)}, & (x, t) \in \Omega \times (0, T), \end{aligned}$$

where  $\|\psi\|_{L^\infty} = \|\psi\|_{L^\infty(\Omega \times (0, T))}$  and  $\lambda > 0$  is a parameter.

*Step 1. CARLEMAN ESTIMATES WITH THE SAME WEIGHT.*

In this step we apply a Carleman inequality for the heat-like equation (4.6) and a Carleman inequality for the ODE (4.7), both with the same weight. We combine such inequalities and obtain a global estimation of  $p$  and  $q$  in terms of local integrals of  $p$  and  $q$ .

For the purpose of the proof, we assume that the following two lemmas are true (their proof are given, respectively, in Appendices B and C).


 FIGURE 6. Sign of the time derivative of  $\psi$ .

**Lemma 4.4.** *There exist some constants  $\lambda_0 > 0$ ,  $s_0 > 0$  and  $C_0 > 0$  such that for all  $\lambda \geq \lambda_0$ , all  $s \geq s_0$  and all  $p \in C([0, T]; L^2(\Omega))$  with  $p_t + \Delta p \in L^2(0, T; L^2(\Omega))$ , the following holds*

$$\begin{aligned} & \int_0^T \int_{\Omega} [(s\theta)^{-1} (|\Delta p|^2 + |p_t|^2) + \lambda^2 (s\theta) |\nabla p|^2 + \lambda^4 (s\theta)^3 |p|^2] e^{-2s\varphi} dx dt \\ & \leq C_0 \left( \int_0^T \int_{\Omega} |p_t + \Delta p|^2 e^{-2s\varphi} dx dt + \int_0^T \int_{X(\omega_1, t, 0)} \lambda^4 (s\theta)^3 |p|^2 e^{-2s\varphi} dx dt \right). \end{aligned} \quad (4.19)$$

**Lemma 4.5.** *There exist some numbers  $\lambda_1 \geq \lambda_0$ ,  $s_1 \geq s_0$  and  $C_1 > 0$  such that for all  $\lambda \geq \lambda_1$ , all  $s \geq s_1$  and all  $q \in H^1(0, T; L^2(\Omega))$ , the following holds*

$$\int_0^T \int_{\Omega} (\lambda^2 s\theta) |q|^2 e^{-2s\varphi} dx dt \leq C_1 \left( \int_0^T \int_{\Omega} |q_t|^2 e^{-2s\varphi} dx dt + \int_0^T \int_{X(\omega, t, 0)} \lambda^2 (s\theta)^2 |q|^2 e^{-2s\varphi} dx dt \right). \quad (4.20)$$

Applying the Carleman inequality given in Lemma 4.4 to the heat-like equation (4.6), we obtain

$$\begin{aligned}
& \int_0^T \int_{\Omega} [(s\theta)^{-1}(|\Delta p|^2 + |p_t|^2) + \lambda^2(s\theta)|\nabla p|^2 + \lambda^4(s\theta)^3|p|^2] e^{-2s\varphi} dxdt \\
& \leq C_0 \left( \int_0^T \int_{\Omega} |(b(x-1)(p-q))^2| e^{-2s\varphi} dxdt + \int_0^T \int_{X(\omega_1, t, 0)} \lambda^4(s\theta)^3|p|^2 e^{-2s\varphi} dxdt \right). \quad (4.21)
\end{aligned}$$

Next, we apply the Carleman inequality given by Lemma 4.5 to the ODE (4.7) and obtain

$$\begin{aligned}
& \int_0^T \int_{\Omega} (\lambda^2 s\theta)|q|^2 e^{-2s\varphi} dxdt \\
& \leq C_1 \left( \int_0^T \int_{\Omega} |q-p|^2 e^{-2s\varphi} dxdt + \int_0^T \int_{X(\omega, t, 0)} \lambda^2(s\theta)^2|q|^2 e^{-2s\varphi} dxdt \right). \quad (4.22)
\end{aligned}$$

Adding (4.21) and (4.22), it is not difficult to see that

$$\begin{aligned}
& \int_0^T \int_{\Omega} [(s\theta)^{-1}(|\Delta p|^2 + |p_t|^2) + \lambda^2(s\theta)|\nabla p|^2 + \lambda^4(s\theta)^3|p|^2] e^{-2s\varphi} dxdt + \int_0^T \int_{\Omega} (\lambda^2 s\theta)|q|^2 e^{-2s\varphi} dxdt \\
& \leq C \left( \int_0^T \int_{X(\omega, t, 0)} \lambda^2(s\theta)^2|q|^2 e^{-2s\varphi} dxdt + \int_0^T \int_{X(\omega_1, t, 0)} \lambda^4(s\theta)^3|p|^2 e^{-2s\varphi} dxdt \right) \quad (4.23)
\end{aligned}$$

for appropriate  $s \geq s_2 \geq s_1$  and  $\lambda \geq \lambda_2 \geq \lambda_1$ .

*Step 2. ARRANGEMENTS AND CONCLUSION.*

In this step we estimate the local integral of  $p$  appearing in (4.23) by a local integral of  $q$  and some small order terms. Finally, using semigroup theory, we finish the proof of Proposition 4.2.

The main result of this step is the following.

**Lemma 4.6.** *There exist some numbers  $\lambda_2 \geq \lambda_1$ ,  $s_2 \geq s_1$  and  $C_2 > 0$  such that for all  $\lambda \geq \lambda_2$ , all  $s \geq s_2$  and all  $(p_0, q_0) \in L^2(\Omega)^2$ , the corresponding solution  $(p, q)$  of system (4.6)-(4.10) fulfills*

$$\begin{aligned}
& \int_0^T \int_{\Omega} [(s\theta)^{-1}(|\Delta p|^2 + |p_t|^2) + \lambda^2(s\theta)|\nabla p|^2 + \lambda^4(s\theta)^3|p|^2] e^{-2s\varphi} dxdt + \int_0^T \int_{\Omega} (\lambda^2 s\theta)|q|^2 e^{-2s\varphi} dxdt \\
& \leq C_2 \int_0^T \int_{X(\omega, t, 0)} \lambda^8(s\theta)^7 e^{-2s\varphi} |q|^2 dxdt. \quad (4.24)
\end{aligned}$$

*Proof of Lemma 4.6.* In order to prove Lemma 4.6, we just need to estimate the  $p$  appearing in the right-hand side of (4.23). For that, we introduce the function

$$\zeta(x, t) := \xi(X(x, 0, t)), \quad (4.25)$$

where  $\xi$  is a cut-off function satisfying

$$\xi \in C_0^\infty(\omega), \quad (4.26)$$

$$0 \leq \xi(x) \leq 1, \quad x \in \mathbb{R}^N, \quad (4.27)$$

$$\xi(x) = 1, \quad x \in \omega_1. \quad (4.28)$$

We have that

$$\int_0^T \int_{X(\omega_1, t, 0)} \lambda^4(s\theta)^3 |p|^2 e^{-2s\varphi} dx dt \leq \int_0^T \int_\Omega \zeta \lambda^4(s\theta)^3 |p|^2 e^{-2s\varphi} dx dt \quad (4.29)$$

and we use (4.7) to write

$$\begin{aligned} \int_0^T \int_\Omega \zeta \lambda^4(s\theta)^3 |p|^2 e^{-2s\varphi} dx dt &= \int_0^T \int_\Omega \zeta \lambda^4(s\theta)^3 p q e^{-2s\varphi} dx dt \\ &\quad + \int_0^T \int_\Omega \zeta \lambda^4(s\theta)^3 p(-q_t) e^{-2s\varphi} dx dt =: M_1 + M_2. \end{aligned} \quad (4.30)$$

It remains to estimate  $M_1$  and  $M_2$ . Using Cauchy-Schwarz inequality and (4.26)-(4.27), we have, for every  $\varepsilon > 0$ ,

$$|M_1| \leq \varepsilon \int_0^T \int_\Omega \lambda^4(s\theta)^3 |p|^2 e^{-2s\varphi} dx dt + \frac{1}{4\varepsilon} \int_0^T \int_{X(\omega, t, 0)} \lambda^4(s\theta)^3 |q|^2 e^{-2s\varphi} dx dt. \quad (4.31)$$

On the other hand, integrating by parts with respect to  $t$  in  $M_2$  yields

$$\begin{aligned} M_2 &= \int_0^T \int_\Omega \zeta \lambda^4(s\theta)^3 p_t q e^{-2s\varphi} dx dt + \int_0^T \int_\Omega \zeta \lambda^4(3s^3 \theta^2 \theta_t - 2s^4 \varphi_t \theta^3) p q e^{-2s\varphi} dx dt \\ &\quad - \int_0^T \int_\Omega \nabla \xi(X(x, 0, t)) \cdot \left(\frac{\partial X}{\partial x}\right)^{-1}(X(x, 0, t), t, 0) f(x, t) \lambda^4(s\theta)^3 p q e^{-2s\varphi} dx dt \\ &=: M_2^1 + M_2^2 - M_2^3. \end{aligned}$$

For  $M_2^1$ , we notice that for every  $\varepsilon > 0$ ,

$$|M_2^1| \leq \varepsilon \int_0^T \int_\Omega (s\theta)^{-1} |p_t|^2 e^{-2s\varphi} dx dt + \frac{1}{4\varepsilon} \int_0^T \int_{X(\omega, t, 0)} \lambda^8(s\theta)^7 |q|^2 e^{-2s\varphi} dx dt. \quad (4.32)$$

Since  $|\theta_t| + |\varphi_t| \leq C\lambda\theta^2$ , we infer that

$$\begin{aligned} |M_2^2| &\leq C \int_0^T \int_\Omega \zeta s^4 (\lambda\theta)^5 |p q| e^{-2s\varphi} dx dt \\ &\leq \varepsilon \int_0^T \int_\Omega \lambda^4(s\theta)^3 |p|^2 e^{-2s\varphi} dx dt + \frac{C}{\varepsilon s^2} \int_0^T \int_{X(\omega, t, 0)} \lambda^6(s\theta)^7 |q|^2 e^{-2s\varphi} dx dt. \end{aligned} \quad (4.33)$$

Finally,  $M_2^3$  is estimated as  $M_1$ :

$$|M_2^3| \leq \varepsilon \int_0^T \int_\Omega \lambda^4(s\theta)^3 |p|^2 e^{-2s\varphi} dx dt + \frac{C}{\varepsilon} \int_0^T \int_{X(\omega, t, 0)} \lambda^4(s\theta)^3 |q|^2 e^{-2s\varphi} dx dt. \quad (4.34)$$

Gathering together (4.23) and (4.29)-(4.34) and taking  $\epsilon$  small enough, we obtain (4.24).

Now we finish the proof of the observability inequality (4.11).

Pick any  $(p_0, q_0) \in L^2(\Omega)^2$ , and denote by  $(p, q)$  the solution of (4.6)-(4.10). Note that  $p \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$  and that  $q \in H^1(0, T; L^2(\Omega))$ . Using classical semigroup estimates, one derives at once (4.11) from (4.24).  $\square$

## 5. FINAL COMMENTS

- *Another decomposition*

As commented in the introduction, there is another splitting of the operator  $\mathcal{L} = \partial_t^2 - \Delta - \Delta \partial_t + \partial_t$ , given by

$$\mathcal{L} = (\partial_t - \Delta)(\partial_t + Id).$$

Thus, letting

$$v(x, t) = y(x, t) + y_t(x, t),$$

we see that (1.1) may be written as

$$v_t - \Delta v = 1_{\omega(t)} h + (1 - b(x))(v - y), \quad (5.1)$$

$$y_t + y = v, \quad (5.2)$$

which is a coupled system of a parabolic equation (5.1) and an ODE (5.2).

This splitting was used to prove Theorem 1.1 with less assumptions on the trajectories (see Section 2).

The control term  $h$  acts directly in the heat equation and indirectly in the ODE through the coupling term  $v$ . The problem can be treated directly as such, with requires further work at the level of the dual observability problem since both components of the adjoint system will be needed to be observed by partial measurements only on one of its components. The problem can also be addressed incorporating in (5.2) an additional auxiliary control acting directly in the ODE. This leads to the system

$$v_t - \Delta v = 1_{\omega(t)} h + (1 - b(x))(v - y) \quad (5.3)$$

$$y_t + y = 1_{\omega(t)} k + v, \quad (5.4)$$

where  $(v, y) \in L^2(\Omega)^2$  is the state function to be controlled, and  $(h, k) \in L^2(0, T; L^2(\Omega)^2)$  is the control input.

Once the controllability of this system is proved, when going back to the original viscoelasticity equation, one gets

$$y_{tt} - \Delta y - \Delta y_t + b(x)y_t = 1_{\omega(t)} [h - (1 - b(x))k] + (\partial_t - \Delta)[1_{\omega(t)}k]. \quad (5.5)$$

But, then, the second control  $1_{\omega(t)}k$  enters under the action of the heat operator. It is then necessary to ensure that the control  $k$  is smooth enough and, furthermore, to replace in (5.4) the cut-off function  $1_{\omega(t)}$  by a regularized version. These are technicalities that can be overcome with further work. To be more precise, the control in (5.5) takes the form  $\mathbf{1}_{X(\omega, t, 0)}(x)\tilde{h}$  with  $\tilde{h} \in L^2(0, T; L^2(\Omega))$ , provided that both  $h, k \in L^2(0, T; L^2(\Omega))$  and

$$k \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$



Therefore special attention has to be paid to obtain smooth controls for the transport equation (see Section 2).

- *Manifolds without boundary*

The lack of propagation properties of the ODE (1.10) in the space variable requires the control to move in time. As we mentioned in the introduction, through a suitable change of variables, this is equivalent to keeping the support of the control fixed but replacing the ODE by a transport equation. Obviously, attention has to be paid to the Dirichlet boundary conditions when performing this change of variables. Of course, this is no longer an issue when the model is considered in a smooth manifold without boundary. As an example of such a situation we consider the periodic case in the torus

$$x \in \mathbb{T}^N := \mathbb{R}^N / \mathbb{Z}^N. \tag{5.6}$$

For a moving control with a constant velocity  $\omega(t) = \{x - ct; x \in \omega\}$ ,  $c \in \mathbb{R}^N \setminus \{0\}$ , system (1.9)-(1.10) can be put in the form of a coupled system of parabolic-hyperbolic equations

$$v_t - \Delta v - c \cdot \nabla v + (b(x + ct) - 1)v = w \tag{5.7}$$

$$w_t - c \cdot \nabla w + w = 1_\omega(x) \tilde{h} + (b(x + ct) - 1)v \tag{5.8}$$

by letting

$$v(x, t) = y(x + ct, t), \tag{5.9}$$

$$w(x, t) = z(x + ct, t), \tag{5.10}$$

$$\tilde{h}(x, t) = h(x + ct, t). \tag{5.11}$$

$$\tag{5.12}$$

The system is now constituted by the coupling between a heat and a transport equation with control  $\tilde{h}$  with fixed support. Once more, the problem now can be treated by means of the classical duality principle between the controllability problem and the observability property of the adjoint system. The later was solved in [25] in  $1 - d$  using Fourier analysis techniques and in this paper we do it using Carleman inequalities.

Note that the Carleman approach developed in this paper cannot be applied as it is to the periodic case. Consider for instance the case of the torus  $\mathbb{T}$ . A weight  $\psi \in C^\infty(\mathbb{T} \times (0, T))$  as in Lemma 4.3 does not exist, because of the periodicity in  $x$  (see Figure 6.) However, it is well known that the periodic case can be deduced from both the Dirichlet case and the Neumann case (using classical extensions by reflection, see e.g. [30]). Even if the Neumann case was not considered in this paper, it is likely that it could be treated in much the same way as we did for the Dirichlet case.

## 6. OPEN PROBLEMS AND FURTHER QUESTIONS

The main result of this paper concerns the controllability of a coupled system consisting on a heat equation and an ODE. By addressing the dual problem of observability and making the controller/observer move in time, this ends being very close to the problem of observability of a coupled system of a heat equation and a first order transport equation. The techniques we

have developed here are inspired in the work [1] where the key point was to use the same weight function for the Carleman inequality in both the heat and the transport equation.

The system under consideration, coupling a heat and a hyperbolic equation, is close to that of thermoelasticity that was considered in [19]. But, there, the problem was only dealt with in the case of manifolds without boundary, by means of spectral decomposition techniques allowing to decouple the system into the parabolic and the hyperbolic components. As far as we know, a complete analysis of the system of thermoelasticity using Carleman type inequalities seems to be not developed so far.

The structure of the parabolic-transport system we consider is also, in some sense, similar to the one considered in [9] for the  $1 - d$  compressible Navier-Stokes equation although, in the latter, the system is of nonlinear nature requiring significant extra analysis beyond the linearized model.

Our analysis is also related to recent works on the control of parabolic equations with memory terms as for instance in [13]. Note that the system (1.9)-(1.10) in the particular case  $b \equiv 1$  and  $z(0) \equiv 0$ , in the absence of the control  $h$  and the addition of a control of the form  $1_{\omega(t)}k$  in the first equation, can be written as an integro-differential equation

$$y_t - \Delta y + (b - 1)[y - \int_0^t e^{s-t} y(x, s) ds] = 1_{\omega(t)}k \quad (6.1)$$

This system is closely related to the one considered in [13]. There it is shown that the system lacks to be null controllable. This is in agreement with our results that, in the particular case under consideration, show also that a moving control could bypass this limitation. It would be interesting to analyze to which extent this idea of controlling by moving the support of the control can be of use for more general parabolic equations with memory terms.

In this paper we have shown the null controllability of a linear system which consists of a parabolic equation and an ordinary differential equation that arise from the identification of the parabolic and hyperbolic parts of system (1.1)-(1.2). Besides, coupled systems consisting of parabolic equations and ode's are important since they appear in biological models of chemotaxis or interactions between cellular process and diffusing growth factors (see [14], [24], [27] and references therein). Systems governing these phenomena are, in general, non linear and have the form

$$u_t = f(u, v), \quad (6.2)$$

$$v_t = D\Delta v + g(u, v), \quad (6.3)$$

where  $v$  and  $u$  are vectors,  $D$  is a diagonal matrix with positive coefficients and  $f$  and  $g$  are real functions.

Other area where coupled parabolic-ode systems play a major role is electrocardiology (see [3], [7], [33] and references therein). Here the cardiac activity is described by the bidomain model, which consists of a system of two degenerate parabolic reaction-diffusion equations, representing the intra and extracellular potential in the cardiac muscle, coupled with a system of ordinary differential equations representing the ionic currents flowing through the cellular membrane.

The bidomain model is given by

$$\chi C_m v_t - \operatorname{div}(D_i \nabla u_i) + \chi I_{ion}(v, w) = I_{app}^i, \quad (6.4)$$

$$-\chi C_m v_t - \operatorname{div}(D_e \nabla u_e) - \chi I_{ion}(v, w) = -I_{app}^e, \quad (6.5)$$

$$w_t - R(v, w) = 0, \quad (6.6)$$

where  $u_i$  and  $u_e$  are the intra and extracellular potentials,  $v$  is the transmembrane potential,  $\chi$  is the ratio of membrane area per tissue volume,  $C_m$  is the surface capacitance of the membrane,  $I_{ion}$  is the ionic current,  $I_{app}^{i,e}$  is an applied current and  $D_{i,e}$  are conductivity tensors.

Concerning controllability of coupled parabolic-ode systems, just a few results for some particular systems are known (see [8] and [23] for the controllability of a simplified one-dimensional model for the motion of a rigid body in a viscous fluid). We believe that ideas presented in this paper can be used for the study of the controllability for other systems of parabolic-ode equations, such as (6.2)-(6.3) and (6.4)-(6.6).

#### APPENDIX A. PROOF OF LEMMA 4.3

*Proof.* Pick any  $\delta < \min(t_1, T - t_2, T/2)$ . We search  $\psi$  (see Figure 6) in the form

$$\psi(x, t) = \psi_1(x, t) + C_2 \psi_2(x, t) + C_3 \quad (\text{A.1})$$

where, roughly,  $\psi_1$  fulfills (4.13),  $\psi_2$  fulfills (4.14)-(4.16) together with  $\nabla \psi_2 \equiv 0$  outside  $X(\omega_1, t, 0)$ , and  $C_2, C_3$  are (large enough) positive constants.

*Step 1. CONSTRUCTION OF  $\psi_1$ .*

Let  $\Gamma \in C^\infty([0, T]; \mathbb{R}^N)$  be as in (1.13), and let  $\varepsilon > 0$  be such that

$$B(\Gamma(t), 3\varepsilon) \subset X(\omega_0, t, 0) \cap \Omega, \quad t \in [0, T].$$

We choose a vector field  $\tilde{f} \in C^\infty(\mathbb{R}^N \times [0, T]; \mathbb{R}^N)$  such that

$$\tilde{f}(x, t) = \begin{cases} \dot{\Gamma}(t) & \text{if } t \in [0, T], x \in B(\Gamma(t), \varepsilon), \\ 0 & \text{if } t \in [0, T], x \in \mathbb{R}^N \setminus B(\Gamma(t), 2\varepsilon). \end{cases}$$

Let  $\tilde{X}$  denote the flow associated with  $\tilde{f}$ ; that is,  $\tilde{X}$  solves

$$\begin{aligned} \frac{\partial \tilde{X}}{\partial t}(x, t, t_0) &= \tilde{f}(\tilde{X}(x, t, t_0), t), & (x, t, t_0) &\in \mathbb{R}^N \times [0, T]^2, \\ \tilde{X}(x, t_0, t_0) &= x, & (x, t_0) &\in \mathbb{R}^N \times [0, T]. \end{aligned}$$

Note that

$$\begin{aligned} \tilde{X}(y + \Gamma(0), t, 0) &= y + \Gamma(t) & \text{if } (y, t) &\in B(0, \varepsilon) \times [0, T], \\ \tilde{X}(x, t, t_0) &= x & \text{if } \operatorname{dist}(x, \partial\Omega) < \varepsilon, (t, t_0) &\in [0, T]^2. \end{aligned}$$

By a well-known result (see [16, Lemma 1.2]), there exists a function  $\tilde{\psi} \in C^\infty(\bar{\Omega})$  such that

$$\begin{aligned} \tilde{\psi}(x) &> 0 & \text{if } x &\in \Omega; \\ \tilde{\psi}(x) &= 0 & \text{if } x &\in \partial\Omega; \\ \nabla \tilde{\psi}(x) &\neq 0 & \text{if } x &\in \bar{\Omega} \setminus B(\Gamma(0), \varepsilon). \end{aligned}$$

Actually, the function  $\tilde{\psi}$  given in [16] is only of class  $C^2$ , but the regularity  $C^\infty$  can be obtained by mollification with a partition of unity (see e.g. [29, Lemma 4.2]). Let us set

$$\psi_1(x, t) = \tilde{\psi}(\tilde{X}(x, 0, t)).$$

Then  $\psi_1 \in C^\infty(\bar{\Omega} \times [0, T])$  and it fulfills

$$\psi_1(x, t) > 0 \quad \text{if } (x, t) \in \Omega \times [0, T], \quad (\text{A.2})$$

$$\psi_1(x, t) = 0 \quad \text{if } (x, t) \in \partial\Omega \times [0, T], \quad (\text{A.3})$$

$$\nabla\psi_1(x, t) = \nabla\tilde{\psi}(\tilde{X}(x, 0, t)) \frac{\partial\tilde{X}}{\partial x}(x, 0, t) \neq 0 \quad \text{if } x \in \bar{\Omega} \setminus X(\omega_0, t, 0). \quad (\text{A.4})$$

For (A.4), we notice that if we write  $x = \tilde{X}(\tilde{x}, t, 0)$ , then  $\tilde{x} = \tilde{X}(x, 0, t)$  hence

$$\nabla\tilde{\psi}(\tilde{X}(x, 0, t)) = \nabla\tilde{\psi}(\tilde{x}) \neq 0$$

if  $\tilde{x} \notin B(\Gamma(0), \varepsilon)$ , which is equivalent to  $x \notin B(\Gamma(t), \varepsilon)$ . The last condition is satisfied when  $x \notin X(\omega_0, t, 0)$ .

*Step 2. CONSTRUCTION OF  $\psi_2$ .*

From (1.15), (1.16), and (1.17), we can pick two curves  $\gamma_1 \in C^0([0, t_2]; \Omega)$  and  $\gamma_2 \in C^0((t_1, T]; \Omega)$  such that

$$\begin{aligned} \gamma_1(t) &\notin \overline{X(\omega_0, t, 0)}, & 0 \leq t < t_2, \\ \gamma_2(t) &\notin \overline{X(\omega_0, t, 0)}, & t_1 < t \leq T. \end{aligned}$$

We infer from (1.17) that for any  $t \in (t_1, t_2)$ ,  $\gamma_1(t)$  and  $\gamma_2(t)$  do not belong to the same connected component of  $\Omega \setminus \overline{X(\omega_0, t, 0)}$ . Let  $\Omega_1(t)$  (resp.  $\Omega_2(t)$ ) denote the connected component of  $\gamma_1(t)$  (resp.  $\gamma_2(t)$ ) for  $0 \leq t < t_2$  (resp. for  $t_1 < t \leq T$ ). Clearly

$$\Omega \setminus \overline{X(\omega_0, t, 0)} = \begin{cases} \Omega_1(t), & \text{if } 0 \leq t \leq t_1, \\ \Omega_1(t) \cup \Omega_2(t), & \text{if } t_1 < t < t_2, \\ \Omega_2(t), & \text{if } t_2 \leq t \leq T. \end{cases}$$

Set  $\Omega_1(t) = \emptyset$  for  $t \in [t_2, T]$ , and  $\Omega_2(t) = \emptyset$  for  $t \in [0, t_1]$ . Let  $\psi_2 \in C^\infty(\bar{\Omega} \times [0, T])$  with

$$\begin{aligned} \psi_2(x, t) &= t(\mathbf{1}_{\Omega_1(t)}(x) - \mathbf{1}_{\Omega_2(t)}(x)) \quad \text{for } t \in [0, T], \quad x \in \Omega \setminus X(\omega_1, t, 0), \\ \frac{\partial\psi_2}{\partial n} &= 0 \quad \text{for } (x, t) \in \partial\Omega \times [0, T]. \end{aligned}$$

Such a function  $\psi_2$  exists, since by (4.12)

$$\inf_{t_1 < t < t_2} \text{dist}(\Omega_1(t) \setminus X(\omega_1, t, 0), \Omega_2(t) \setminus X(\omega_1, t, 0)) > 0.$$

Then

$$\frac{\partial\psi_2}{\partial t} = \begin{cases} 1 & \text{if } 0 \leq t < t_2 \text{ and } x \in \Omega_1(t) \setminus X(\omega_1, t, 0), \\ -1 & \text{if } t_1 < t \leq T \text{ and } x \in \Omega_2(t) \setminus X(\omega_1, t, 0). \end{cases}$$

and

$$\nabla\psi_2(x, t) = 0 \quad \text{if } x \in \Omega \setminus X(\omega_1, t, 0).$$

Note that (4.14)-(4.16) are satisfied for  $\psi_2$ . Note also that for any pair  $(\tau_1, \tau_2)$  with  $0 \leq \tau_1 < \tau_2 \leq T$ , the set

$$K_{\tau_1, \tau_2} := \{(x, t) \in \mathbb{R}^{N+1}; \tau_1 \leq t \leq \tau_2, x \in \bar{\Omega} \setminus X(\omega_1, t, 0)\}$$

is compact.

*Step 3. CONSTRUCTION OF  $\psi$ .*

Let  $\psi$  be defined as in (A.1), with  $C_2 > 0$  and  $C_3$  to be determined. Then (4.13) and (4.17) are satisfied. We pick  $C_2$  large enough for (4.14)-(4.16) to be satisfied. Finally, (4.18) is satisfied for  $C_3$  large enough.  $\square$

## APPENDIX B. PROOF OF LEMMA 4.4

*Proof of Lemma 4.4.* The method of the proof is widely inspired from [12], and the computations are presented as in [29, Proof of Proposition 4.3].

Let  $v = e^{-s\varphi}p$  and  $P = \partial_t + \Delta$ . Then

$$e^{-s\varphi}Pp = e^{-s\varphi}P(e^{s\varphi}v) = P_s v + P_a v$$

where

$$P_s v = \Delta v + (s\varphi_t + s^2|\nabla\varphi|^2)v, \quad (\text{B.1})$$

$$P_a v = v_t + 2s\nabla\varphi \cdot \nabla v + s(\Delta\varphi)v \quad (\text{B.2})$$

denote the (formal) self-adjoint and skew-adjoint parts of  $e^{-s\varphi}P(e^{s\varphi}\cdot)$ , respectively. It follows that

$$\|e^{-s\varphi}Pp\|^2 = \|P_s v\|^2 + \|P_a v\|^2 + 2(P_s v, P_a v) \quad (\text{B.3})$$

where  $(f, g) = \int_0^T \int_{\Omega} fg \, dxdt$ ,  $\|f\|^2 = (f, f)$ . In the sequel,  $\int_0^T \int_{\Omega} f(x, t) \, dxdt$  is denoted  $\iint f$  for the sake of shortness. We have

$$\begin{aligned} (P_s v, P_a v) &= (\Delta v, v_t) + (\Delta v, 2s\nabla\varphi \cdot \nabla v) + (\Delta v, s(\Delta\varphi)v) + (s\varphi_t v + s^2|\nabla\varphi|^2 v, v_t) \\ &+ (s\varphi_t v + s^2|\nabla\varphi|^2 v, 2s\nabla\varphi \cdot \nabla v) + (s\varphi_t v + s^2|\nabla\varphi|^2 v, s(\Delta\varphi)v) =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \quad (\text{B.4})$$

First, observe that

$$I_1 = - \iint \nabla v \cdot \nabla v_t = 0. \quad (\text{B.5})$$

Using the convention of repeated indices and denoting  $\partial_i = \partial/\partial x_i$ , we obtain that

$$\begin{aligned} I_2 &= 2s \iint \partial_j^2 v \partial_i \varphi \partial_i v \\ &= -2s \iint \partial_j v (\partial_j \partial_i \varphi \partial_i v + \partial_i \varphi \partial_j \partial_i v) + 2s \int_0^T \int_{\partial\Omega} (\partial_j v) n_j \partial_i \varphi \partial_i v \, d\sigma. \end{aligned}$$

Since  $v = 0$  for  $(x, t) \in \partial\Omega \times (0, T)$ ,  $\nabla v = (\partial v/\partial n)n$ , so that  $\nabla\varphi \cdot \nabla v = (\partial\varphi/\partial n)(\partial v/\partial n)$  and

$$\int_0^T \int_{\partial\Omega} (\partial_j v) n_j \partial_i \varphi \partial_i v \, d\sigma = \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n) |\partial v/\partial n|^2 \, d\sigma.$$

It follows that

$$\begin{aligned} I_2 &= -2s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v - s \iint \partial_i \varphi \partial_i (|\partial_j v|^2) + 2s \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n) |\partial v/\partial n|^2 d\sigma \\ &= -2s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + s \iint \Delta\varphi |\nabla v|^2 + s \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n) |\partial v/\partial n|^2 d\sigma \end{aligned} \quad (\text{B.6})$$

On the other hand, integrations by parts in  $x$  yields

$$I_3 = -s \iint \nabla v \cdot (v \nabla(\Delta\varphi) + (\Delta\varphi) \nabla v) = s \iint \Delta^2 \varphi \frac{|v|^2}{2} - s \iint \Delta\varphi |\nabla v|^2 \quad (\text{B.7})$$

and integration by parts with respect to  $t$  gives

$$I_4 = - \iint (s\varphi_{tt} + s^2 \partial_t |\nabla\varphi|^2) \frac{|v|^2}{2}.$$

Integrating by parts with respect to  $x$  in  $I_5$  yields

$$I_5 = - \iint s^2 \nabla \cdot (\varphi_t \nabla\varphi) |v|^2 - \iint s^3 \nabla \cdot (|\nabla\varphi|^2 \nabla\varphi) |v|^2. \quad (\text{B.8})$$

Gathering (B.4)-(B.8), we infer that

$$\begin{aligned} 2(P_s v, P_a v) &= -4s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + 2s \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n) |\partial v/\partial n|^2 d\sigma \\ &\quad + \iint |v|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^2 \partial_t |\nabla\varphi|^2 - 2s^3 \nabla\varphi \cdot \nabla |\nabla\varphi|^2]. \end{aligned}$$

Consequently, (B.3) may be rewritten

$$\begin{aligned} \|e^{-s\varphi} Pp\|^2 &= \|P_s v\|^2 + \|P_a v\|^2 - 4s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + 2s \int_0^T \int_{\partial\Omega} (\partial\varphi/\partial n) |\partial v/\partial n|^2 d\sigma \\ &\quad + \iint |v|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^2 \partial_t |\nabla\varphi|^2 - 2s^3 \nabla\varphi \cdot \nabla |\nabla\varphi|^2]. \end{aligned}$$

CLAIM 1. There exist some numbers  $\lambda_1 > 0$ ,  $s_1 > 0$  and  $A \in (0, 1)$  such that for all  $\lambda \geq \lambda_1$  and all  $s \geq s_1$ ,

$$\begin{aligned} &\iint |v|^2 [s(\Delta^2 \varphi - \varphi_{tt}) - 2s^2 \partial_t |\nabla\varphi|^2 - 2s^3 \nabla\varphi \cdot \nabla |\nabla\varphi|^2] \\ &\quad + A^{-1} \lambda s^3 \int_0^T \int_{X(\omega_1, t, 0)} (\lambda\theta)^3 |v|^2 \geq A \lambda s^3 \iint (\lambda\theta)^3 |v|^2. \end{aligned} \quad (\text{B.9})$$

*Proof of Claim 1.* Easy computations show that

$$\partial_i \varphi = -\lambda g(t) e^{\lambda\psi} \partial_i \psi, \quad \partial_j \partial_i \varphi = -g(t) e^{\lambda\psi} (\lambda^2 \partial_i \psi \partial_j \psi + \lambda \partial_j \partial_i \psi) \quad (\text{B.10})$$

and

$$-\nabla |\nabla\varphi|^2 \cdot \nabla\varphi = -2(\partial_j \partial_i \varphi) \partial_i \varphi \partial_j \varphi = 2(\lambda g e^{\lambda\psi})^3 (\lambda |\nabla\psi|^4 + \partial_j \partial_i \psi \partial_i \psi \partial_j \psi).$$

It follows from (4.13) that for  $\lambda$  large enough, say  $\lambda \geq \lambda_1$ , we have that

$$-\nabla|\nabla\varphi|^2 \cdot \nabla\varphi \geq A\lambda(\lambda\theta)^3, \quad t \in [0, T], \quad x \in \bar{\Omega} \setminus X(\omega_1, t, 0) \quad (\text{B.11})$$

$$|\nabla|\nabla\varphi|^2 \cdot \nabla\varphi| \leq A^{-1}\lambda(\lambda\theta)^3, \quad t \in [0, T], \quad x \in X(\omega_1, t, 0) \quad (\text{B.12})$$

for some constant  $A \in (0, 1)$ . According to (4.18), we have for some constant  $C > 0$

$$|\Delta^2\varphi| + |\varphi_{tt}| + |\partial_t|\nabla\varphi|^2| \leq C\lambda(\lambda\theta)^3, \quad t \in [0, T], \quad x \in \bar{\Omega}.$$

Therefore, we infer that for  $s$  large enough, say  $s \geq s_1$ , and for all  $\lambda \geq \lambda_1$  we have that

$$\begin{aligned} s(\Delta^2\varphi - \varphi_{tt}) - 2s^2\partial_t|\nabla\varphi|^2 - 2s^3\nabla\varphi \cdot \nabla|\nabla\varphi|^2 &\geq A\lambda s^3(\lambda\theta)^3, \quad t \in [0, T], \quad x \in \bar{\Omega} \setminus X(\omega_1, t, 0) \\ |s(\Delta^2\varphi - \varphi_{tt}) - 2s^2\partial_t|\nabla\varphi|^2 - 2s^3\nabla\varphi \cdot \nabla|\nabla\varphi|^2| &\leq 3A^{-1}\lambda s^3(\lambda\theta)^3, \quad t \in [0, T], \quad x \in X(\omega_1, t, 0). \end{aligned}$$

This gives (B.9) with a possibly decreased value of  $A$ .  $\square$

Thus, using the fact that  $\partial\varphi/\partial n \geq 0$  on  $\partial\Omega$  by (4.17), we conclude that

$$\begin{aligned} \|P_s v\|^2 + \|P_a v\|^2 + A\lambda s^3 \iint (\lambda\theta)^3 |v|^2 \\ \leq \|e^{-s\varphi} P p\|^2 + 4s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + A^{-1} \lambda s^3 \int_0^T \int_{X(\omega_1, t, 0)} (\lambda\theta)^3 |v|^2. \end{aligned} \quad (\text{B.13})$$

CLAIM 2. There exist some numbers  $\lambda_2 \geq \lambda_1$  and  $s_2 \geq s_1$  such that for all  $\lambda \geq \lambda_2$  and all  $s \geq s_2$ ,

$$\lambda s \iint (\lambda\theta) |\nabla v|^2 + \lambda s^{-1} \iint (\lambda\theta)^{-1} |\Delta v|^2 \leq C \left( s^{-1} \|P_s v\|^2 + \lambda s^3 \iint (\lambda\theta)^3 |v|^2 \right). \quad (\text{B.14})$$

*Proof of Claim 2.* By (B.1), we have

$$\begin{aligned} s^{-1} \iint (\lambda\theta)^{-1} |\Delta v|^2 &= s^{-1} \iint (\lambda\theta)^{-1} |P_s v - s\varphi_t v - s^2 |\nabla\varphi|^2 v|^2 \\ &\leq C s^{-1} \iint (\lambda\theta)^{-1} (|P_s v|^2 + s^2 |\varphi_t|^2 |v|^2 + s^4 (\lambda\theta)^4 |v|^2) \\ &\leq C \left( \frac{\|P_s v\|^2}{\lambda s} + s^3 \iint (\lambda\theta)^3 |v|^2 \right) \end{aligned} \quad (\text{B.15})$$

provided that  $s$  and  $\lambda$  are large enough, where we used (4.18) in the last line to bound  $\varphi_t$ . On the other hand,

$$\begin{aligned} \lambda s \iint (\lambda\theta) |\nabla v|^2 &= \lambda s \{ \iint (\lambda\theta) (-\Delta v) v - \iint (\nabla(\lambda\theta) \cdot \nabla v) v \} \\ &\leq \frac{\lambda}{2s} \iint (\lambda\theta)^{-1} |\Delta v|^2 + \frac{\lambda s^3}{2} \iint (\lambda\theta)^3 |v|^2 + \frac{\lambda s}{2} \iint \Delta(\lambda\theta) |v|^2 \\ &\leq C \left( s^{-1} \|P_s v\|^2 + \lambda s^3 \iint (\lambda\theta)^3 |v|^2 \right) \end{aligned} \quad (\text{B.16})$$

by (B.15), provided that  $s \geq s_2 \geq s_1$  and  $\lambda \geq \lambda_2 \geq \lambda_1$ . Then (B.14) follows from (B.15)-(B.16).  $\square$

We infer from (B.13)-(B.14) that

$$\begin{aligned} & \|P_a v\|^2 + \lambda s \iint (\lambda \theta) |\nabla v|^2 + \lambda s^{-1} \iint (\lambda \theta)^{-1} |\Delta v|^2 + \lambda s^3 \iint (\lambda \theta)^3 |v|^2 \\ & \leq C \left( \|e^{-s\varphi} P p\|^2 + 4s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v + A^{-1} \lambda s^3 \int_0^T \int_{X(\omega_1, t, 0)} (\lambda \theta)^3 |v|^2 \right). \end{aligned} \quad (\text{B.17})$$

By (B.10),

$$s \iint \partial_j \partial_i \varphi \partial_j v \partial_i v \leq -s \lambda \iint g(t) e^{\lambda \psi} \partial_j \partial_i \psi \partial_j v \partial_i v \leq C s \iint (\lambda \theta) |\nabla v|^2.$$

Therefore, for  $\lambda$  large enough and  $s \geq s_2$ ,

$$\begin{aligned} & \|P_a v\|^2 + \lambda s^3 \iint (\lambda \theta)^3 |v|^2 + \lambda s \iint (\lambda \theta) |\nabla v|^2 + \lambda s^{-1} \iint (\lambda \theta)^{-1} |\Delta v|^2 \\ & \leq C \left( \|e^{-s\varphi} P p\|^2 + \lambda s^3 \int_0^T \int_{X(\omega_1, t, 0)} (\lambda \theta)^3 |v|^2 \right). \end{aligned} \quad (\text{B.18})$$

Using (B.2) and (B.18), we see that for  $\lambda$  large enough and  $s \geq s_2$

$$\begin{aligned} \lambda s^{-1} \iint (\lambda \theta)^{-1} |v_t|^2 & \leq C \lambda s^{-1} \iint (\lambda \theta)^{-1} (|P_a v|^2 + s^2 |\nabla \varphi|^2 |\nabla v|^2 + s^2 |\Delta \varphi|^2 |v|^2) \\ & \leq C \left( \|e^{-s\varphi} P p\|^2 + \lambda s^3 \int_0^T \int_{X(\omega_1, t, 0)} (\lambda \theta)^3 |v|^2 \right). \end{aligned}$$

Hence, there exists some number  $\lambda_3 \geq \lambda_2$  such that for all  $\lambda \geq \lambda_3$  and all  $s \geq s_2$ , we have

$$\begin{aligned} & \lambda s^3 \iint (\lambda \theta)^3 |v|^2 + \lambda s \iint (\lambda \theta) |\nabla v|^2 + \lambda s^{-1} \iint (\lambda \theta)^{-1} (|\Delta v|^2 + |v_t|^2) \\ & \leq C \left( \|e^{-s\varphi} P p\|^2 + \lambda s^3 \int_0^T \int_{X(\omega_1, t, 0)} (\lambda \theta)^3 |v|^2 \right). \end{aligned} \quad (\text{B.19})$$

Replacing  $v$  by  $e^{-s\varphi} p$  in (B.19) gives at once (4.19). The proof of Lemma 4.4 is complete.  $\square$

### APPENDIX C. PROOF OF LEMMA 4.5

*Proof of Lemma 4.5.* The proof is divided into three parts corresponding to the estimates for  $t \in [0, \delta]$ , for  $t \in [\delta, T - \delta]$  and for  $t \in [T - \delta, T]$ . The estimates for  $t \in [0, \delta]$  and for  $t \in [T - \delta, T]$  being similar, we shall prove only the first ones.

Let  $v = e^{-s\varphi} q$ . Then

$$e^{-s\varphi} q_t = e^{-s\varphi} (e^{s\varphi} v)_t = s\varphi_t v + v_t =: P_s v + P_a v. \quad (\text{C.1})$$



CLAIM 3.

$$\begin{aligned} \int_0^\delta \int_\Omega \lambda (s\theta)^2 |v|^2 dx dt &\leq C \left( \int_0^\delta \int_\Omega \lambda^{-1} |e^{-s\varphi} q_t|^2 dx dt \right. \\ &\quad \left. + \int_\Omega [(1-\zeta)^2 (s\theta) |v|^2]_{t=\delta} dx + \int_0^\delta \int_{X(\omega, t, 0)} \lambda (s\theta)^2 |v|^2 dx dt \right), \end{aligned} \quad (\text{C.2})$$

where  $\zeta$  is the function introduced in (4.25).

To prove the claim, we compute in several ways

$$I := \int_0^\delta \int_\Omega (e^{-s\varphi} q_t) (1-\zeta)^2 s\theta v dx dt.$$

We split  $I$  into

$$I = \int_0^\delta \int_\Omega (P_s v) (1-\zeta)^2 s\theta v dx dt + \int_0^\delta \int_\Omega (P_a v) (1-\zeta)^2 s\theta v dx dt =: I_1 + I_2.$$

Then

$$\begin{aligned} I_1 &= \int_0^\delta \int_\Omega (1-\zeta)^2 s^2 \varphi_t \theta v^2 dx dt \\ &= \int_0^\delta \int_\Omega [g'(e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty}} - e^{\lambda\psi}) - g\lambda\psi_t e^{\lambda\psi}] (1-\zeta)^2 s^2 g e^{\lambda\psi} v^2 dx dt. \end{aligned}$$

On the other hand

$$\begin{aligned} I_2 &= \int_0^\delta \int_\Omega (1-\zeta)^2 (s g e^{\lambda\psi} v v_t) dx dt \\ &= \frac{1}{2} \int_\Omega [(1-\zeta)^2 s\theta |v|^2]_{t=\delta} dx - \int_0^\delta \int_\Omega s [g' e^{\lambda\psi} + g\lambda\psi_t e^{\lambda\psi}] (1-\zeta)^2 \frac{v^2}{2} dx dt \\ &\quad - \int_0^\delta \int_\Omega (1-\zeta) \nabla \xi(X(x, 0, t)) \cdot \left(\frac{\partial X}{\partial x}\right)^{-1}(X(x, 0, t), t, 0) f(x, t) s\theta v^2 dx dt, \end{aligned}$$

where we used the fact that  $[\theta |v|^2]_{t=0} = 0$ . Clearly, since  $\theta \geq 1$ , for  $s \geq 1$

$$\begin{aligned} \left| \int_0^\delta \int_\Omega (1-\zeta) \nabla \xi(X(x, 0, t)) \cdot \left(\frac{\partial X}{\partial x}\right)^{-1}(X(x, 0, t), t, 0) f(x, t) s\theta v^2 dx dt \right| \\ \leq C \int_0^\delta \int_{X(\omega, t, 0)} (s\theta)^2 |v|^2 dx dt. \end{aligned}$$

On the other hand, using (4.15), we see that there exist some constants  $C > 0$  and  $s_1 \geq s_0$  such that for all  $s \geq s_1$  and all  $\lambda \geq \lambda_0 > 0$ , it holds

$$\begin{aligned} g\lambda\psi_t e^{\lambda\psi} \left(s^2 g e^{\lambda\psi} + \frac{s}{2}\right) &\geq C\lambda (s\theta)^2, & t \in (0, \delta), x \in \bar{\Omega} \setminus X(\omega_1, t, 0) \\ -g'(t) \left((e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty}} - e^{\lambda\psi}) s^2 g e^{\lambda\psi} - \frac{s}{2} e^{\lambda\psi}\right) &> 0 & t \in (0, \delta), x \in \bar{\Omega} \setminus X(\omega_1, t, 0). \end{aligned}$$

It follows that for some positive constant  $C' > C$

$$C \int_0^\delta \int_\Omega \lambda (s\theta)^2 |v|^2 dx dt \leq -I + \frac{1}{2} \int_\Omega [(1-\zeta)^2 (s\theta)^2 |v|^2]_{t=\delta} dx + C' \int_0^\delta \int_{X(\omega, t, 0)} \lambda (s\theta)^2 |v|^2 dx dt. \quad (\text{C.3})$$

Finally, by Cauchy-Schwarz inequality, we have for any  $\kappa > 0$

$$|I| \leq (4\kappa)^{-1} \int_0^\delta \int_\Omega |e^{-s\varphi} q_t|^2 dx dt + \kappa \int_0^\delta \int_\Omega (s\theta)^2 |v|^2 dx dt. \quad (\text{C.4})$$

Combining (C.3) with (C.4) gives (C.2) for  $\kappa/\lambda > 0$  small enough. Therefore, Claim 3 is proved.  $\square$

We can prove in the same way the following estimate for  $t \in [T - \delta, T]$ :

$$\begin{aligned} \int_{T-\delta}^T \int_\Omega \lambda (s\theta)^2 |v|^2 dx dt &\leq C \left( \int_{T-\delta}^T \int_\Omega \lambda^{-1} |e^{-s\varphi} q_t|^2 dx dt \right. \\ &\quad \left. + \int_\Omega [(1-\zeta)^2 (s\theta)^2 |v|^2]_{t=T-\delta} dx + \int_{T-\delta}^T \int_{X(\omega, t, 0)} \lambda (s\theta)^2 |v|^2 dx dt \right). \end{aligned} \quad (\text{C.5})$$

Let us now consider the estimate for  $t \in [\delta, T - \delta]$ .

CLAIM 4.

$$\begin{aligned} \int_\delta^{T-\delta} \int_\Omega \lambda^2 (s\theta)^2 |v|^2 dx dt + \int_\Omega [(1-\zeta)^2 (\lambda s\theta)^2 |v|^2]_{t=\delta} dx + \int_\Omega [(1-\zeta)^2 (\lambda s\theta)^2 |v|^2]_{t=T-\delta} dx \\ \leq C \left( \int_\delta^{T-\delta} \int_\Omega |e^{-s\varphi} q_t|^2 dx dt + \int_\delta^{T-\delta} \int_{X(\omega, t, 0)} \lambda^2 (s\theta)^2 |v|^2 dx dt \right). \end{aligned} \quad (\text{C.6})$$

$\|\cdot\|$  and  $(\cdot, \cdot)$  denoting here the Euclidean norm and scalar product in  $L^2(\Omega \times (\delta, T - \delta))$ , we have that

$$\|e^{-s\varphi} q_t\|^2 \geq \|(1-\zeta)(s\varphi_t v + v_t)\|^2 \geq 2((1-\zeta)s\varphi_t v, (1-\zeta)v_t). \quad (\text{C.7})$$

Next, we compute

$$\begin{aligned} ((1-\zeta)s\varphi_t v, (1-\zeta)v_t) &= \int_\Omega (1-\zeta)^2 s\varphi_t \frac{v^2}{2} dx \Big|_{t=\delta}^{T-\delta} - \frac{s}{2} \int_\delta^{T-\delta} \int_\Omega (1-\zeta)^2 \varphi_{tt} |v|^2 dx dt \\ &\quad - \int_\delta^{T-\delta} \int_\Omega (1-\zeta) \nabla \xi(X(x, 0, t)) \cdot \left(\frac{\partial X}{\partial x}\right)^{-1}(X(x, 0, t), t, 0) f(x, t) s\varphi_t v^2 dx dt. \end{aligned} \quad (\text{C.8})$$

Since  $g(t) = 1$  for  $t \in [\delta, T - \delta]$ , we have that  $\varphi_t = -\lambda\psi_t e^{\lambda\psi}$ . From (4.15)-(4.16), we infer that

$$\begin{aligned} s\varphi_t(x, T - \delta) &\geq C\lambda s e^{\lambda\psi} & x \in \overline{\Omega} \setminus X(\omega_1, T - \delta, 0), \\ -s\varphi_t(x, \delta) &\geq C\lambda s e^{\lambda\psi} & x \in \overline{\Omega} \setminus X(\omega_1, \delta, 0). \end{aligned}$$

Therefore, using (4.28),

$$\int_\Omega (1-\zeta)^2 s\varphi_t \frac{v^2}{2} dx \Big|_\delta^{T-\delta} \geq C \left( \int_\Omega [(1-\zeta)^2 (\lambda s\theta)^2 |v|^2]_{t=\delta} dx + \int_\Omega [(1-\zeta)^2 (\lambda s\theta)^2 |v|^2]_{t=T-\delta} dx \right). \quad (\text{C.9})$$

Next, with  $\varphi_{tt} = -\{(\lambda\psi_t)^2 + \lambda\psi_{tt}\}e^{\lambda\psi}$  and (4.14), we obtain for  $\lambda \geq \lambda_1 > \lambda_0$

$$-\frac{s}{2} \int_{\delta}^{T-\delta} \int_{\Omega} (1-\zeta)^2 \varphi_{tt} |v|^2 dx dt \geq C \int_{\delta}^{T-\delta} \int_{\Omega} (1-\zeta)^2 \lambda^2 s \theta |v|^2 dx dt. \quad (\text{C.10})$$

Finally

$$\left| \int_{\delta}^{T-\delta} \int_{\Omega} (1-\zeta) \nabla \xi(X(x, 0, t)) \cdot \left(\frac{\partial X}{\partial x}\right)^{-1}(X(x, 0, t), t, 0) f(x, t) s \varphi_t v^2 dx dt \right| \leq C \int_{\delta}^{T-\delta} \int_{X(\omega, t, 0)} \lambda s \theta |v|^2. \quad (\text{C.11})$$

Claim 4 follows from (C.7)-(C.11).

We infer from (C.2), (C.5) and (C.6) that for some constants  $\lambda_1 \geq \lambda_0$ ,  $s_1 \geq s_0$  and  $C_1 > 0$  we have for all  $\lambda \geq \lambda_1$  and all  $s \geq s_1$

$$\int_0^T \int_{\Omega} \lambda^2 (s\theta) |v|^2 dx dt \leq C_1 \left( \int_0^T \int_{\Omega} |e^{-s\varphi} q_t|^2 dx dt + \int_0^T \int_{X(\omega, t, 0)} \lambda^2 (s\theta)^2 |v|^2 dx dt \right). \quad (\text{C.12})$$

Replacing  $v$  by  $e^{-s\varphi} q$  in (C.12) gives at once (4.20). The proof of Lemma 4.5 is complete.  $\square$

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