On the null controllability of the one-dimensional heat equation with non-smooth coefficients

by

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Dedicated to the memory of Jacques-Louis LIONS.

Abstract

In this paper, we analyze the null controllability of the one-dimensional heat equation with non-smooth time-independent coefficients. We prove that this equation with BV coefficients is null-controllable at any positive time in the context of boundary control. The argument used in the proof relies on the exact controllability of the one-dimensional wave equation with BV coefficients and Russell's general method (which provides the null controllability of a parabolic equation at any positive time as a consequence of the exact controllability, for large time, of the corresponding wave equation).

*Supported by grants PB98-1134 and BFM2000-1317 of the D.G.E.S. (Spain).
†Supported by grant PB96-0663 of the D.G.E.S. (Spain) and the E.C. TMR network HMS2000.

1 Introduction. The main result

Let us consider the following system for the one-dimensional heat equation with variable coefficients

$$\begin{cases} \rho(x)y_t - (a(x)y_x)_x = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = v(t), \quad y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y_0(x), & 0 < x < 1. \end{cases} \tag{1.1}$$

In (1.1), $y = y(x,t)$ is the state and $v = v(t)$ is a control that acts on the system through the extreme $x = 0$. The coefficients ρ and a are assumed to be (at least) measurable, bounded and uniformly positive, i.e. such that

$$0 < \rho_0 \le \rho(x) \le \rho_1\,, \quad 0 < a_0 \le a(x) \le a_1\,, \quad \text{a.e. in } (0,1). \tag{1.2}$$

For any given $y_0 \in L^2(0,1)$ and any $v \in C^0([0,T])$, there exists exactly one solution $y = y(x,t)$ to (1.1), with

$$y \in C^0([0,T]; L^2(0,1)). \tag{1.3}$$

The solution y is defined by transposition and its precise form is given below, see section 4.

The main goal of this paper is to analyze the null controllability of (1.1). Specifically, we want to solve the following problem:

Given $T > 0$ and $y_0 \in L^2(0,1)$, to find $v \in C^0([0,T])$ such that the corresponding solution y satisfies

$$y(x,T) = 0 \quad \text{a.e. in } (0,1). \tag{1.4}$$

According to the results in [6], system (1.1) is null-controllable at any time $T > 0$ provided the coefficients ρ and a satisfy

$$\rho, a \in W^{1,\infty}(0,1) \tag{1.5}$$

and (1.2). In that reference, the proof of null controllability is based on an appropriate observability inequality for the associate adjoint system. This is implied by some particular (global) Carleman estimates that hold for the adjoint states.

When the coefficients ρ and a are sufficiently smooth, the observability inequality can also be proved introducing Fourier series and using high frequency asymptotic formulæ for the eigenvalues of the corresponding Sturm-Liouville problem and classical results on the sums of real exponentials, see [10]. We also refer to [7] and [8] for some results in this direction in several space dimensions.

As far as we know, there is no evidence in the literature of lack of null controllability for (1.1) with bounded measurable coefficients satisfying (1.2). It is then natural to try to relax the Lipschitz regularity assumption (1.5). This is the main goal in this paper.

We will adopt here the approach introduced by D.L. Russell in [12]. There, the main underlying idea is to obtain the null controllability of a parabolic equation as a consequence of the exact controllability of the corresponding "similar" wave equation.

Thus, assume that, in addition to (1.2), the coefficients satisfy

$$\rho, a \in \mathrm{BV}(0,1) \tag{1.6}$$

and consider the following system for the wave equation:

$$\begin{cases} \rho(x) z_{tt} - (a(x) z_x)_x = 0, & 0 < x < 1,\ 0 < t < T_0\,, \\ z(0,t) = w(t), \quad z(1,t) = 0, & 0 < t < T_0\,, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & 0 < x < 1. \end{cases} \tag{1.7}$$

Now, $z = z(x,t)$ is the state and the control is $w = w(t)$. For any (z_0, z_1) with $z_0 \in L^2(0,1)$ and $\rho z_1 \in H^{-1}(0,1)$ and any $w \in L^2(0,T_0)$, system (1.7) possesses exactly one solution z, with

$$z \in C^0([0,T_0]; L^2(0,1)), \quad \rho z_t \in C^0([0,T_0]; G),$$

where G is an appropriate Hilbert space such that $L^2(0,1) \hookrightarrow G$ with a dense embedding. Again, z is defined by transposition (see more details in section 3).

The problem of the null controllability of (1.7) can be formulated as follows:

Given $T_0 > 0$ and (z_0, z_1) with $z_0 \in L^2(0,1)$ and $\rho z_1 \in H^{-1}(0,1)$, to find $w \in L^2(0,T_0)$ such that the corresponding solution z satisfies

$$z(x,T_0) = 0, \quad z_t(x,T_0) = 0, \quad \text{a.e. in } (0,1).$$

Notice that, in view of the fact that (1.7) is linear and reversible in time, this property holds if and only if the system can be driven exactly at time T_0 to any final state (Z_0, Z_1) in an appropriate space. In other words, the null controllability and the exact controllability of (1.7) are equivalent concepts.

The null controllability of (1.7) is known to hold under "reasonable" assumptions. In fact, by means of J.L. Lions' *Hilbert uniqueness method* [9], the proof of this controllability property can be reduced to the proof of the following observability inequality:

To find a positive constant $C_1 = C_1(\rho, a, T_0)$ such that

$$\|\varphi_0\|^2_{H^1_0} + \|\varphi_1\|^2_{L^2} \leq C_1 \int_0^{T_0} |\varphi_x(0,t)|^2\, dt \tag{1.8}$$

for every solution of the adjoint system

$$\begin{cases} \rho(x)\varphi_{tt} - (a(x)\varphi_x)_x = 0, & 0 < x < 1,\ 0 < t < T_0\,, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T_0\,, \\ \varphi(x,T_0) = \varphi_0(x), \quad \varphi_t(x,T_0) = \varphi_1(x), & 0 < x < 1. \end{cases}$$

Inequality (1.8) can be easily proved when (1.2) and (1.6) are satisfied and T_0 is sufficiently large. In particular, this is the case if $T_0 > 2\ell$, where

$$\ell = \operatorname*{ess\,sup}_{[0,1]} \sqrt{\frac{\rho}{a}}$$

(for completeness, we will sketch the proof below, in section 3; see [1] for more details).

It has been recently proved in [1] that there exist Hölder-continuous coefficients ρ and a not in $\mathrm{BV}(0,1)$ for which (1.8) and therefore the exact controllability property of the wave equation (1.7) fail. Thus, the BV assumption on the coefficients is sharp in the context of the exact controllability of the wave equation and it is a minimal requirement to apply the method employed in the present paper. However, the analysis in [1] does not provide any counter-example to the null controllability of the heat equation with non-smooth coefficients.

The main result in this paper is the following:

Theorem 1.1 *Assume that the coefficients $\rho = \rho(x)$ and $a = a(x)$ in* (1.1) *satisfy* (1.6) *and* (1.2)*. Then* (1.1) *is null-controllable at time T for all $T > 0$, with controls $v \in C^0([0,T])$.*

As we mentioned above, it will be seen that this result is a consequence of the exact controllability of the linear one-dimensional wave equation with coefficients in $\mathrm{BV}(0,1)$ and the general method developed by D.L. Russell in [12].

Let us repeat that there is no result in the literature asserting the lack of null controllability of the linear heat equation with non-smooth coefficients. In particular, whether the BV assumption on the coefficients in Theorem 1.1 is sharp or not is an open problem.

It will be seen below that we can find controls that drive the heat equation in (1.1) to zero and depend continuously on the initial data $y_0 \in L^2(0,1)$. A consequence is that the following observability inequality holds:

Corollary 1.1 *There exists a positive constant $C_2 = C_2(\rho, a, T)$ such that*

$$\|\psi(\cdot,0)\|^2_{L^2} \leq C_2 \int_0^T |\psi_x(0,t)|^2\, dt \tag{1.9}$$

for every solution of the adjoint system

$$\begin{cases} -\rho(x)\psi_t - (a(x)\psi_x)_x = 0, & 0 < x < 1,\ 0 < t < T, \\ \psi(0,t) = \psi(1,t) = 0, & 0 < t < T, \\ \psi(x,T) = \psi_0(x), & 0 < x < 1, \end{cases} \tag{1.10}$$

where $\psi_0 \in L^2(0,1)$.

Properties of this kind have been recently investigated by A. Doubova, A. Osses and J.-P. Puel in [2], in the context of the N-dimensional heat equation with non-smooth coefficients. There, the authors use appropriate Carleman estimates to show that inequalities like (1.9) hold under suitable monotonicity conditions on the coefficients. Our result is satisfied for general time-independent coefficients in $\mathrm{BV}(0,1)$ and consequently seems to indicate that, at least in the one-dimensional case, the monotonicity conditions in [2] are not necessary.

In several space dimensions the situation is much more complex and will not be considered here. It will be the subject of a forthcoming paper. Let us however describe a particular case, two-dimensional in space, for which the desired null controllability result can still be established.

Let Ω be the open set

$$\Omega = (0,1) \times (0,1)$$

and consider the following system:

$$\begin{cases} \rho(x_1)y_t - \nabla \cdot (\rho(x_1)\nabla y) = 0, & (x,t) \in \Omega \times (0,T), \\ y(x,t) = v(x_2,t)1_\gamma, & (x,t) \in \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{1.11}$$

Here, $\gamma = \{\, (0,x_2) : x_2 \in (0,1) \,\}$ and 1_γ is the characteristic function of γ. In (1.11), we assume that $y_0 \in L^2(\Omega)$ and $v \in C^0(\overline{\gamma} \times [0,T])$. As a consequence, this system possesses exactly one solution $y \in C^0([0,T]; L^2(\Omega))$. The following result holds for (1.11) (see [4] for the proof):

Theorem 1.2 *Assume that the coefficient* $\rho = \rho(x_1)$ *in* (1.11) *satisfies*

$$\rho \in \mathrm{BV}(0,1), \quad 0 < \rho_0 \le \rho(x_1) \le \rho_1 \,.$$

Then (1.11) *is null-controllable at time* T *for all* $T > 0$*, with controls* $v \in C^0(\overline{\gamma} \times [0,T])$.

The rest of this paper is organized as follows. In section 2, we will recall Russell's method and the key points of its proof. In section 3, we will briefly sketch the proof of the null controllability of the one-dimensional wave equation with BV coefficients. Then, in section 4, the main result of this paper (Theorem 1.1) will be proved. We will also include in this section several additional comments.

2 Russell's principle

In this section, it will be assumed that $L : D(L) \subset L^2(0,1) \mapsto L^2(0,1)$ is a densely defined, self-adjoint, maximal monotone operator with compact resolvent. Let $\rho = \rho(x)$ be a measurable function satisfying

$$0 < \rho_0 \le \rho(x_1) \le \rho_1 \,, \quad \text{a.e. in } (0,1).$$

We will denote by λ_n and φ_n the following eigenvalues and associate eigenfunctions:

$$\begin{cases} L\varphi_n = \lambda_n \rho(x)\varphi_n \,, \quad 0 < x < 1, \\ \varphi_n \in D(L), \quad 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots, \\ \displaystyle\int_0^1 \rho(x)\varphi_n(x)\varphi_m(x)\,dx = \delta_{n,m} \quad \forall n, m \ge 1 \end{cases}$$

($\delta_{n,m}$ is the usual Kronecker's symbol). Note that, in the context we will be working in sections 3 and 4, the eigenvalues of L will be simple. But this is irrelevant for the application of Russell's principle.

We will consider the following two systems:

$$\begin{cases} \rho(x)y_t + Ly = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = v(t), \quad y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y_0(x), & 0 < x < 1 \end{cases} \tag{2.1}$$

and

$$\begin{cases} \rho(x)z_{tt} + Lz = 0, & 0 < x < 1,\ 0 < t < T_0 \,, \\ z(0,t) = w(t), \quad z(1,t) = 0, & 0 < t < T_0 \,, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & 0 < x < 1. \end{cases} \tag{2.2}$$

We will assume that exact controllability holds for (2.2) at large time:

System (2.2) *is null-controllable at time T_0 with controls $w \in L^2(0,T_0)$ such that*

$$\|w\|_{L^2} \le C_1 \left(\|z_0\|_{L^2} + \|z_1\|_{H^{-1}} \right). \tag{2.3}$$

Then, the following is satisfied:

Theorem 2.1 *Under the previous hypotheses, system* (2.1) *is null-controllable at time T for all $T > 0$.*

As mentioned above, this result says that, when the wave equation is exactly controllable *at some finite time*, then the heat equation is null controllable *at any positive time.*

Sketch of the proof: The idea is to rewrite the exact controllability property of (2.2) as a first moment problem that we are able to solve. Then, this moment problem is transformed into a second (solvable) moment problem and, finally, the latter is shown to be in fact equivalent to the null controllability problem for the heat equation (2.1). Accordingly, there are three main steps in the proof.

STEP 1: From our assumption (2.3), it is clear that, for each $n \geq 1$, there exists a control $w_n \in L^2(0, T_0)$ such that

$$\|w_n\|_{L^2(0,T_0)} \leq C_1 \lambda_n^{-1/2}$$

and the solution to (2.2) associated to w_n and the initial data

$$z_0 = 0, \quad z_1 = \varphi_n$$

satisfies

$$z(x, T_0) = 0, \quad z_t(x, T_0) = 0, \quad \text{a.e. in } (0, 1). \tag{2.4}$$

On the other hand, we can also find controls $w_n^* \in L^2(0, T_0)$ such that

$$\|w_n^*\|_{L^2(0,T_0)} \leq C_1$$

and the solutions to (2.2) associated to the w_n^* and the initial data

$$z_0 = \varphi_n \,, \quad z_1 = 0$$

satisfy (2.4). Let us set

$$W_n = \frac{1}{2}\left(w_n - \frac{i}{\sqrt{\lambda_n}} w_n^*\right), \quad W_{-n} = \frac{1}{2}\left(w_n + \frac{i}{\sqrt{\lambda_n}} w_n^*\right)$$

for each $n \geq 1$. Then the following properties hold:

$$W_{\pm n} \in L^2(0, T_0), \quad \|W_{\pm n}\|_{L^2} \leq C_1 \lambda_n^{-1/2}$$

and

$$\int_0^{T_0} h_{\pm n}(t) W_{\pm m}(t) \, dt = \delta_{n,m} \quad \forall n, m \geq 1. \tag{2.5}$$

Here, we have introduced the following functions:

$$h_n(t) = -\varphi_n'(0) e^{i\sqrt{\lambda_n} t}, \quad h_{-n}(t) = -\varphi_n'(0) e^{-i\sqrt{\lambda_n} t}, \quad \text{for } t \text{ a.e. in } (0, T_0) \text{ and } n \geq 1.$$

Step 2: From (2.5), we notice that

$$-\varphi_n'(0)Q_m(i\lambda_n) = \delta_{n,m} \quad \forall n, m \geq 1,$$

where the functions Q_n are given as follows:

$$Q_n(z) = \tilde{W}_n(-i\sqrt{z}), \qquad \tilde{W}_n(\zeta) = W_n^*(\zeta) + W_n^*(-\zeta) \tag{2.6}$$

and

$$W_n^*(\zeta) = \int_0^{T_0} e^{i\zeta t} W_n(t)\, dt \quad \forall \zeta \in \mathbf{C}.$$

Obviously, we have

$$|Q_n(z)| \leq 2C_1\sqrt{T_0}\ \lambda_n^{-1/2}$$

for all $n \geq 1$.

Let $T > 0$ be given (arbitrary but fixed). A result by R.M. Redheffer [11] ensures the existence of an entire function $E = E(z)$ that is even, real for real z, has all its zeros in $\mathbf{R}_+$ and satisfies

$$|E(\xi)| \leq \frac{C(T, T_0)}{1 + |\xi|} e^{-2T_0|\xi|^{1/2}} \quad \forall \xi \in \mathbf{R} \tag{2.7}$$

and also

$$|E(-i\eta)| \leq C(T, T_0)e^{T\eta}, \qquad C(T, T_0)e^{-K\eta^{1/2}} \leq |E(i\eta)| \leq 1 \quad \forall \eta \in \mathbf{R}_+\,. \tag{2.8}$$

Here and in the sequel, $C(T, T_0)$ stands for a generic positive constant that depends only on T and T_0. In (2.8), K only depends on T. Let us introduce the functions v_n^*, given by

$$v_n^*(z) = \frac{1}{E(i\lambda_n)} E(z)Q_n(z) \quad \forall z \in \mathbf{C}.$$

It is then clear from (2.6), (2.7) and (2.8) that

$$|v_n^*(\xi)| \leq C_1 C(T, T_0)e^{K\lambda_n^{1/2}} \lambda_n^{-1/2} e^{-T_0|\xi|^{1/2}} \quad \forall \xi \in \mathbf{R}. \tag{2.9}$$

We also have

$$|v_n^*(i\eta)| \leq C_1 C(T, T_0)e^{K\lambda_n^{1/2}} \lambda_n^{-1/2} \tag{2.10}$$

and

$$|v_n^*(-i\eta)| \leq C_1 C(T, T_0)e^{K\lambda_n^{1/2}} \lambda_n^{-1/2} e^{T\eta} \tag{2.11}$$

for all $\eta \in \mathbf{R}_+$. Let us finally set

$$v_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\xi t} v_n^*(\xi)\, d\xi \quad \forall t \in \mathbf{R}.$$

For each $n \geq 1$, v_n is the Fourier transform of v_n^* (a real-valued C^∞ function, since $v_n^*(-\xi) \equiv v_n^*(\xi)$ and (2.9) is satisfied). In view of (2.10), (2.11) and Paley-Wiener's Theorem, the support of v_n satisfies

$$\operatorname{Supp} v_n \subset [0, T].$$

On the other hand,

$$|v_n(t)| \leq C_1 C(T, T_0) e^{K\lambda_n^{1/2}} \lambda_n^{-1/2} \quad \forall t \in \mathbf{R} \tag{2.12}$$

and

$$-\varphi_n'(0) \int_0^T e^{-\lambda_n t} v_m(t)\, dt = -\varphi_n'(0) v_m^*(i\lambda_n) = -\varphi_n'(0) \frac{E(i\lambda_n) Q_m(i\lambda_n)}{E(i\lambda_m)} = \delta_{n,m} \tag{2.13}$$

for all $n, m \geq 1$.

Step 3: For any $y_0 \in L^2(0,1)$ and any $v \in C^0([0,T])$, there exists exactly one solution $y = y(x,t)$ to (2.1), with

$$y \in C^0([0,T]; L^2(0,1)).$$

This is justified by the argument we present at the beginning of section 4. At least formally, we can put

$$y(\cdot, t) = \sum_{n\geq 1} e^{-\lambda_n t} (y_0, \varphi_n)_\rho \varphi_n + \sum_{n \geq 1} \left(-\varphi_n'(0) \int_0^t e^{-\lambda_n (t-s)} v(s)\, ds \right) \varphi_n\,, \tag{2.14}$$

where $(\cdot\,,\cdot)_\rho$ denotes the following scalar product in $L^2(0,1)$:

$$(\varphi, \psi)_\rho = \int_0^1 \rho(x) \varphi(x) \psi(x)\, dx \quad \forall \varphi, \psi \in L^2(0,1).$$

The series in (2.14) converge, at least in the L^2 sense, for all t. Let us see that v can be chosen in such a way that the function in (2.14) satisfies (1.4).

It will suffice to find $v \in C^0([0,T])$ such that

$$-\varphi_n'(0) \int_0^T e^{-\lambda_n (T-s)} v(s)\, ds = -e^{-\lambda_n T} (y_0, \varphi_n)_\rho \quad \forall n \geq 1. \tag{2.15}$$

But this is now easy in view of the properties satisfied by the functions v_n. Indeed, let us set

$$v = \sum_{n \geq 1} \beta_n v_n(T - t), \quad \text{with} \quad \beta_n = -e^{-\lambda_n T} (y_0, \varphi_n)_\rho \quad \forall n \geq 1.$$

Taking into account (2.12) and (2.13), we see that this series converges in $C^0([0,T])$ to a function v satisfying (2.15). Furthermore, the following estimate of the L^∞-norm of the control v is guaranteed:

$$\|v\|_{L^\infty} \le C_1 C(T,T_0) \sum_{n\ge 1} |(y_0,\varphi_n)_\rho|\, e^{-T\lambda_n + K\lambda_n^{1/2}} \lambda_n^{-1/2}\,. \tag{2.16}$$

This ends the proof of Theorem 2.1.

■

Remark 2.1 Let $\mathcal{F}$ be the space

$$\mathcal{F} = \{\, f \in L^2(0,1) : \sum_{n\ge 1} |(f,\varphi_n)_\rho|\, e^{K\lambda_n^{1/2}} \lambda_n^{(r-1)/2} < +\infty \,\}, \tag{2.17}$$

where K is the constant (only depending on T) that we have found in the previous proof. It is not difficult to see that, under the assumptions of Theorem 2.1, (2.1) can be controlled exactly to all final states $y_1 \in \mathcal{F}$. On the other hand, it is clear that the assumption $y_0 \in L^2(0,1)$ can be considerably enlarged. We refer to [3] for other similar results in several space dimensions.

■

3 Exact controllability of the one-dimensional wave equation with BV coefficients

In this section, we will consider again the following system for the one-dimensional wave equation

$$\begin{cases} \rho(x) z_{tt} - (a(x) z_x)_x = 0, & 0 < x < 1,\ 0 < t < T_0\,, \\ z(0,t) = w(t), \quad z(1,t) = 0, & 0 < t < T_0\,, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & 0 < x < 1. \end{cases} \tag{3.1}$$

We will assume that

$$\rho, a \in \mathrm{BV}(0,1) \tag{3.2}$$

and

$$0 < \rho_0 \le \rho(x) \le \rho_1\,, \quad 0 < a_0 \le a(x) \le a_1\,, \quad \text{a.e. in } (0,1). \tag{3.3}$$

For any (z_0, z_1) with $z_0 \in L^2(0,1)$ and $\rho z_1 \in H^{-1}(0,1)$ and any $w \in L^2(0,T_0)$, system (3.1) possesses exactly one solution z, with

$$z \in C^0([0,T_0]; L^2(0,1)) \quad \rho z_t \in C^0([0,T_0]; X^{-1-\varepsilon}) \quad \forall \varepsilon > 0. \tag{3.4}$$

Here, we have introduced the notation $X^\alpha = D(L^{\alpha/2})$, where L is the maximal monotone operator defined by

$$\begin{cases} D(L) = \{\, \psi \in H_0^1(0,1) : (a\psi_x)_x \in L^2(0,T) \,\}, \\ L\psi = -\,(a\psi_x)_x \quad \forall \psi \in D(L). \end{cases} \tag{3.5}$$

The solution z to (3.1) is defined by transposition. More precisely, z is the unique function in $L^\infty(0,T_0;L^2(0,1))$ satisfying

$$\begin{cases} \displaystyle\int_0^{T_0}\int_0^1 zg\,dx\,dt = \int_0^{T_0} (a\varphi_x)\,(0,t)w(t)\,dt \\ \displaystyle\qquad - \int_0^1 \rho(x)z_0(x)\varphi_t(x,0)\,dx + \langle \rho(x)z_1, \varphi(\cdot,0)\rangle \end{cases} \tag{3.6}$$

for all $g \in L^1(0,T_0;L^2(0,1))$, where $\langle \cdot\,,\cdot\rangle$ is the duality pairing associated to $H^{-1}(0,1)$ and $H_0^1(0,1)$ and φ is the solution to

$$\begin{cases} \rho(x)\varphi_{tt} - (a(x)\varphi_x)_x = g(x,t), & 0 < x < 1,\ 0 < t < T_0\,, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T_0\,, \\ \varphi(x,T_0) = 0, \quad \varphi_t(x,T_0) = 0, & 0 < x < 1. \end{cases}$$

Notice that, if $g \in L^1(0,T_0;L^2(0,1))$, then φ satisfies

$$\varphi \in C^0([0,T_0];H_0^1(0,1)), \quad \varphi_t \in C^0([0,T_0];L^2(0,1))$$

and

$$(a\varphi_x)\,(0,\cdot) \in L^2(0,T_0).$$

Thus, all terms in (3.6) have a sense and the definition of z is correct. It can also be seen that z satisfies (3.4).

Our main goal in this section is to recall the main steps of the proof of the following controllability result:

Theorem 3.1 *Assume that $T_0 > 2\ell$, with*

$$\ell = \operatorname*{ess\,sup}_{[0,1]} \sqrt{\frac{\rho}{a}}\,. \tag{3.7}$$

Then, for every (z_0,z_1) with $z_0 \in L^2(0,1)$ and $\rho z_1 \in H^{-1}(0,1)$, there exist controls $w \in L^2(0,T_0)$ such that the corresponding solution z of (3.1) satisfies

$$z(x,T_0) = 0, \quad z_t(x,T_0) = 0, \quad \text{a.e. in } (0,1).$$

Furthermore, there exists a positive constant $C_1 = C_1(\rho,a,T_0)$, independent of the initial data to be controlled, such that w can be chosen satisfying

$$\|w\|_{L^2} \le C_1\left(\|z_0\|_{L^2} + \|z_1\|_{H^{-1}}\right). \tag{3.8}$$

Proof: It will be sufficient to prove this result when the coefficients ρ and a are smooth, provided the constant C_1 in (3.8) only depends on the coercivity constants in (3.3) and the BV-norms of ρ and a. Indeed, the result can be later extended to cover the case of general BV coefficients satisfying (3.3) by means of a classical regularization argument.

Secondly, observe that, applying J.L. Lions' Hilbert uniqueness method [9], the proof of the controllability property stated here is equivalent to the proof of the observability inequality

$$\|\varphi_0\|^2_{H^1_0} + \|\varphi_1\|^2_{L^2} \leq C_1 \int_0^{T_0} |\varphi_x(0,t)|^2 \, dt \tag{3.9}$$

for the solutions to the adjoint system

$$\begin{cases} \rho(x)\varphi_{tt} - (a(x)\varphi_x)_x = 0, & 0 < x < 1,\ 0 < t < T_0\,, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T_0\,, \\ \varphi(x,T_0) = \varphi_0(x), \quad \varphi_t(x,T_0) = \varphi_1(x), & 0 < x < 1. \end{cases} \tag{3.10}$$

In order to prove (3.9), we will use "sidewise energy estimates". Thus, assume that $(\varphi_0, \varphi_1) \in H^1_0(0,1) \times L^2(0,1)$ and $T_0 > 2\ell$ (ℓ is given by (3.7)) and set

$$F(x) = \frac{1}{2} \int_{\ell x}^{T_0 - \ell x} \left(\rho(x)|\varphi_t(x,t)|^2 + a(x)|\varphi_x(x,t)|^2 \right) \, dt.$$

For each $x \in [0,1]$, $F(x)$ represents the amount of energy concentrated at x during the time interval $[\ell x, T_0 - \ell x]$. Obviously, because of the Dirichlet boundary conditions satisfied by φ at $x = 0$, we have

$$F(0) = \frac{a(0)}{2} \int_0^{T_0} |\varphi_x(0,t)|^2 \, dt.$$

We can compute the derivative of F with respect to x. In fact, we have:

$$\begin{cases} \dfrac{dF}{dx}(x) = -\dfrac{\ell}{2} \displaystyle\sum_{t=\ell x, T_0 - \ell x} \left[\rho(x)|\varphi_t(x,t)|^2 + a(x)|\varphi_x(x,t)|^2 \right] \\ \qquad + \displaystyle\int_{\ell x}^{T_0 - \ell x} \left(\rho(x)\varphi_t\varphi_{tx} + a(x)\varphi_x\varphi_{xx} + \frac{\rho'(x)}{2}|\varphi_t(x,t)|^2 + \frac{a'(x)}{2}|\varphi_x(x,t)|^2 \right) \, dt. \end{cases} \tag{3.11}$$

Integrating by parts and using the equation satisfied by φ, we have

$$\begin{cases} \displaystyle\int_{\ell x}^{T_0 - \ell x} \left(\rho(x)\varphi_t\varphi_{tx} + a(x)\varphi_x\varphi_{xx} \right) \, dt \\ \qquad = -\displaystyle\int_{\ell x}^{T_0 - \ell x} a'(x)|\varphi_x|^2 \, dt + \rho(x)\varphi_t\varphi_x\big|_{t = T_0 - \ell x} - \rho(x)\varphi_t\varphi_x\big|_{t = \ell x}\ . \end{cases} \tag{3.12}$$

Combining (3.11) and (3.12), together with the fact that

$$|\rho(x)\varphi_t\varphi_x| \le \frac{\ell}{2}\left[\rho(x)|\varphi_t|^2 + a(x)|\varphi_x|^2\right],$$

we deduce that

$$\begin{cases} \dfrac{dF}{dx}(x) \le \dfrac{1}{2}\displaystyle\int_{\ell x}^{T_0-\ell x} \left(\rho'(x)|\varphi_t|^2 - a'(x)|\varphi_x|^2\right) dt \\ \qquad \le \dfrac{1}{2}\max\left[\dfrac{|\rho'|}{\rho}, \dfrac{|a'|}{a}\right]\displaystyle\int_{\ell x}^{T_0-\ell x} \left(\rho(x)|\varphi_t|^2 + a(x)|\varphi_x|^2\right) dt \\ \qquad = \max\left[\dfrac{|\rho'|}{\rho}, \dfrac{|a'|}{a}\right] F(x). \end{cases}$$

Integrating this differential inequality with respect to x, we deduce that

$$\begin{cases} F(x) \le \exp\left(\displaystyle\int_0^x \max\left[\dfrac{|\rho'(s)|}{\rho(s)}, \dfrac{|a'(s)|}{a(s)}\right] ds\right) F(0) \\ \qquad = \dfrac{a(0)}{2} e^{h(x)} \displaystyle\int_0^{T_0} |\varphi_x(0,t)|^2\, dt, \end{cases}$$

where

$$h(x) = \int_0^x \max\left[\frac{|\rho'(s)|}{\rho(s)}, \frac{|a'(s)|}{a(s)}\right] ds \le \frac{\mathrm{TV}(\rho)}{\rho_0} + \frac{\mathrm{TV}(a)}{a_0}.$$

Hence,

$$F(x) \le \frac{a(0)}{2}\exp\left[\frac{\mathrm{TV}(\rho)}{\rho_0} + \frac{\mathrm{TV}(a)}{a_0}\right]\int_0^{T_0} |\varphi_x(0,t)|^2\, dt \qquad \forall x \in (0,1). \tag{3.13}$$

Let us now integrate (3.13) with respect to x in $(0,1)$ and let us remember that $T_0 > 2\ell$. The following is found:

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_{\ell}^{T_0-\ell}\int_0^1 \left[\rho(x)|\varphi_t|^2 + a(x)|\varphi_x|^2\right] dx\, dt \\ \qquad \le \dfrac{a(0)}{2}\exp\left[\dfrac{\mathrm{TV}(\rho)}{\rho_0} + \dfrac{\mathrm{TV}(a)}{a_0}\right]\displaystyle\int_0^{T_0} |\varphi_x(0,t)|^2\, dt. \end{cases}$$

Finally, taking into account that the energy

$$E(t) = \frac{1}{2}\int_0^1 \left(\rho(x)|\varphi_t|^2 + a(x)|\varphi_x|^2\right) dx$$

is constant in time for the solutions of (3.10), we find that

$$(T_0 - 2\ell)E(T_0) \le \frac{a(0)}{2}\exp\left[\frac{\mathrm{TV}(\rho)}{\rho_0} + \frac{\mathrm{TV}(a)}{a_0}\right]\int_0^{T_0} |\varphi_x(0,t)|^2\, dt. \tag{3.14}$$

It is now immediate to deduce from (3.14) the observability inequality (3.9) with

$$C_1 = \frac{a(0)}{(T_0 - 2\ell)\min(\rho_0, a_0)} \exp\left[\frac{\mathrm{TV}(\rho)}{\rho_0} + \frac{\mathrm{TV}(a)}{a_0}\right].$$

Thus, Theorem 3.1 is proved.

■

Remark 3.1 Arguing as in the previous proof, a similar exact controllability result can be established for more general wave equations with variable coefficients depending both of space and time. This is the case in particular for the slightly more general system

$$\begin{cases} \rho(x)z_{tt} - (a(x)z_x)_x + m(x)z = 0, & 0 < x < 1,\ 0 < t < T_0\,, \\ z(0,t) = w(t), \quad z(1,t) = 0, & 0 < t < T_0\,, \\ z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), & 0 < x < 1, \end{cases}$$

where ρ and a are as above and $m \in L^\infty(0,1)$. In this case, the requirement on T_0 is the same ($T_0 > 2\ell$), but the constant C_1 in (3.9) also depends on $\|m\|_{L^\infty}$.

■

4 Null controllability of the one-dimensional heat equation with BV coefficients

In this section, we will come back to the following system for the one-dimensional heat equation:

$$\begin{cases} \rho(x)y_t - (a(x)y_x)_x = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = v(t), \quad y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y_0(x), & 0 < x < 1. \end{cases} \tag{4.1}$$

We will assume again that the coefficients ρ and a satisfy (3.2) and (3.3).

As mentioned above, for any $y_0 \in L^2(0,1)$ and any $v \in C^0([0,T])$, this system possesses exactly one solution $y = y(x,t)$, with

$$y \in C^0([0,T]; L^2(0,1)). \tag{4.2}$$

The solution is defined by transposition as follows: y is the unique function in $L^2(0,T;L^2(0,1))$ satisfying

$$\int_0^T \int_0^1 yf\,dx\,dt = \int_0^T (a\psi_x)(0,t)v(t)\,dt + \int_0^1 \rho(x)y_0(x)\psi(x,0)\,dx \tag{4.3}$$

for all $f \in L^2(0,T;L^2(0,1))$, where ψ is the solution of the system

$$\begin{cases} -\rho(x)\psi_t - (a(x)\psi_x)_x = f(x,t), & 0 < x < 1,\ 0 < t < T, \\ \psi(0,t) = \psi(1,t) = 0, & 0 < t < T, \\ \psi(x,T) = 0, & 0 < x < 1. \end{cases}$$

It can be shown that, for any such f, the associated solution ψ satisfies

$$\psi \in C^0([0,T];L^2(0,1)), \quad (a\psi_x)(0,\cdot) \in L^2(0,T).$$

Hence, y is well defined by (4.3). It can also be seen that (4.2) holds.

Combining the exact controllability result in section 3 and Russell's general principle described in section 2, we obtain the following:

Theorem 4.1 *Assume that $T > 0$. Then, for every $y_0 \in L^2(0,1)$, there exist controls $v \in C^0([0,T])$ such that the corresponding solution y of* (4.1) *satisfies*

$$y(x,T) = 0, \quad a.e.\ in\ (0,1).$$

Furthermore, there exists a positive constant $C_3 = C_3(\rho,a,T)$ such that v can be chosen satisfying

$$\|v\|_{L^\infty} \le C_3\|y_0\|_{L^2}\,. \tag{4.4}$$

Actually, the unique information we find in this result and not in Theorem 2.1 is the estimate (4.4). But this is an almost straightforward consequence of (2.16) written for instance for $T_0 = 3\ell$ and the fact that $C_1 = C_1(\rho,a,T_0)$.

Remark 4.1 We can argue as in Remark 2.1 and deduce the exact controllability of (4.1) to final states in a (small) space. More precisely, let λ_n and φ_n be the eigenvalues and associate eigenfunctions corresponding to the operator L in (3.5), that is,

$$\begin{cases} -\left(a(x)\varphi_{n,x}\right)_x = \lambda_n\rho(x)\varphi_n\,, \quad 0 < x < 1, \\ \varphi_n(0) = \varphi_n(1) = 0, \\ \displaystyle\int_0^1 \rho(x)\varphi_n(x)\varphi_m(x)\,dx = \delta_{n,m} \quad \forall n,m \ge 1. \end{cases}$$

Let $\mathcal{F}$ be the space in (2.17), where K is the constant (only depending on T) found in the proof of Theorem 2.1. Then, (4.1) can be controlled exactly to all final states $y_1 \in \mathcal{F}$.

■

Remark 4.2 As stated in Corollary 1.1, we can use Theorem 4.1 to prove the observability inequality (1.9) for the solutions of the adjoint system (1.10). In fact, since we have found controls in $C^0([0,T])$ and we are able to estimate their L^∞-norms, (1.9) can be improved: there exists a positive constant $C_4 = C_4(\rho, a, T)$ such that

$$\|\psi(\cdot,0)\|_{L^2}^2 \leq C_4 \left(\int_0^T |\psi_x(0,t)|\, dt \right)^2$$

for all $\psi_0 \in L^2(0,1)$, where ψ is the solution to (1.10).

■

Remark 4.3 Up to our knowledge, the null controllability of the one-dimensional heat equation with non-smooth coefficients ρ and a that may depend on x and t is an open problem. The case of Lipschitz-continuous coefficients has been treated in [6]; remember that, under suitable monotonicity assumptions, the problem has been analyzed in [2].

■

References

[1] C. Castro, E. Zuazua, *Concentration and lack of observability of waves in highly heterogeneous media*, preprint.

[2] A. Doubova, A. Osses, J.-P. Puel, *Exact controllability to the trajectories for a transmission problem*, preprint.

[3] E. Fernández-Cara, E. Zuazua, *The cost of approximate controllability for heat equations: The linear case*, Advances in Differential Equations, **5** (4-6) (2000), 465–514.

[4] E. Fernández-Cara, E. Zuazua, *Some null controllability results for the heat equation with non-smooth coefficients in several space dimensions*, in preparation.

[5] A. Fursikov, O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes #34, Seoul National University, Korea, 1996.

[6] O. Yu Imanuvilov, M. Yamamoto, *Carleman estimate for a parabolic equation in Sobolev spaces of negative order and its applications*, in "Control of Nonlinear Distributed Parameter Systems", G. Chen et al. eds., Marcel-Dekker, 2000, pp. 113-137.

[7] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. P.D.E., **20** (1995), 335-356.

[8] G. Lebeau, E. Zuazua, *Null controllability of a system of linear thermoelasticity*, Archive Rat. Mech. Anal, **141** (4) (1998), 297–329.

[9] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués, Tomes 1 & 2*. Masson, RMA **8** & **9**, Paris 1988.

[10] A. López, E. Zuazua, *Uniform null controllability for heat equations with rapidly oscillating coefficients*, Ann. I.H.P. - Analyse non linéaire, to appear.

[11] R.M. Redheffer, *Elementary remarks on completeness*, Duke Math. Jour. **35** (1968), pp. 1103–116.

[12] D.L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*, Studies in Appl. Math., **52** (1973), 189–221.