

# Averaged controllability of parameter dependent wave equations\*

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## Abstract

We consider the problem of averaged controllability for parameter depending (either in a discrete or continuous fashion) control systems, the aim being to find a control, independent of the unknown parameters, so that the average of the states is controlled. We do it in the context of conservative models, both in an abstract setting and also analysing the specific example of the wave equation.

Our first result is of perturbative nature. Assuming the averaging probability measure to be a small parameter-dependent perturbation (in a sense that we make precise) of an atomic measure given by a Dirac mass corresponding to a specific realisation of the system, we show that the averaged controllability property is achieved whenever the system corresponding to the support of the Dirac is controllable.

Similar tools can be employed to obtain averaged versions of the so-called *Ingham inequalities*.

Particular attention is devoted to 1d wave and Schrödinger equations in which the time-periodicity of solutions can be exploited to obtain more precise results, provided the parameters involved satisfy Diophantine conditions ensuring the lack of resonances.

**Key words:** Parameter dependent systems, averaged control, perturbation arguments, Ingham inequalities, non-harmonic Fourier series, wave equations.

**Mathematical Subject Classification (MSC2010):** 49J55, 93C20, 42A70.

## 1 Introduction and main results

### 1.1 Problem formulation

This paper is devoted to analyze the following question: *Given a system depending on a random variable, is it possible to find a control such that the average or expected value of the output of the system is controlled?*

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In this paper, we address this issue motivated by the prototypical model represented by the wave equation, but our methods and results apply to a wide class of time-reversible models.

Let us explain the problem in the context of the string equation with Dirichlet boundary control:

$$\ddot{y}_\zeta(t, x) = \partial_x (a_\zeta(x) \partial_x y_\zeta(t, x)) \quad ((t, x) \in \mathbb{R}_+^* \times (0, 1)), \quad (1.1a)$$

$$y_\zeta(t, 0) = u(t) \quad (t \in \mathbb{R}_+^*), \quad (1.1b)$$

$$y_\zeta(t, 1) = 0 \quad (t \in \mathbb{R}_+^*), \quad (1.1c)$$

$$y_\zeta(0, x) = y_\zeta^{i,0}(x) \quad \text{and} \quad \dot{y}_\zeta(0, x) = y_\zeta^{i,1}(x) \quad (x \in (0, 1)), \quad (1.1d)$$

$\zeta \in \mathbb{R}$  being the unknown parameter and  $a_\zeta \in L^\infty(0, 1)$  a coefficient bounded from below by a positive constant independent of  $\zeta$ .

Notice that this system fits in the abstract frame:

$$\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u, \quad y_\zeta(0) = y_\zeta^i, \quad (1.2)$$

where  $\zeta$  is a random variable following the probability law  $\eta$ . As we shall see, the averaged controllability properties will significantly depend on the nature of the averaging measure  $\eta$ .

Given  $T > 0$ , the problem of *exact averaged controllability* consists in analysing whether, for every set of parameter dependent initial conditions  $(y_\zeta^{i,0}, y_\zeta^{i,1}) \in L^2(0, 1) \times H^{-1}(0, 1)$  and every final target  $(y^{f,0}, y^{f,1}) \in L^2(0, 1) \times H^{-1}(0, 1)$ , there exists a control  $u \in L^2(0, T)$  (independent of the parameter  $\zeta$ ) such that:

$$\int_{\mathbb{R}} y_\zeta(T) d\eta_\zeta = y^{f,0} \quad \text{and} \quad \int_{\mathbb{R}} \dot{y}_\zeta(T) d\eta_\zeta = y^{f,1}. \quad (1.3)$$

One can also address the weaker *approximate averaged control problem*, in which, for every  $\varepsilon > 0$ , one aims to find a control  $u \in L^2(0, T)$  such that:

$$\left\| \int_{\mathbb{R}} y_\zeta(T) d\eta_\zeta - y^{f,0} \right\|_{L^2(0,1)}^2 \leq \varepsilon \quad \text{and} \quad \left\| \int_{\mathbb{R}} \dot{y}_\zeta(T) d\eta_\zeta - y^{f,1} \right\|_{H^{-1}(0,1)}^2 \leq \varepsilon. \quad (1.4)$$

In both (1.3) and (1.4),  $y_\zeta$  is the solution of (1.1) with initial Cauchy condition  $(y_\zeta^{i,0}, y_\zeta^{i,1})$  and control  $u$ .

This paper is devoted to address these questions both in the abstract version (1.2) in which the generator of the semigroup  $A$  is anti-adjoint and some particular instances as the 1d wave equation above or the corresponding Schrödinger analog. But our results apply in the multi-dimensional context too.

## 1.2 Main results

We address the problem of averaged control analyzing the equivalent one of averaged observability for the corresponding adjoint system. We do it in two complementary contexts that we briefly describe below. We first show the stability of the observability inequality under small enough perturbations, to later derive a much more specific result for Fourier series expansions, using its periodicity properties.

**Perturbation argument:** Consider the general abstract parameter dependent system (1.2), depending on the unknown parameter  $\zeta \in \mathbb{R}$ .

We focus on the case where the uncontrolled dynamics, i.e. the one associated with  $u = 0$ , is time-conservative. Our results apply also on a slightly larger context (for instance, involving bounded damping

terms) but, for instance, cannot be applied directly for heat-like equations because of its time irreversibility.

In order to tackle the averaged controllability problem, we consider a probability measure of the form  $\eta = (1 - \theta)\delta_{\zeta_0} + \theta\tilde{\eta}$ , where  $\tilde{\eta}$  is a probability measure on  $\mathbb{R}$  and  $\theta \in [0, 1]$  a small parameter so that, in practice, we deal with a small perturbation of an atomic measure concentrated at  $\zeta_0$ . Our result ensures that, under suitable smallness conditions, averaged observability holds provided the realization of the system for  $\zeta = \zeta_0$  is observable.

To be more precise, proving the exact averaged controllability in time  $T > 0$  is equivalent to the averaged observability inequality:

$$\int_0^T \left\| \int_{\mathbb{R}} B_{\zeta}^* z_{\zeta}(t) d\eta_{\zeta} \right\|_U^2 dt \geq c(T) \|z^f\|_X^2 \quad (z^f \in X), \quad (1.5)$$

with  $c(T) > 0$  and where  $X$  (resp.  $U$ ) is the state (resp. control) space and  $z_{\zeta}$  is solution of the adjoint system:

$$-\dot{z}_{\zeta} = A_{\zeta}^* z_{\zeta}, \quad z_{\zeta}(T) = z^f.$$

Assuming that for the parameter  $\zeta = \zeta_0$  the system is exactly controllable/observable, i.e. that we have:

$$\int_0^T \|B_{\zeta_0}^* z_{\zeta_0}(t)\|_U^2 dt \geq c_{\zeta_0}(T) \|z^f\|_X^2 \quad (z^f \in X),$$

with  $c_{\zeta_0}(T) > 0$ , we prove that, for  $\theta \in (0, 1)$  small enough, the inequality (1.5) holds, i.e. the parameter dependent system (1.2) is exactly controllable in average for the probability measure  $\eta$ .

This result is the core of Theorem 2.1 and it can be applied in many situations such as wave, Schrödinger or plate equations with internal or boundary control.

A similar result holds in the context of Ingham inequalities (see Proposition 3.1), an issue that we discuss now in more detail.

**Averaged Ingham inequalities:** In the context of one dimensional equations such as string or Schrödinger equations, the problem of averaged controllability can be reduced to the analysis of averages of non-harmonic Fourier series and the recovery of its coefficients out of its  $L^2(0, T)$ -norm.

To be more precise, we introduce the following parameter-dependent family of non-harmonic Fourier series:

$$\sum_{n \in \mathbb{Z}} [L_{\zeta} a]_n e^{2i\pi\lambda_n \varsigma(\zeta)t} \quad (t \in \mathbb{R}), \quad (1.6)$$

and its average:

$$f(t) = \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} [L_{\zeta} a]_n e^{2i\pi\lambda_n \varsigma(\zeta)t} d\eta_{\zeta} \quad (t \in \mathbb{R}), \quad (1.7)$$

$(a_n)_n \in \ell^2$  being a square summable sequence,  $(\lambda_n)_n$  a given sequence of real numbers, both independent of the unknown parameter  $\zeta$ , and  $L_{\zeta} \in \mathcal{L}(\ell^2)$  and  $\varsigma(\zeta)$  depending on  $\zeta \in \mathbb{R}^{\mathbb{R}}$ .

The problem of averaged controllability is then reduced to the obtention of results of the form:

- *Averaged admissibility:*

There exists  $C(T) > 0$  independent of  $(a_n)_n$  such that,

$$\int_0^T |f(t)|^2 dt \leq C(T) \sum_{n \in \mathbb{Z}} |a_n|^2; \quad (1.8)$$

- *Exact averaged observability:*

There exists  $c(T) > 0$  independent of  $(a_n)_n$  such that,

$$\int_0^T |f(t)|^2 dt \geq c(T) \sum_{n \in \mathbb{Z}} |a_n|^2; \quad (1.9)$$

- *Approximate averaged observability:*

$$\int_0^T |f(t)|^2 dt = 0 \quad \implies \quad a_n = 0 \quad (n \in \mathbb{Z}). \quad (1.10)$$

Notice that in the case where  $\eta = \delta_{\zeta_0}$  is a single Dirac mass, these results are true (provided some gap condition is satisfied for the eigenfrequencies  $\lambda_n$  and  $T$  is large enough) as a consequence of Ingham's inequality (see, for instance, the original paper of A. E. Ingham [14]), that has played an important role when dealing with one-dimensional control problems.

But here we are interested on averaged versions of these Ingham inequalities. They can be achieved, as described above, by perturbation arguments.

But in some specific situations, more precise results can be obtained, combining a periodicity properties and classical Ingham inequalities.

To show how these arguments can be applied, we consider the particular case in which :

1.  $\eta$  is a sum of Dirac masses located at points  $\zeta_k$ ;
2. the parameters  $\varsigma(\zeta_k)$  satisfy the non-resonance condition guaranteeing the irrationality with respect to all other ones.
3. there exists  $\gamma > 0$  such that  $\lambda_n \in \gamma\mathbb{Z}$  for every  $n$ .

Notice that the last condition is fulfilled for the string or one dimensional Schrödinger equation but that it is much stronger than the gap condition required to apply Ingham's inequality.

Under these conditions, in Theorem 3.1 we derive the unique continuation property (1.10) (see Corollary 3.1) and a weighted Ingham inequality (see corolaries 3.2 and 3.3) of the form:

$$\int_0^T |f(t)|^2 dt \geq c(T) \sum_{n \in \mathbb{Z}} \rho_n |a_n|^2, \quad (1.11)$$

where the weights  $\rho_n$  depend of the Diophantine properties of the parameters  $\zeta_k$ . This weighted averaged Ingham inequality allows deriving averaged controllability results for 1d wave and Schrödinger equations in weighted spaces (see § 3.3, Theorem 3.1, propositions 3.3 and 3.4).

These results are related to those (see R. Dáger and E. Zuazua [10, § 5.8.2]) on the simultaneous controllability of strings and networks of string equations. But, while in the context of simultaneous controllability all the velocities of propagation need to be mutually irrational, for averaged control it suffices one of them to be non-resonant with all the other ones.

### 1.3 Bibliographical comments

The notion of averaged controllability was introduced in [37] where, also, necessary and sufficient rank conditions were given in the finite dimensional context.

The works of J.-S. Li and N. Khaneja [25] and J.-S. Li [24] on *ensemble control* are also worth mentioning. In the context of the control of nuclear spins the ensemble control notion is introduced to steer, with a control independent of the parameter, all the parameter dependent trajectories in an arbitrary small ball around a desired target.

In the PDE context, in [23], the problem of averaged control was considered for two different wave equations by means of a common interior control, using  $H$ -measures techniques. In [36], other situations were also considered when, for instance, the solution of a given PDE is perturbed in an additive way by the solution of another one. Furthermore, in [29], the authors considered one-parameter families of Schrödinger and heat equations in the multi-dimensional case, with controls distributed in some interior sub-domain, showing that, depending on the averaging measure, one can obtain either the controllability results corresponding to time-reversible or parabolic-like equations.

The present paper is the first contribution for PDEs depending on the unknown parameter in a rather general manner which are also of application in the context of boundary control.

As we mentioned above, the results we obtain for the string equation are related but different to previous ones on the simultaneous controllability, a notion that was first introduced by D. L. Russell [31] (see also J.-L. Lions [26, Chapter 5]) and that has been extensively analyzed in the literature (see R. Dáger and E. Zuazua [10], C. Baiocchi, V. Komornik and P. Loreti [2], S. A. Avdonin and W. Moran [1] and the references therein).

In the case  $L_\zeta = \text{Id}$  and  $\zeta(\zeta) = \zeta$ , the issues we discussed in the previous paragraph on the averages of non-harmonic Fourier series can be recast in terms of the property of Riesz sequence stability of the family  $\{t \mapsto \hat{\eta}(-\lambda_n t)\}_n$  ( $\hat{\eta}$  being the Fourier-Stieltjes transform of the density of probability  $\eta$ ), in the closed subspace of  $L^2(0, T)$  they generate. This is so since the function  $f$  introduced by (1.7) becomes:

$$f(t) = \sum_{n \in \mathbb{Z}} a_n \hat{\eta}(-\lambda_n t) \quad (t \in \mathbb{R}).$$

This problem is related to frame theory. However, even if the literature on this subject is huge (see for instance I. Joó [16], N. Bary [4], G. Chistyakov and Y. Lyubarskii [9], A. González and R. A. Zalik [11], D. Han, W. Jing and R. N. Mohapatra [12], P. G. Casazza and O. Christensen [8, 5], Y. Y. Koo and J. K. Lim [22]), the results we needed, and that, accordingly, we prove in this article, did not seem to be available.

There are several other possible natural paths to extend the results of this paper. In particular, it would be natural to address similar issues for wave equations in networks. We refer to the book R. Dáger and E. Zuazua [10] and to I. Joó [17], S. Nicaise and J. Valein [30] and J. Valein and E. Zuazua [35] for some of the main existing results on the control and stabilization of networks of 1d wave equations.

Averaged controllability can be seen also as a first step to achieve simultaneous controllability. Obviously, the later requires also the control of the differences of all possible states for the various different realizations of the unknown parameters, and not only of their average. In the concluding section, we will show the link between these two notions via penalized optimization problems, an issue that is treated in more detail in [28]. This procedure, quickly explained in this paper, is similar to the one implemented by J.-L. Lions in [27] to link approximate controllability to exact controllability for the heat equation.

## 1.4 Structure of the paper

The core of this work is devoted to the obtention of averaged observability inequalities.

More precisely, in section 2, after having defined a functional setting and some abstract duality results in § 2.1, in § 2.2 we give some general averaged admissibility conditions. Then, in § 2.3, we use a perturbation argument in order to derive some exact averaged observability results. Finally, we apply this result in § 2.4 for the averaged controllability of wave equations, comparing our result with the one in [23].

Then, in section 3 we give some averaged Ingham inequalities. We start with § 3.1 where we apply the results of section 2 to non-harmonic Fourier series. Then in § 3.2, we prove an approximate averaged observability result in the particular context of Fourier series expansions (see (1.10)) and a weighted Ingham inequality of type (1.9), with weights depending on Diophantine approximation properties. Then, in § 3.3, we apply the results obtained in § 3.1 and § 3.2 to the string equation with Dirichlet boundary control. Connections with simultaneous controllability will also be discussed.

We conclude with some remarks and open questions in section 4.

## 2 An abstract perturbation result

### 2.1 Functional setting and duality

In this paragraph, we present some basic notations, the abstract functional setting and some well-known key duality results.

Let us introduce two Hilbert spaces, namely the state space  $X$  and the control space  $U$ , each of them being identified with its dual.

For every  $\zeta \in \mathbb{R}$ , we define the operator  $A_\zeta$ , given by  $A_\zeta \in \mathcal{L}(\mathcal{D}(A_\zeta), X)$ , with  $\mathcal{D}(A_\zeta)$  a dense linear subspace of  $X$ , with non empty resolvent  $\rho(A_\zeta)$ . We define  $X_{\zeta,1} = \mathcal{D}(A_\zeta)$  the Hilbert space endowed with the norm:

$$\|y\|_{X_{\zeta,1}} = \|(\beta I - A_\zeta)y\|_X \quad (y \in X_{\zeta,1})$$

and  $X_{\zeta,-1}$  the completion of  $X$  with respect to the norm:

$$\|y\|_{X_{\zeta,-1}} = \|(\beta I - A_\zeta)^{-1}y\|_X \quad (y \in X),$$

where, in the above, we have chosen  $\beta \in \rho(A_\zeta)$ . We refer to [33, § 2.10] for those definitions.

In addition, we assume that  $A_\zeta$  is skew-adjoint and generates a strongly continuous group  $\mathbb{T}_\zeta$  of isometries on  $X$ . We also denote by  $A_\zeta$  and  $\mathbb{T}_\zeta$  their extensions to  $X_{\zeta,-1}$

Consider the Cauchy problems:

$$\dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u, \quad y_\zeta(0) = y_\zeta^i, \quad (2.1)$$

with  $y_\zeta$  the state variable,  $u$  the control,  $A_\zeta$  the operator for the free system,  $B_\zeta \in \mathcal{L}(U, X_{\zeta,-1})$  the control operator and  $\zeta \in \mathbb{R}$  the random variable following the probability law  $\eta$ . In addition,  $y_\zeta^i \in X$  is the parameter dependent initial condition for which we assume:

$$\int_{\mathbb{R}} \|y_\zeta^i\|_X \, d\eta_\zeta < \infty. \quad (2.2)$$

In this abstract setting it is easy to see that averaged control problems cannot be handled by classical methods. Indeed, the average  $Y(t) = \int_{\mathbb{R}} y_{\zeta}(t) d\eta_{\zeta}$  satisfies:

$$\dot{Y} = \int_{\mathbb{R}} A_{\zeta} y_{\zeta} d\eta_{\zeta} + \left( \int_{\mathbb{R}} B_{\zeta} d\eta_{\zeta} \right) u, \quad Y(0) = \int_{\mathbb{R}} y_{\zeta}^i d\eta_{\zeta}.$$

This shows that the dynamics of the average is not governed by an abstract differential equation.

Despite of this, classical duality theory can be developed, and averaged controllability can be shown to be equivalent to averaged observability.

For every  $\zeta \in \mathbb{R}^*$ , one can classically define the *input to state map*:

$$\Phi_t^{\zeta} u = \int_0^t \mathbb{T}_{\zeta}(t-s) B_{\zeta} u(s) ds \quad (t > 0, u \in L^2(\mathbb{R}_+, U)), \quad (2.3)$$

so that, for every  $\zeta \in \mathbb{R}$ , the solution of (2.1) is formally:

$$y_{\zeta}(t) = \mathbb{T}_{\zeta}(t) y_{\zeta}^i + \Phi_t^{\zeta} u \quad (t > 0, u \in L^2(\mathbb{R}_+, U)). \quad (2.4)$$

Taking the average of (2.4) with respect to  $\zeta$ , we obtain (formally):

$$\int_{\mathbb{R}} y_{\zeta}(t) d\eta_{\zeta} = \int_{\mathbb{R}} \mathbb{T}_{\zeta}(t) y_{\zeta}^i d\eta_{\zeta} + \mathcal{F}_t u \quad (t > 0, u \in L^2(\mathbb{R}_+, U)), \quad (2.5)$$

where  $y_{\zeta}$  is the solution of (2.1) and where we have defined the *averaged input to state map*:

$$\mathcal{F}_t u = \int_{\mathbb{R}} \Phi_t^{\zeta} u d\eta_{\zeta} \quad (t > 0, u \in L^2(\mathbb{R}_+, U)). \quad (2.6)$$

Finally, let us define for every  $\zeta \in \mathbb{R}$  the observability map:

$$(\psi_t^{\zeta} z)(s) = \begin{cases} B_{\zeta}^* \mathbb{T}_{\zeta}^*(s) z & \text{if } s \leq t, \\ 0 & \text{if } s > t \end{cases} \quad (z \in X_{\zeta,1}, t, s > 0) \quad (2.7)$$

and the *averaged observability map*:

$$(\Psi_t z)(s) = \int_{\mathbb{R}} (\psi_t^{\zeta} z)(s) d\eta_{\zeta} = \begin{cases} \int_{\mathbb{R}} B_{\zeta}^* \mathbb{T}_{\zeta}^*(s) z d\eta_{\zeta} & \text{if } s \leq t, \\ 0 & \text{if } s > t \end{cases} \quad (t, s > 0), \quad (2.8)$$

with  $z \in X_{\zeta,1}$  for almost every  $\zeta \in \mathbb{R}$  with respect to the measure  $\eta$ .

Let us also define the time reflection operator:

$$(\mathfrak{A}_t f)(s) = f(t-s) \quad (0 < s < t, f \text{ defined almost every where on } [0, t]). \quad (2.9)$$

With these notations we are in position to define the admissibility, controllability and observability concepts.

**Definition 2.1** (Averaged admissibility). *The sequence of control operators  $(B_{\zeta})_{\zeta}$  is said to be admissible in average for the family of semi-groups  $(\mathbb{T}_{\zeta})_{\zeta}$  if there exists a time  $T > 0$  such that the map  $\mathcal{F}_T$  is bounded.*

**Remark 2.1.** If  $(B_\zeta)_\zeta$  is admissible in average for  $(\mathbb{T}_\zeta)_\zeta$  and if the initial conditions satisfy (2.2), then the averaged solution  $\int_{\mathbb{R}} y_\zeta(t) d\eta_\zeta$  defined by (2.6) is well defined for every  $t \in \mathbb{R}_+$ .

Let us now introduce the following averaged controllability concepts.

**Definition 2.2** (Exact/Approximate averaged controllability). *Let  $T > 0$ . The sequence of pairs  $(A_\zeta, B_\zeta)_\zeta$  is said to be exactly (resp. approximatively) controllable in average in time  $T$  if  $\mathcal{F}_T(L^2([0, T], U))$  is equal to (resp. dense in)  $X$ .*

As in classical control theory (see for instance [33, §4.4]), we have the following duality results:

**Proposition 2.1.** *Let  $t \in (0, T]$ . Then, we have:*

$$\mathcal{F}_t \in \mathcal{L}(L^2([0, T], U), X) \iff \Psi_t \in \mathcal{L}(X, L^2([0, T], U))$$

and  $\mathcal{F}_t^* = \mathfrak{H}_t \Psi_t$ , where  $\mathcal{F}_t$ ,  $\Psi_t$  and  $\mathfrak{H}_t$  are defined by (2.6), (2.8) and (2.9).

Moreover, if  $\mathcal{F}_t \in \mathcal{L}(L^2([0, T], U), X)$ , then:

1.  $\mathcal{F}_t(L^2([0, T], U))$  is dense in  $X$  if and only if  $\text{Ker } \Psi_t = \{0\}$ ;
2.  $\mathcal{F}_t(L^2([0, T], U)) = X$  if and only if  $\Psi_t \in \mathcal{L}(X, L^2([0, T], U))$  is bounded from below.

In the next paragraphs, following this general abstract path, we prove admissibility and exact averaged observability results for the corresponding adjoint systems.

## 2.2 A general admissibility condition

In this paragraph, we give a general condition on the measure  $\eta$  such that the averaged admissibility condition is satisfied.

**Proposition 2.2.** *Let us assume that for almost every  $\zeta \in \mathbb{R}$  with respect to the measure  $\eta$ , the control operator  $B_\zeta$  is admissible for the semi-group  $\mathbb{T}_\zeta$ . That is to say, for every  $T > 0$ , there exists a constant  $C_\zeta(T) > 0$  such that:*

$$\left\| \psi_{Tz}^\zeta \right\|_{L^2([0, T], U)}^2 \leq C_\zeta(T) \|z\|_X^2 \quad (z \in X, \zeta \in \mathbb{R} \text{ a.e.}), \quad (2.10)$$

with  $\psi_t^\zeta$  defined by (2.7).

Let  $\eta$  be a probability measure on  $\mathbb{R}$  and assume:

$$C(T) := \left( \int_{\mathbb{R}} \sqrt{C_\zeta(T)} d\eta_\zeta \right)^2 < \infty. \quad (2.11)$$

Then  $(B_\zeta)_\zeta$  is admissible in average for  $(\mathbb{T}_\zeta)_\zeta$  and we have:

$$\|\Psi_{Tz}\|_{L^2([0, T], U)}^2 \leq C(T) \|z\|_X^2 \quad (z \in X, T > 0), \quad (2.12)$$

with  $\Psi_T$  defined by (2.8)

*Proof.* Using Minkowski inequality, we have:

$$\|\Psi_{Tz}\|_{L^2([0, T], U)} \leq \int_{\mathbb{R}} \left\| \psi_{Tz}^\zeta \right\|_{L^2([0, T], U)} d\eta_\zeta.$$

This last inequality together with (2.10), gives the result.  $\square$



### 2.3 A general perturbation argument

Using the admissibility condition given in the previous paragraph, one can easily develop a perturbation argument leading to averaged controllability.

**Theorem 2.1.** *Set  $T > 0$ , let  $\tilde{\eta}$  be a probability measure and  $\zeta_0 \in \mathbb{R}$ . Assume that,*

1. *For almost every  $\zeta \in \mathbb{R}$  with respect to the measure  $\tilde{\eta}$  and for  $\zeta = \zeta_0$ ,  $B_\zeta$  is an admissible control operator for the semi-group  $\mathbb{T}_\zeta$ , i.e. there exists  $\tilde{C}_\zeta(T) > 0$  for which*

$$\left\| \psi_T^\zeta z \right\|_{L^2([0,T],U)}^2 \leq \tilde{C}_\zeta(T) \|z\|_X^2 \quad (z \in X), \quad (2.13)$$

with  $\psi_t^\zeta$  defined by (2.7).

2. *The measure  $\tilde{\eta}$  satisfies (2.10), i.e.:*

$$\int_{\mathbb{R}} \sqrt{\tilde{C}_\zeta(T)} d\tilde{\eta}_\zeta < \infty. \quad (2.14)$$

3. *The pair  $(A_{\zeta_0}, B_{\zeta_0})$  is exactly controllable in time  $T$ , i.e., there exists  $c_{\zeta_0}(T) > 0$  such that:*

$$c_{\zeta_0}(T) \|z\|_X^2 \leq \left\| \psi_T^{\zeta_0} z \right\|_{L^2([0,T],U)}^2 \quad (z \in X), \quad (2.15)$$

with  $\psi_t^\zeta$  defined by (2.7).

Set  $\theta_0 = \left( 1 + \int_{\mathbb{R}} \sqrt{\frac{\tilde{C}_\zeta(T)}{c_{\zeta_0}(T)}} d\tilde{\eta}_\zeta \right)^{-1}$ . Then for every  $\theta \in [0, \theta_0)$ ,  $(A_\zeta, B_\zeta)_\zeta$  is exactly controllable in average in time  $T$  with respect to the probability measure  $\eta$  given by:

$$\eta = (1 - \theta)\delta_{\zeta_0} + \theta\tilde{\eta}. \quad (2.16)$$

In addition, for every  $\theta \in [0, \theta_0)$ , we have:

$$c^\theta(T) \|z\|_X^2 \leq \|\Psi_T z\|_{L^2([0,T],U)}^2 \leq C^\theta(T) \|z\|_X^2 \quad (z \in X), \quad (2.17)$$

with  $\Psi_T$  defined by (2.8),  $C^\theta(T) > 0$  and  $c^\theta(T) = \left( (1 - \theta)\sqrt{c_{\zeta_0}(T)} - \theta \int_{\mathbb{R}} \sqrt{\tilde{C}_\zeta(T)} d\tilde{\eta}_\zeta \right)^2$ .

**Remark 2.2.** 1. *This result can be applied in many different examples such as wave equations, the Schödinger and plate equations, etc. with boundary or internal controls of different nature. However, the proof, which is rather straightforward, is based on a smallness argument and, hence, it does not cover the sharp results in [23] for the averaged controllability of two wave equations with internal control, or the ones in [36] for the additive superposition of wave and a heat equations.*

2. *Similar results can be obtained in more general probability spaces.*

*Proof.* The assumption made on  $\tilde{\eta}$  and the admissibility of  $B_{\zeta_0}$  for  $\mathbb{T}_{\zeta_0}$  ensure the averaged admissibility of  $(B_{\zeta})_{\zeta}$  for  $(\mathbb{T}_{\zeta})_{\zeta}$  with respect to the measure  $\eta$  given by (2.16) for every  $\theta \in [0, 1]$ .

For every  $\theta \in [0, 1]$ , let us notice that for every  $z \in X$ , we have:

$$\|\Psi_{TZ}\|_{L^2([0,T],U)} = \left\| (1-\theta)\psi_T^{\zeta_0}z + \theta\tilde{\Psi}_{TZ} \right\|_{L^2([0,T],U)} \geq (1-\theta) \left\| \psi_T^{\zeta_0}z \right\|_{L^2([0,T],U)} - \theta \left\| \tilde{\Psi}_{TZ} \right\|_{L^2([0,T],U)},$$

with  $\Psi_T$  given by (2.8) (with  $\eta$  given by (2.16)) and with  $\tilde{\Psi}_{TZ} = \int_{\mathbb{R}} \psi_T^{\zeta}z \, d\tilde{\eta}_{\zeta}$ .

Thus, from Proposition 2.2 and (2.15), we easily obtain:

$$\|\Psi_{TZ}\|_{L^2([0,T],U)} \geq \left( (1-\theta)\sqrt{c_{\zeta_0}(T)} - \theta \int_{\mathbb{R}} \sqrt{\tilde{C}_{\zeta}(T)} \, d\tilde{\eta}_{\zeta} \right) \|z\|_X \quad (z \in X).$$

This ends the proof.  $\square$

## 2.4 Averaged control of parameter depending wave systems

For every  $\zeta \in \mathbb{R}$ , let us consider the controlled wave equation:

$$\ddot{y}_{\zeta} = \operatorname{div}(a_{\zeta}(x)\nabla y_{\zeta}) + \chi_{\omega}u \quad \text{in } (0, T) \times \Omega, \quad (2.18a)$$

$$y_{\zeta} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (2.18b)$$

$$y_{\zeta}(0, x) = y_{\zeta}^{i,0}(x) \quad \text{and} \quad \dot{y}_{\zeta}(0, x) = y_{\zeta}^{i,1}(x) \quad (x \in \Omega), \quad (2.18c)$$

where  $\Omega$  is a smooth domain of  $\mathbb{R}^d$ ,  $\omega$  an open subset of  $\Omega$ ,  $a_{\zeta} \in L^{\infty}(\Omega)$  is uniformly strictly positive and bounded and the parameter-dependent initial data  $(y_{\zeta}^{i,0}, y_{\zeta}^{i,1}) \in L^2(\Omega) \times H^{-1}(\Omega)$  are such that:

$$\int_{\mathbb{R}} \left\| (y_{\zeta}^{i,0}, y_{\zeta}^{i,1}) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)} \, d\eta_{\zeta} < \infty.$$

Applying Theorem 2.1 to this system, we obtain the following:

**Proposition 2.3.** *Assume, for every  $\zeta \in \mathbb{R}$ ,  $a_{\zeta}$  is bounded from below by a positive constant independent of  $\zeta$  and  $a_1 \in C^2(\overline{\Omega})$ . Assume in addition that  $(0, T) \times \omega$  satisfies the geometric control condition (see [3]) for the equation (2.18) indexed by  $\zeta = 1$ .*

*Then there exists  $\theta_0 \in (0, 1]$  such that system (2.18) fulfils the exact averaged control property for every  $\theta \in [0, \theta_0]$  with measure  $\eta^{\theta} = (1-\theta)\delta_1 + \theta\tilde{\eta}$ .*

*Proof.* From [3], the geometric control condition for the control system indexed by  $\zeta = 1$  ensures that this system is exactly controllable in time  $T$ .

In addition, for every  $(z_{\zeta}^{f,0}, z_{\zeta}^{f,1}) \in L^2(\Omega) \times H^{-1}(\Omega)$  and every  $T > 0$ , we have:

$$\int_0^T \left| \int_{\omega} \dot{z}_{\zeta}(t, x) \, dx \right|^2 \, dt \leq 2T \left\| (z_{\zeta}^{f,0}, z_{\zeta}^{f,1}) \right\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2,$$

where  $z_{\zeta}$  is solution of the adjoint system:

$$\ddot{z}_{\zeta} = \operatorname{div}(a_{\zeta}(x)\nabla z_{\zeta}) \quad \text{in } (0, T) \times \Omega, \quad (2.19a)$$

$$z_{\zeta} = 0 \quad \text{on } (0, T) \times \partial\Omega, \quad (2.19b)$$

$$z_{\zeta}(T, x) = z_{\zeta}^{f,0}(x) \quad \text{and} \quad \dot{z}_{\zeta}(T, x) = z_{\zeta}^{f,1}(x) \quad (x \in \Omega). \quad (2.19c)$$

Thus the condition (2.14) of Theorem 2.1 is automatically satisfied for every probability measure  $\tilde{\eta}$ . Hence, applying Theorem 2.1, we obtain the result.  $\square$

**Remark 2.3.** *This result holds in the particular case  $\eta^\theta = (1 - \theta)\delta_1 + \theta\delta_2$  where two wave equations with different velocities of propagation are averaged.*

*This case was addressed in [23, Theorem 2.1] where it was proved that, if the coefficients  $a_1$  and  $a_2$  satisfy:*

$$a_1(x) \neq a_2(x) \quad (x \in \omega), \quad (2.20)$$

*then the system satisfies the averaged control property for every  $\theta \in [0, 1]$  (see [23, Theorem 2.1]). The proof in [23, Theorem 2.1] employs microlocal defect measures and the fact that the characteristic manifolds of the two wave equations involved are disjoint. This example shows that the smallness condition we impose on the perturbations is not always sharp.*

### 3 Averaged Ingham inequalities

Let us define the Hilbert space of square summable sequences:

$$\ell^2 = \left\{ (a_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \sum_{n \in \mathbb{Z}} |a_n|^2 < \infty \right\}.$$

We also consider a real sequence  $\lambda = (\lambda_n)_{n \in \mathbb{Z}}$ , which is assumed to satisfy the following gap condition: there exists  $\gamma > 0$  such that

$$\inf_{\substack{(m,n) \in \mathbb{Z}^2 \\ m \neq n}} |\lambda_m - \lambda_n| \geq \gamma. \quad (3.1)$$

Our goal in this section is to find conditions on  $(\lambda_n)$ , the measure  $\eta$  and the time  $T$  such that, for every  $a \in \ell^2$ , the function  $f$  defined by (1.7) satisfies (1.8) and (1.9) or (1.10).

Notice that when  $\eta$  is the atomic mass located in  $\zeta_0$ ,  $L_{\zeta_0} = \text{Id}$  and  $\varsigma(\zeta_0) \neq 0$ , according to Ingham's inequality, (1.8) and (1.9) are valid for  $T > 1/|\varsigma(\zeta_0)|\gamma$ . More precisely,

$$c(T) \sum_{n \in \mathbb{Z}} |a_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t} \right|^2 dt \quad ((a_n)_n \in \ell^2, T > \frac{1}{|\varsigma(\zeta_0)|\gamma}), \quad (3.2a)$$

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t} \right|^2 dt \leq C(T) \sum_{n \in \mathbb{Z}} |a_n|^2 \quad ((a_n)_n \in \ell^2, T \geq 0), \quad (3.2b)$$

with  $c(T) = \frac{2(\varsigma(\zeta_0)\gamma T)^2 - 1}{\pi(\varsigma(\zeta_0)\gamma T)^2} T$  and  $C(T) = \frac{10T}{\pi \min(1, 2|\varsigma(\zeta_0)|\gamma T)}$ .

This classical result can be found in the original paper by A. E. Ingham, [14, Theorem 1 and 2]. For its relation with control theory, we refer, for instance, to [15, 20, 21] and the books [33, 10].

#### 3.1 Perturbation of Ingham inequalities

In this paragraph we apply our perturbation argument developed in §2.3 in the context of non-harmonic Fourier series.

**Proposition 3.1.** *Let  $\lambda = (\lambda_n)_{n \in \mathbb{Z}}$  be a sequence of real numbers satisfying the gap condition (3.1). Let  $L_\zeta \in \mathcal{L}(\ell^2)$ ,  $\tilde{\eta}$  a probability measure on  $\mathbb{R}$ ,  $\varsigma \in \mathbb{R}^{\mathbb{R}}$  and  $\zeta_0 \in \mathbb{R}$ .*

Assume  $\varsigma(\zeta_0) \neq 0$  and let  $T > \frac{1}{\gamma|\varsigma(\zeta_0)|}$ . Assume in addition:

$$\int_{\mathbb{R}} \|L_\zeta\|_{\mathcal{L}(\ell^2)} d\tilde{\eta}_\zeta < \infty \quad \text{and} \quad \int_{\mathbb{R}} \frac{\|L_\zeta\|_{\mathcal{L}(\ell^2)}}{\sqrt{|\varsigma(\zeta)|}} d\tilde{\eta}_\zeta < \infty$$

$L_{\zeta_0}$  being bounded from below, i.e. there exists  $\Lambda_{\zeta_0} > 0$  such that:

$$\Lambda_{\zeta_0} \|a\|_{\ell^2} \leq \|L_{\zeta_0} a\|_{\ell^2} \quad (a \in \ell^2).$$

$$\text{Set } \theta_0 = \left( 1 + \frac{1}{\Lambda_{\zeta_0}} \sqrt{\frac{5(\gamma\varsigma(\zeta_0)T)^2}{(\gamma\varsigma(\zeta_0)T)^2 - 1}} \int_{\mathbb{R}} \frac{\|L_\zeta\|_{\mathcal{L}(\ell^2)}}{\min(1, \sqrt{2\gamma|\varsigma(\zeta)|}T)} d\tilde{\eta}_\zeta \right)^{-1}.$$

Then for every  $\theta \in [0, \theta_0)$ , there exists  $c^\theta(T) > 0$  and  $C^\theta(T) > 0$  such that:

$$c^\theta(T) \|a\|_{\ell^2}^2 \leq \int_0^T \left| \theta \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t} d\tilde{\eta}_\zeta + (1 - \theta) \sum_{n \in \mathbb{Z}} [L_{\zeta_0} a]_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t} \right|^2 dt \leq C^\theta(T) \|a\|_{\ell^2}^2 \quad (a \in \ell^2),$$

*Proof.* First of all, we have from (3.2):

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} [L_\zeta a]_n e^{2i\pi\lambda_n\varsigma(\zeta)t} \right|^2 dt \leq \|L_\zeta\|_{\mathcal{L}(\ell^2)}^2 C_\zeta(T) \|a\|_{\ell^2}^2 \quad (\zeta \in \mathbb{R}^*, a = (a_n)_n \in \ell^2, T > 0),$$

and

$$\Lambda_{\zeta_0}^2 c_{\zeta_0}(T) \|a\|_{\ell^2}^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}} [L_{\zeta_0} a]_n e^{2i\pi\lambda_n\varsigma(\zeta_0)t} \right|^2 dt \quad (a = (a_n)_n \in \ell^2, T > \frac{1}{|\varsigma(\zeta_0)|\gamma}),$$

$$\text{with } c_\zeta(T) = \frac{2(\gamma\varsigma(\zeta)T)^2 - 1}{\pi(\gamma\varsigma(\zeta)T)^2} T \text{ and } C(T) = \frac{10T}{\pi \min(1, 2\gamma|\varsigma(\zeta)|T)}.$$

We conclude as in the proof of Theorem 2.1.  $\square$

**Remark 3.1.** The condition  $T > \frac{1}{|\varsigma(\zeta_0)|\gamma}$  is only required in view of the fact that we have employed the classical formulation of Ingham's inequality. But, for instance, if the sequence  $(\lambda_n)_{n \in \mathbb{N}^*}$  is nondecreasing and satisfies the asymptotic gap condition  $\liminf_{n \rightarrow \infty} \lambda_{n+1} - \lambda_n = +\infty$  then, employing generalised versions of Ingham's inequalities (see [18]), our result can be shown to hold true for every  $T > 0$ .

**Remark 3.2.** We have shown that averaged versions of Ingham's inequalities hold true under a suitable smallness condition on the perturbing measures, that is necessary in some sense as the example below shows.

Let us build an example where averaged Ingham fails. Consider the case  $L_\zeta = \text{Id}$ ,  $\lambda_n = n$ ,  $\zeta_0 = 1$ ,  $\zeta_1 = 2$ ,  $\varsigma(\zeta) = \zeta$  and the measure  $\eta = (1 - \theta)\delta_{\zeta_0} + \theta\delta_{\zeta_1}$ . In view of the proposition above, for every  $T > 1/\zeta_0$ , there exists  $\theta_0 \in [0, 1)$  such that for every  $\theta \in [0, \theta_0)$ , there exist constants  $c^\theta(T), C^\theta(T) > 0$  such that:

$$c^\theta(T) \|a\|_{\ell^2}^2 \leq \int_0^T \left| \sum_{n \in \mathbb{Z}} a_n \left( (1 - \theta)e^{2i\pi nt} + \theta e^{4i\pi nt} \right) \right|^2 dt \leq C^\theta(T) \|a\|_{\ell^2}^2 \quad (a \in \ell^2).$$

But the smallness condition on  $\theta$  is necessary. In particular, for  $\theta = \frac{1}{2}$ , no such  $c^{\frac{1}{2}}(T) > 0$  exists. Indeed, more precisely, consider:

$$f(t) = \sum_{n \in \mathbb{Z}} a_n \left( \frac{1}{2} e^{2i\pi n t} + \frac{1}{2} e^{4i\pi n t} \right) = \frac{1}{2} \left( \sum_{n=1}^{\infty} (a_{2n} + a_n) e^{4i\pi n t} + \sum_{n \in \mathbb{Z}} a_{2n+1} e^{2i\pi(2n+1)t} \right),$$

with

$$a_n = \begin{cases} (-1)^k & \text{if } n = 2^k \text{ and } k \in \{0, \dots, N\}, \\ 0 & \text{otherwise,} \end{cases} \quad (n \in \mathbb{Z}),$$

with  $N \in \mathbb{N}^*$  given, so that

$$f(t) = \frac{(-1)^N}{2} e^{4i\pi 2^{N+1} t}.$$

Thus for every  $T > 0$ ,  $\int_0^T |f(t)|^2 dt = \frac{T}{4}$ , whereas,  $\|a\|_{\ell^2} = N + 1$ .

Letting  $N$  tend to infinity we see that the observability inequality fails and this for all  $T > 0$ .

### 3.2 Discrete averages

In §3.1, we used a perturbation argument to prove, roughly speaking, the stability of Ingham inequalities when the measure  $\eta$  is a Dirac mass plus a small enough perturbation. Obviously, there are many other cases of interest that do not enter on that setting.

In this paragraph we consider another interesting particular case, in which a finite number of equations are involved. In other words, we address the case in which the unknown parameter varies on a finite set.

In order to handle this case and to prove the needed averaged Ingham inequalities we will use a different argument. Instead of arguing through a perturbation principle, we shall rather use a method inspired on [10] and [36] whose key tool is to use the fact the solutions of the model under consideration, for given values of the parameter, are annihilated by a given linear bounded operator commuting with all other equations. This is the case for the 1d wave and Schrödinger equations with Dirichlet boundary conditions, for which the solutions are time-periodic.

In order to present these cases we consider a sequence  $(\lambda_n)_{n \in \mathbb{Z}}$  satisfying:

$$\lambda_m \neq \lambda_n, \text{ for } m \neq n \quad \text{and} \quad \lambda_n \in \gamma \mathbb{Z} \quad (m, n \in \mathbb{Z}), \quad (3.3)$$

with  $\gamma > 0$ . Of course, in this case the Ingham gap condition holds.

In order to make things more clear, let us write  $\lambda_n = \mu_n \gamma$ , with  $\mu_n \in \mathbb{Z}$ .

As in the previous paragraph, we also consider a operator  $L_\zeta \in \mathcal{L}(\ell^2)$  and a function  $\varsigma \in \mathbb{R}^{\mathbb{R}}$ , and we will consider the function  $f$  defined by:

$$f(t) = \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi \mu_n \gamma \varsigma(\zeta_k) t} \quad (a \in \ell^2, t > 0), \quad (3.4)$$

with  $K > 0$ ,  $\theta_k \in [0, 1)$  such that  $\sum_{k=0}^K \theta_k = 1$ , and  $\zeta_k \in \mathbb{R}$ .

Since we are averaging on a finite number of parameters, it is easy to see from (3.2) that, for every  $T > 0$ , there exists a constant  $C(T) > 0$  such that:

$$\|f\|_{L^2(0, T)}^2 \leq C(T) \|a\|_{\ell^2}^2 \quad (a \in \ell^2).$$

Let us now recover an observability inequality.

**Theorem 3.1.** Let  $(\mu_n)_{n \in \mathbb{Z}}$  be a sequence of integers and  $\gamma > 0$ .

Let  $\varsigma \in \mathbb{R}^{\mathbb{R}}$ ,  $K \in \mathbb{N}^*$ , and for every  $k \in \{0, \dots, K\}$ , let  $\theta_k \in [0, 1]$  be the weights (so that  $\sum_{k=0}^K \theta_k = 1$ ),  $(a_n^k)_n \in \ell^2$  and  $\zeta_k \in \mathbb{R}$ . Assume for every  $k \in \{0, \dots, K\}$ ,  $\varsigma(\zeta_k) \neq 0$ .

Then, if

$$T > \frac{1}{\gamma} \sum_{k=0}^K \frac{1}{|\varsigma(\zeta_k)|},$$

there exists a constant  $c(T) > 0$  independent of  $(a_n^k)_n$  such that:

$$\int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} a_n^k e^{2i\pi\mu_n \gamma \varsigma(\zeta_k) t} \right|^2 dt \geq \theta_0^2 c(T) \sum_{n \in \mathbb{Z}} |a_n^0|^2 \prod_{l=1}^K \sin^2 \left( \pi \mu_n \frac{\varsigma(\zeta_0)}{\varsigma(\zeta_l)} \right). \quad (3.5)$$

*Proof.* First of all, by changing  $\gamma \varsigma(\zeta_k)$  in  $\zeta_k$ , we can assume without loss of generality that  $\gamma = 1$  and  $\varsigma(\zeta) = \zeta$ .

Considering the function  $f$  defined by:

$$f(t) = \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} a_n^k e^{2i\pi\mu_n \zeta_k t} \quad (t \in \mathbb{R}),$$

one can notice that:

$$f(t + |\zeta_K|^{-1}) - f(t) = \sum_{k=0}^{K-1} \theta_k \sum_{n \in \mathbb{Z}} a_n^k \left( e^{2i\pi\mu_n \frac{\zeta_k}{|\zeta_K|}} - 1 \right) e^{2i\pi\mu_n \zeta_k t} \quad (t \in \mathbb{R}).$$

Iterating this argument it is easy to see that:

$$F_0(t) = \theta_0 \sum_{n \in \mathbb{Z}} a_n^0 \prod_{l=0}^{K-1} \left( e^{2i\pi\mu_n \frac{\zeta_0}{|\zeta_{K-l}|}} - 1 \right) e^{2i\pi\mu_n \zeta_0 t} \quad (t \in \mathbb{R}),$$

where  $F_0$  is defined recursively by:

$$\begin{aligned} F_K(t) &= f(t), \\ F_{k-1}(t) &= F_k(t + |\zeta_k|^{-1}) - F_k(t) \quad (k \in \{1, \dots, K\}). \end{aligned} \quad (t \in \mathbb{R}). \quad (3.6)$$

Then for any  $\tau > 0$ , by the classical Ingham inequality [14, Theorem 1] we deduce the existence of a constant  $c_\tau$  such that

$$\int_0^{\frac{1}{\zeta_0} + \tau} |F_0(t)|^2 dt \geq c_\tau \theta_0^2 \sum_{n \in \mathbb{Z}} |a_n^0|^2 \prod_{l=0}^{K-1} \left| e^{2i\pi\mu_n \frac{\zeta_0}{|\zeta_{K-l}|}} - 1 \right|^2, \quad (3.7)$$

with  $c_\tau > 0$  a constant depending only on  $\tau$  and  $\zeta_0$ .

But, we have:

$$\int_0^{\frac{1}{|\zeta_0|} + \tau} |F_0(t)|^2 dt = \int_0^{\frac{1}{|\zeta_0|} + \tau} \left| F_1(t + \frac{1}{|\zeta_0|}) - F_1(t) \right|^2 dt \leq 2 \int_0^{\frac{1}{|\zeta_0|} + \frac{1}{|\zeta_1|} + \tau} |F_1(t)|^2 dt \quad (\tau > 0)$$

and by iteration,

$$\|F_0\|_{L^2(0, \frac{1}{|\zeta_0|} + \tau)}^2 \leq 2^K \|f\|_{L^2(0, \tau + \sum_{k=0}^K |\zeta_k|^{-1})}^2 \quad (\tau > 0). \quad (3.8)$$

Consequently, (3.7) together with (3.8) gives:

$$\|f\|_{L^2(0, \tau + \sum_{k=0}^K |\zeta_k|^{-1})}^2 \geq \frac{c_\tau \theta_0^2}{2^K} \sum_{n \in \mathbb{Z}} |a_n^0|^2 \prod_{l=0}^{K-1} \left| e^{2i\pi\mu_n \frac{\zeta_0}{\zeta_{K-l}}} - 1 \right|^2 \quad (\tau > 0).$$

□

**Corollary 3.1.** *Let  $(\mu_n)_{n \in \mathbb{Z}}$  be a sequence of integers and  $\gamma > 0$ .*

*Let  $\varsigma \in \mathbb{R}^{\mathbb{R}}$ ,  $K \in \mathbb{N}^*$ , and for every  $k \in \{0, \dots, K\}$ , let  $\theta_k \in [0, 1]$  be the weights (so that  $\sum_{k=0}^K \theta_k = 1$ ),  $(a_n^k)_n \in \ell^2$ ,  $\zeta_k \in \mathbb{R}$  and  $L_{\zeta_k} \in \mathcal{L}(\ell^2)$ .*

*Assume  $\theta_0 \neq 0$ ,  $L_{\zeta_0}$  is bounded from below,  $\varsigma(\zeta_0) \neq 0$  and*

$$\varsigma(\zeta_0)^{-1} \varsigma(\zeta_k) \notin \mathbb{Q} \quad (k \in \{1, \dots, K\}). \quad (3.9)$$

*Then, for every  $T > \frac{1}{\gamma} \sum_{k=0}^K \frac{1}{|\varsigma(\zeta_k)|}$ ,*

$$\int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n \gamma \varsigma(\zeta_k) t} \right|^2 dt = 0 \quad \implies \quad \forall n \in \mathbb{Z}, a_n = 0.$$

*Proof.* Setting  $a^k = L_{\zeta_k} a$  in Theorem 3.1, we obtain from (3.5):

$$\int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n \gamma \varsigma(\zeta_k) t} \right|^2 dt \geq \theta_0^2 c(T) \sum_{n \in \mathbb{Z}} |[L_{\zeta_0} a]_n|^2 \prod_{l=1}^K \sin^2 \left( \pi \mu_n \frac{\varsigma(\zeta_0)}{\varsigma(\zeta_l)} \right),$$

with  $C(T) > 0$ . Since  $\theta_0 \neq 0$ ,  $\varsigma(\zeta_0)^{-1} \varsigma(\zeta_k) \notin \mathbb{Q}$  for every  $k \in \{1, \dots, K\}$  and, hence,  $\prod_{l=1}^K \sin^2 \left( \pi \mu_n \frac{\varsigma(\zeta_0)}{\varsigma(\zeta_l)} \right) \neq 0$ , we obtain  $[L_{\zeta_0} a]_n = 0$  for every  $n \in \mathbb{Z}$  and hence  $a = 0$  since  $L_{\zeta_0}$  is bounded from below. □

**Remark 3.3.** *Let us discuss the optimality of Corollary 3.1.*

1. *Inequality (3.5) is similar to the one in (5.87), p. 139 of [10] which can be applied to the simultaneous control of finitely many strings (see § 5.8.2 of that book).*
2. *The irrationality condition (3.9) is sharp as the example presented in Remark 3.2 shows.*

In corollaries 3.1 and 3.3, we present a unique continuation result. However, with some more restrictive conditions on the parameters  $\zeta_k$ , we can obtain an observability inequality.

**Corollary 3.2.** *Under the conditions of Corollary 3.1, let  $\varepsilon > 0$  and assume in addition, for every  $\alpha > 0$ , there exists  $\Lambda_{\zeta_0, \alpha} > 0$  such that:*

$$\sum_{n \in \mathbb{Z}} \frac{|[L_{\zeta_0} a]_n|^2}{|\mu_n|^{2\alpha}} \geq \Lambda_{\zeta_0, \alpha}^2 \sum_{n \in \mathbb{Z}} \frac{|a_n|^2}{|\mu_n|^{2\alpha}} \quad (a \in \ell^2) \quad (3.10)$$

and

$$\varsigma(\zeta_0)^{-1} \varsigma(\zeta_k) \in \mathbf{B}_\varepsilon \quad (k \in \{1, \dots, K\}), \quad (3.11)$$

with  $\mathbf{B}_\varepsilon$  defined, as in [6, p. 120], so that the Lebesgue measure of  $\mathbb{R} \setminus \mathbf{B}_\varepsilon$  vanishes and there exists a constant  $\rho_\varepsilon > 0$  so that, if  $\zeta \in \mathbf{B}_\varepsilon$  then, for every  $m \in \mathbb{N}^*$ , we have:  $\min_{r \in \mathbb{Z}} |r - m\zeta| \geq \frac{\rho_\varepsilon}{m^{1+\varepsilon}}$ .

Then, for every  $T > \frac{1}{\gamma} \sum_{k=0}^K \frac{1}{|\varsigma(\zeta_k)|}$ , there exists a constant  $C_\varepsilon(T) > 0$  such that:

$$\int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n \gamma \varsigma(\zeta_k) t} \right|^2 dt \geq C_\varepsilon(T) \sum_{n \in \mathbb{Z}} \frac{|a_n|^2}{|\mu_n|^{2K(1+\varepsilon)}} \quad (a \in \ell^2).$$

*Proof.* The proof follows directly from (3.5) and [6, p. 120].  $\square$

**Corollary 3.3.** *Under the conditions of Corollary 3.1 and assume, for every  $\alpha > 0$ , there exists  $\Lambda_{\zeta_0, \alpha} > 0$  such that (3.10) holds. Assume in addition that  $\varsigma(\zeta_0)^{-1}\varsigma(\zeta_1), \dots, \varsigma(\zeta_0)^{-1}\varsigma(\zeta_K)$  are algebraic and  $\varsigma(\zeta_0), \dots, \varsigma(\zeta_K)$  are  $\mathbb{Q}$ -linearly independent.*

Then for every  $T > \frac{1}{\gamma} \sum_{k=0}^K \frac{1}{|\varsigma(\zeta_k)|}$  and every  $\varepsilon > 0$ , there exists  $C_\varepsilon(T) > 0$  such that:

$$\int_0^T \left| \sum_{k=0}^K \theta_k \sum_{n \in \mathbb{Z}} [L_{\zeta_k} a]_n e^{2i\pi\mu_n \gamma \varsigma(\zeta_k) t} \right|^2 dt \geq C_\varepsilon(T) \sum_{n \in \mathbb{Z}} \frac{|a_n|^2}{|\mu_n|^{2(1+\varepsilon)}} \quad (a \in \ell^2).$$

*Proof.* The proof follows directly from (3.5) and [32].  $\square$

### 3.3 Application to the string equation

We are now in position to derive the main consequences concerning averaged controllability of the string equation with Dirichlet boundary control:

$$\ddot{y}_\zeta(t, x) = \zeta^2 \partial_x^2 y_\zeta(t, x) \quad ((t, x) \in \mathbb{R}_+^* \times (0, 1)), \quad (3.12a)$$

$$y_\zeta(t, 0) = u(t) \quad (t \in \mathbb{R}_+^*), \quad (3.12b)$$

$$y_\zeta(t, 1) = 0 \quad (t \in \mathbb{R}_+^*), \quad (3.12c)$$

$$y_\zeta(0, x) = y_\zeta^{i,0}(x) \quad \text{and} \quad \dot{y}_\zeta(0, x) = y_\zeta^{i,1}(x) \quad (x \in (0, 1)). \quad (3.12d)$$

Let us briefly describe how the string equation with Dirichlet boundary control enters in the abstract formalism. We refer the interested reader to [33, Sections 10.9 and 11.6] for further details.

Notice that since we have a second order operator, it is more convenient to assume that the parameter  $\zeta$  enters in a quadratic manner in (3.12a). So that the adjoint problem can be expressed in terms of non-harmonic Fourier series of the form (1.7). In addition, one can assume  $\zeta \in \mathbb{R}_+$  or equivalently, the averaging measure  $\eta$  satisfies  $\text{supp } \eta \subset \mathbb{R}_+$ .

Let us first introduce the one dimensional Dirichlet-Laplacian operator,  $A_0$ :

$$\mathcal{D}(A_0) = H^2(0, 1) \cap H_0^1(0, 1) \quad \text{and} \quad A_0 f = -\partial_x^2 f \quad (f \in \mathcal{D}(A_0)).$$

We also introduce the Hilbert spaces  $H = L^2(0, 1)$ ,  $H_1 = \mathcal{D}(A)$ ,  $H_{\frac{1}{2}} = H_0^1(0, 1)$  and  $H_{-1}$  (resp.  $H_{-\frac{1}{2}}$ ) the dual space of  $H_1$  (resp.  $H_{\frac{1}{2}}$ ) with respect to the pivot space  $H$ . Then  $A_0$  can be seen as an unitary operator from  $H_1$  to  $H$ ,  $H_{\frac{1}{2}}$  to  $H_{-\frac{1}{2}}$  and  $H$  to  $H_{-1}$ .



In addition, let us remind that the Dirichlet-Laplacian operator  $A_0$  can be diagonalized in an orthonormal basis  $(\varphi_n)_{n \in \mathbb{N}^*}$  of  $L^2(0, 1)$ . More precisely, we have:

$$\varphi_n(x) = \sqrt{2} \sin(n\pi x) \quad \text{and} \quad A_0 \varphi_n = (n\pi)^2 \varphi_n \quad (x \in [0, 1], n \in \mathbb{N}^*). \quad (3.13)$$

With these notations in mind, let us define the state space  $X = H \times H_{-\frac{1}{2}}$ , the control space  $U = \mathbb{R}$ , the operator  $A = \begin{bmatrix} 0 & \text{Id} \\ -A_0 & 0 \end{bmatrix}$  with domain  $\mathcal{D}(A) = H_{\frac{1}{2}} \times H := X_1$  and the control operator  $B = \begin{bmatrix} 0 \\ A_0 D \end{bmatrix} \in \mathcal{L}(U, X_{-1})$ , with  $D$  the Dirichlet map [33, Proposition 10.6.1]. Notice that we have  $A^* = -A$  and  $B^* \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} = \partial_x (A_0^{-1} z^1)(0)$ . Let us denote by  $\mathbb{T}$  the semi-group generated by  $A$ . It is classical that  $B$  is an admissible control operator for  $\mathbb{T}$ .

Let us now define  $f_\zeta(s, x) = y_\zeta(\frac{s}{\zeta}, x)$ , thus  $f$  is solution of:

$$\begin{aligned} \ddot{f}_\zeta(s, x) &= \partial_x^2 f_\zeta(s, x) && ((s, x) \in \mathbb{R}_+^* \times (0, 1)), \\ f_\zeta(s, 0) &= u(\frac{s}{\zeta}) && (s \in \mathbb{R}_+^*), \\ f_\zeta(s, 1) &= 0 && (t \in \mathbb{R}_+^*), \\ f_\zeta(0, x) &= y_\zeta^{i,0}(x) \quad \text{and} \quad \dot{f}_\zeta(0, x) = \frac{1}{\zeta} y_\zeta^{i,1}(x) && (x \in (0, 1)). \end{aligned}$$

Thus, setting  $I_\zeta = \begin{bmatrix} \text{Id} & 0 \\ 0 & \zeta \text{Id} \end{bmatrix}$ ,  $F_\zeta(s) = \begin{bmatrix} f_\zeta(s) \\ \dot{f}_\zeta(s) \end{bmatrix} = I_\zeta^{-1} \begin{bmatrix} y_\zeta(\frac{s}{\zeta}) \\ \dot{y}_\zeta(\frac{s}{\zeta}) \end{bmatrix}$ ,  $F_\zeta$  is solution of:

$$\dot{F}_\zeta = A F_\zeta + B u(\frac{\cdot}{\zeta}), \quad F_\zeta(0) = I_\zeta^{-1} \begin{bmatrix} y_\zeta^{i,0} \\ y_\zeta^{i,1} \\ y_\zeta \end{bmatrix}.$$

Using Duhamel formula, we obtain:

$$F_\zeta(\zeta T) = \mathbb{T}(\zeta T) I_\zeta^{-1} \begin{bmatrix} y_\zeta^{i,0} \\ y_\zeta^{i,1} \\ y_\zeta \end{bmatrix} + \int_0^{\zeta T} \mathbb{T}(\zeta T - s) B u(\frac{s}{\zeta}) ds = \mathbb{T}(\zeta T) I_\zeta^{-1} \begin{bmatrix} y_\zeta^{i,0} \\ y_\zeta^{i,1} \\ y_\zeta \end{bmatrix} + \int_0^T \mathbb{T}(\zeta(T - t)) \zeta B u(t) dt$$

and hence,

$$\begin{bmatrix} y_\zeta(T) \\ \dot{y}_\zeta(T) \end{bmatrix} = I_\zeta \mathbb{T}(\zeta T) I_\zeta^{-1} \begin{bmatrix} y_\zeta^{i,0} \\ y_\zeta^{i,1} \\ y_\zeta \end{bmatrix} + \int_0^T I_\zeta \mathbb{T}(\zeta(T - t)) I_\zeta^{-1} \zeta^2 B u(t) dt.$$

Let us then define the averaged input to state map:

$$\mathcal{F}_T u = \int_{\mathbb{R}} \int_0^T I_\zeta \mathbb{T}(\zeta(T - t)) I_\zeta^{-1} \zeta^2 B u(t) dt d\eta_\zeta \quad (u \in L^2(0, T))$$

and the averaged observability map is:

$$\begin{aligned} \left( \Psi_T \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right) (t) &= \int_{\mathbb{R}} \zeta^2 B^* I_\zeta^{-1} \mathbb{T}(-\zeta t) I_\zeta \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} d\eta_\zeta \\ &= \int_{\mathbb{R}} \zeta B^* \mathbb{T}(-\zeta t) \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} d\eta_\zeta \quad \left( \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \in X_1, t \in (0, T) \right). \end{aligned}$$

Let  $z_\zeta$  be the solution of:

$$\begin{aligned} \ddot{z}_\zeta &= \zeta^2 \partial_x^2 z_\zeta, \\ 0 &= z_\zeta(t, 0) = z_\zeta(t, 1) \end{aligned} \quad (t \geq 0)$$

with initial conditions:

$$z_\zeta(0, \cdot) = z^0 \quad \text{and} \quad \dot{z}_\zeta(0, \cdot) = -\zeta^2 z^1.$$

Thus,

$$\mathbb{T}(-\zeta t) \begin{bmatrix} z^0 \\ \zeta z^1 \end{bmatrix} = \begin{bmatrix} z_\zeta(t) \\ \frac{-1}{\zeta} \dot{z}_\zeta(t) \end{bmatrix}$$

and hence,

$$\left( \Psi_T \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right) (t) = - \int_{\mathbb{R}} \partial_x (A_0^{-1} \dot{z}_\zeta(t, \cdot)) (0) \zeta d\eta_\zeta,$$

Expanding the initial conditions  $z^0 = \sum \alpha_n \varphi_n$  and  $z^1 = \sum \beta_n \varphi_n$  on the eigenvector basis  $\{\varphi_n\}_n$  of  $A_0$  defined by (3.13) leads to:

$$z_\zeta(t, x) = \sum_{n \in \mathbb{N}^*} \left( \alpha_n \cos(n\pi\zeta t) - \zeta \frac{\beta_n}{n\pi} \sin(n\pi\zeta t) \right) \varphi_n(x).$$

Thus, the observation operator is:

$$\begin{aligned} \left( \Psi_T \begin{bmatrix} z^0 \\ z^1 \end{bmatrix} \right) (t) &= -\sqrt{2} \int_{\mathbb{R}} \sum_{n=0}^{\infty} \left( \alpha_n \sin(n\pi\zeta t) + \zeta \frac{\beta_n}{n\pi} \cos(n\pi\zeta t) \right) \zeta d\eta_\zeta \\ &= \frac{-\sqrt{2}}{2} \int_{\mathbb{R}} \left( \sum_{n \in \mathbb{N}^*} \left( -i\alpha_n + \zeta \frac{\beta_n}{n\pi} \right) e^{in\pi\zeta t} + \sum_{n \in \mathbb{N}^*} \left( i\alpha_n + \zeta \frac{\beta_n}{n\pi} \right) e^{-in\pi\zeta t} \right) \zeta d\eta_\zeta. \end{aligned}$$

Let us also notice that  $\left\| \begin{bmatrix} z_0^i \\ z_1^i \end{bmatrix} \right\|_X^2 = \sum_{n \in \mathbb{N}^*} \left( \alpha_n^2 + \frac{\beta_n^2}{(n\pi)^2} \right)$ . Thus setting:

$$\begin{aligned} \lambda_n &= \frac{n}{2}, \quad a_n = \begin{cases} \alpha_n & \text{if } n > 0, \\ \frac{\beta_{-n}}{n\pi} & \text{if } n < 0 \end{cases} \quad (n \in \mathbb{Z}^*) \quad \text{and} \\ [L_\zeta a]_n &= \begin{cases} \frac{-\sqrt{2}}{2} (-ia_n + \zeta a_{-n}) \zeta & \text{if } n > 0, \\ \frac{-\sqrt{2}}{2} (ia_{-n} + \zeta a_n) \zeta & \text{if } n < 0, \end{cases} \quad (a \in \ell^2(\mathbb{Z}^*), n \in \mathbb{Z}^*, \zeta \in \mathbb{R}), \end{aligned}$$

the observation operator is  $\int_{\mathbb{R}} \sum_{n \in \mathbb{Z}^*} [L_\zeta a]_n e^{2i\pi\lambda_n \zeta t} d\eta_\zeta$ .

Let us notice that:

$$\zeta^2 (1 + (\zeta^2 - 1) \mathbf{1}_{[0,1]}(|\zeta|)) \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{|n|^{2\alpha}} \leq \sum_{n \in \mathbb{Z}^*} \frac{|[L_\zeta a]_n|^2}{|n|^{2\alpha}} \leq \zeta^2 (1 + (\zeta^2 - 1) \mathbf{1}_{(1,\infty)}(|\zeta|)) \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{|n|^{2\alpha}} \quad (a \in \ell^2(\mathbb{Z}^*), \zeta \in \mathbb{R}, \alpha \geq 0). \quad (3.14)$$

Consequently, using Proposition 3.1 together with (3.14) and the duality result, Proposition 2.1, we obtain this perturbation result:

**Proposition 3.2** (Exact averaged controllability with a perturbed measure). *Let  $\zeta_0 \in \mathbb{R}^*$  and  $\tilde{\eta}$  be a probability measure on  $\mathbb{R}$  with:*

$$\int_{\mathbb{R}} |\zeta|^\alpha d\tilde{\eta}_\zeta < \infty \quad (\alpha \in [\frac{1}{2}, 2]).$$

Set  $T > \frac{2}{|\zeta_0|}$  and

$$\theta_0 = \begin{cases} \left( 1 + \frac{\sqrt{5}T}{\zeta_0 \sqrt{(\zeta_0 T)^2 - 4}} \int_{\mathbb{R}} \frac{|\zeta| |1 + (\zeta^2 - 1)\mathbf{1}_{(1,\infty)}(|\zeta|)|^{\frac{1}{2}}}{\min(1, \sqrt{|\zeta|T})} d\tilde{\eta}_\zeta \right)^{-1} & \text{if } 0 < \zeta_0 < 1, \\ \left( 1 + \frac{\sqrt{5}T}{\zeta_0^2 \sqrt{(\zeta_0 T)^2 - 4}} \int_{\mathbb{R}} \frac{|\zeta| |1 + (\zeta^2 - 1)\mathbf{1}_{(1,\infty)}(|\zeta|)|^{\frac{1}{2}}}{\min(1, \sqrt{|\zeta|T})} d\tilde{\eta}_\zeta \right)^{-1} & \text{if } \zeta_0 \geq 1. \end{cases}$$

Then, for every  $\theta \in [0, \theta_0)$ , every target  $\begin{bmatrix} y^{f,0} \\ y^{f,1} \end{bmatrix} \in L^2(0,1) \times H^{-1}(0,1)$  and every initial conditions

$$\begin{bmatrix} y_\zeta^{i,0} \\ y_\zeta^{i,1} \end{bmatrix} \in L^2(0,1) \times H^{-1}(0,1) \text{ satisfying:}$$

$$\int_{\mathbb{R}} \|(y_\zeta^{i,0}, y_\zeta^{i,1})\|_{L^2(0,1) \times H^{-1}(0,1)} d\tilde{\eta}_\zeta < \infty,$$

there exists a control  $u \in L^2(0,T)$  so that:

$$(1 - \theta)y_{\zeta_0}(T) + \theta \int_{\mathbb{R}} y_\zeta(T) d\tilde{\eta}_\zeta = y^{f,0} \quad \text{and} \quad (1 - \theta)\dot{y}_{\zeta_0}(T) + \theta \int_{\mathbb{R}} \dot{y}_\zeta(T) d\tilde{\eta}_\zeta = y^{f,1},$$

where, for every  $\zeta \in \mathbb{R}$ ,  $y_\zeta$  is solution of (3.12).

Moreover, there exists a constant  $C^\theta(T) > 0$  independent of the initial and final conditions such that:

$$\|u\|_{L^2(0,T)}^2 \leq C^\theta(T) \left( \left\| (1 - \theta)y_{\zeta_0}^{i,0} + \theta \int_{\mathbb{R}} y_\zeta^{i,0} d\tilde{\eta}_\zeta \right\|_{L^2(0,1)}^2 + \left\| (1 - \theta)y_{\zeta_0}^{i,1} + \theta \int_{\mathbb{R}} y_\zeta^{i,1} d\tilde{\eta}_\zeta \right\|_{H^{-1}(0,1)}^2 + \|y^{f,0}\|_{L^2(0,1)}^2 + \|y^{f,1}\|_{H^{-1}(0,1)}^2 \right).$$

In the same way, from Corollary 3.1 we can derive an averaged approximate controllability results.

**Proposition 3.3** (Approximate averaged controllability with a discrete measure). *Let  $K \in \mathbb{N}^*$ , and for every  $k \in \{0, \dots, K\}$ , define the weight  $\theta_k \in (0, 1)$  (so that  $\sum_{k=0}^K \theta_k = 1$ ) and the parameter  $\zeta_k \in \mathbb{R}_*$  and assume:*

$$\zeta_0^{-1} \zeta_k \notin \mathbb{Q} \quad (k \in \{1, \dots, K\} \setminus \{k_0\}).$$

Then for every  $T > 2 \sum_{k=0}^K \frac{1}{|\zeta_k|}$ , every  $\varepsilon > 0$ , every target  $\begin{bmatrix} y^{f,0} \\ y^{f,1} \end{bmatrix} \in L^2(0,1) \times H^{-1}(0,1)$  and every

initial conditions  $\begin{bmatrix} y_{\zeta_k}^{i,0} \\ y_{\zeta_k}^{i,1} \end{bmatrix} \in L^2(0,1) \times H^{-1}(0,1)$ , there exists a control  $u \in L^2(0,T)$  for which we have:

$$\left\| y^{f,0} - \sum_{k=0}^K \theta_k y_{\zeta_k}(T) \right\|_{L^2(0,1)}^2 \leq \varepsilon \quad \text{and} \quad \left\| y^{f,1} - \sum_{k=0}^K \theta_k \dot{y}_{\zeta_k}(T) \right\|_{H^{-1}(0,1)}^2 \leq \varepsilon,$$

where for every  $\zeta \in \mathbb{R}^*$ ,  $y_\zeta$  solves (3.12).

Moreover, there exists a constant  $C_\varepsilon(T) > 0$  independent of the initial and final conditions such that:

$$\|u\|_{L^2(0,T)}^2 \leq C_\varepsilon(T) \left( \left\| \sum_{k=0}^K \theta_k y_{\zeta_k}^{i,0} \right\|_{L^2(0,1)}^2 + \left\| \sum_{k=0}^K \theta_k y_{\zeta_k}^{i,1} \right\|_{H^{-1}(0,1)}^2 + \|y^{f,0}\|_{L^2(0,1)}^2 + \|y^{f,1}\|_{H^{-1}(0,1)}^2 \right).$$

Now using Diophantine approximations, see Corollary 3.3, we obtain:

**Proposition 3.4** (Exact average controllability with a discrete measure). *Let  $\varepsilon > 0$  and let  $K \in \mathbb{N}^*$ ,  $\theta_k$  and  $\zeta_k$  be defined by Proposition 3.3 and assume,  $\zeta_0, \dots, \zeta_K$  are  $\mathbb{Q}$ -linearly independent and*

$$\zeta_0^{-1}\zeta_1, \dots, \zeta_0^{-1}\zeta_K \text{ are algebraic.} \quad (3.15)$$

Then, if  $(y_{\zeta_0}^{i,0}, y_{\zeta_0}^{i,1}), \dots, (y_{\zeta_K}^{i,0}, y_{\zeta_K}^{i,1}), (y^{f,0}, y^{f,1}) \in X_{1+\varepsilon} \times X_\varepsilon$  with:

$$X_\alpha = \left\{ \varphi : x \in (0, 1) \mapsto \sum_{n=1}^{\infty} a_n \sin(n\pi x), \sum_{n \in \mathbb{N}^*} n^{2\alpha} |a_n|^2 < \infty \right\} \quad (\alpha \in \mathbb{R}), \quad (3.16)$$

for every  $T > 2 \sum_{k=0}^K \frac{1}{|\zeta_k|}$  there exists a control  $u \in L^2(0, T)$  such that for every  $k \in \{1, \dots, K\}$ , the solution  $y_{\zeta_k}$  of (3.12) (with parameter  $\zeta = \zeta_k$ ) satisfy:

$$\sum_{k=0}^K \theta_k y_{\zeta_k}(T) = y^{f,0} \quad \text{and} \quad \sum_{k=0}^K \theta_k \dot{y}_k(T) = y^{f,1}.$$

A similar result could have been obtained from Corollary 3.2,

Let us also notice that applying directly [10, Corollary 5.43], we obtain the following exact simultaneous controllability result:

**Proposition 3.5.** *Let  $\varepsilon > 0$ ,  $K \in \mathbb{N}^*$  and  $\zeta_k \in \mathbb{R}^*$  for every  $k \in \{0, \dots, K\}$  and assume,  $\zeta_0, \dots, \zeta_K$  are  $\mathbb{Q}$ -linearly independent and*

$$\zeta_k^{-1}\zeta_l \text{ is algebraic for every } k, l \in \{0, \dots, K\}. \quad (3.17)$$

Let  $(y_{\zeta_k}^{i,0}, y_{\zeta_k}^{i,1})$  and choose final conditions  $(y^{f,0}, y^{f,1})$  satisfying the assumption given in Proposition 3.4.

Then, for every  $T \geq 2 \sum_{k=0}^K \frac{1}{|\zeta_k|}$  there exists a control  $u \in L^2(0, T)$  such that:

$$y_{\zeta_k}(T) = y^{f,0} \quad \text{and} \quad \dot{y}_{\zeta_k}(T) = y^{f,1} \quad (k \in \{0, \dots, K\}).$$

This result ensures that all the parameter dependent trajectories, and, of course, consequently, their average, can be steered to a prescribed target with an input independent of the parameter.

**Remark 3.4.** *As expected, the assumption (3.15) needed to obtain averaged controllability is weaker than the one for simultaneous controllability, (3.17).*

**Example 3.1.** Let us now summarise the main results on the averaged control of two string equations, one parametrised by  $\zeta_0 = 1$  and the other one by  $\zeta_1 = \sqrt{2}$ , with averaging measure:

$$\eta^\theta = (1 - \theta)\delta_{\zeta_0} + \theta\delta_{\zeta_1}. \quad (3.18)$$

The following holds:

- From Proposition 3.2, if  $T > 2$ , the system (3.12) is controllable in average with averaging measure  $\eta^\theta$  for  $\theta \in \left[0, \left(1 + \frac{2\sqrt{5}T}{\sqrt{T^2-4}}\right)^{-1}\right]$ ;
- From Proposition 3.2, if  $T > \sqrt{2}$ , the system (3.12) is controllable in average with averaging measure  $\eta^\theta$  for  $\theta \in \left(1 - \left(1 + \frac{\sqrt{5}T}{2\sqrt{2T^2-4}}\right)^{-1}, 1\right]$ ;
- From Proposition 3.4, if  $T > 2\left(1 + \frac{\sqrt{2}}{2}\right)$ , the system (3.12) is controllable in average with averaging measure  $\eta^\theta$  in some weighted space for  $\theta \in [0, 1]$ .

This leads to the time-dependent set of parameters  $\theta$  for which we have averaged controllability, see Figure 1 below.

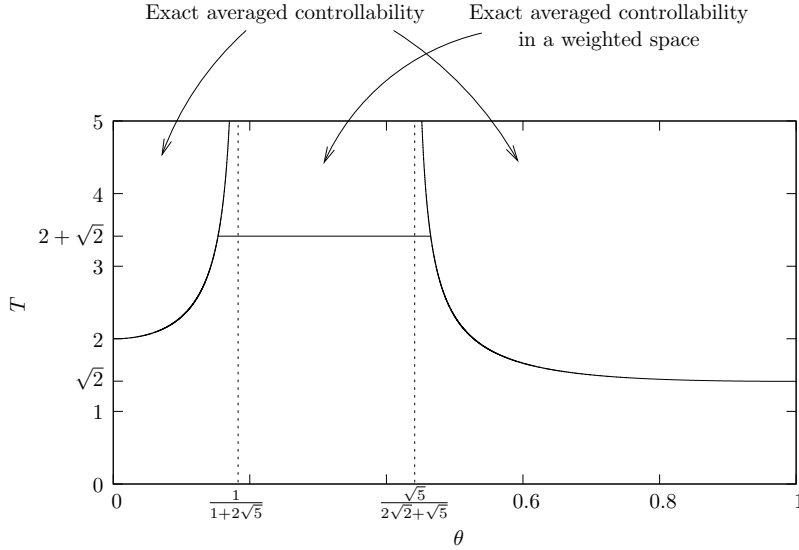


Figure 1: Time dependent set of parameters  $\theta$  for which averaged controllability holds, for two strings driven by the system (3.12) with parameters  $\zeta_0 = 1$  and  $\zeta_1 = \sqrt{2}$  and averaging measure  $\eta^\theta$  given by (3.18).

## 4 Concluding remarks

The aim of this article was to give a systematic result, based on perturbation arguments, on the averaged controllability and observability of parameter-dependent families of equations, mainly in the context of time-reversible groups of isometries.

There are several interesting open problems that arise in this context. This is so even for the one dimensional case, where Fourier series representations can be used. Let us point out some of them:

- In § 2.3, we gave an averaged observability inequality. However, this result only holds when the measure is the sum of a Dirac mass and a small enough perturbation measure. This condition can be used for absolutely continuous (with respect to the Lebesgue measure) averaging measures. But it would be natural to consider more general cases as well.

In the context of Ingham inequalities and in the particular case where  $L_\zeta = \text{Id}$  and  $\varsigma(\zeta) = \zeta$  the main problem can be recast as follows: *Given  $\eta$  a probability measure and  $(\lambda_n)_{n \in \mathbb{Z}}$ , is  $\{t \mapsto \hat{\eta}(-\lambda_n t)\}_{n \in \mathbb{Z}}$  a Riesz sequence of  $L^2(0, T)$ ?*

One of the simplest case to be considered is when  $\eta^\varepsilon$  is given by  $d\eta_\zeta^\varepsilon = \frac{1}{2\varepsilon} \mathbf{1}_{[1-\varepsilon, 1+\varepsilon]}(\zeta) d\zeta$  for  $\varepsilon > 0$ . Then the sequence  $(\eta^\varepsilon)_{\varepsilon > 0}$  converges in the sense of measures to the Dirac mass  $\delta_1$  when  $\varepsilon$  goes to 0. Assuming that the sequence  $(\lambda_n)_n$  satisfies the Ingham gap condition (3.1) for some  $\gamma > 0$ , we know from [14] there exists a constant  $c(T) > 0$  such that:

$$\int_0^T \left| \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n t} \right|^2 dt \geq c(T) \sum_{n \in \mathbb{Z}} |a_n|^2 \quad \left( T > \frac{1}{\gamma}, (a_n)_{n \in \mathbb{Z}} \in \ell^2 \right).$$

It is then natural to wonder if there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon < \varepsilon_0$ , we have:

$$\int_0^T \left| \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n \zeta t} d\eta_\zeta^\varepsilon \right|^2 dt \geq c_\varepsilon(T) \sum_{n \in \mathbb{Z}} |a_n|^2 \quad \left( T > \frac{1}{\gamma}, (a_n)_{n \in \mathbb{Z}} \in \ell^2 \right) \quad (4.1)$$

and if it is so, whether  $c_\varepsilon(T)$  converges to  $c(T)$  as  $\varepsilon$  tends to 0?

A way to prove this result is to look at the quantity:

$$\left| \int_0^T \left( \left| \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n \zeta t} d\eta_\zeta^\varepsilon \right|^2 - \left| \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n t} \right|^2 \right) dt \right|$$

One can easily get the upper bound

$$\varepsilon T^2 C \sqrt{\sum_{n \in \mathbb{Z}} |\lambda_n| |a_n|^2} \sqrt{\sum_{n \in \mathbb{Z}} |a_n|^2}.$$

It goes to 0 as  $\varepsilon$  tends to 0, but it does not ensure inequality (4.1) to hold.

One can also proceed with a direct computation and, in this case, from § 3.2 one can derive a weighted averaged Ingham inequality when the eigenvalues  $\lambda_n$  satisfy (3.3). Indeed, let us consider a measure  $\eta$  given by  $d\eta_\zeta = \frac{1}{\zeta_1 - \zeta_0} \mathbf{1}_{[\zeta_0, \zeta_1]}(\zeta) d\zeta$  for  $\zeta_0 < \zeta_1$  and  $\zeta_0, \zeta_1 \neq 0$ . Writing  $\lambda_n = \gamma \mu_n$  with  $\mu_n \in \mathbb{Z}^*$ , we obtain:

$$\int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n \zeta t} d\eta_\zeta = \frac{1}{2i\pi\gamma(\zeta_1 - \zeta_0)t} \sum_{n \in \mathbb{Z}} \frac{a_n}{\mu_n} \left( e^{2i\pi\gamma\zeta_1\mu_n t} - e^{2i\pi\gamma\zeta_0\mu_n t} \right).$$

Thus, we have:

$$\int_0^T \left| \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} a_n e^{2i\pi\lambda_n \zeta t} d\eta_\zeta \right|^2 dt \geq \frac{1}{(2\pi\gamma)^2 (\zeta_1 - \zeta_0)^2 T^2} \int_0^T \left| \sum_{n \in \mathbb{Z}} \frac{a_n}{\mu_n} \left( e^{2i\pi\gamma\zeta_1\mu_n t} - e^{2i\pi\gamma\zeta_0\mu_n t} \right) \right|^2 dt.$$

Now, assuming that  $T > \frac{1}{\gamma} \left( \frac{1}{\zeta_0} + \frac{1}{\zeta_1} \right)$  and  $\zeta_0^{-1}\zeta_1 \notin \mathbb{Q}$ , we obtain the unique continuation from Corollary 3.1 and, under the assumption of corollary 3.2 or 3.3 on  $\zeta_0$  and  $\zeta_1$ , we end up with a weighted Ingham inequality.

- When dealing with the control system (2.18), in [23], the condition (2.20) was required to ensure averaged controllability. However, according to Proposition 2.3 (see Remark 2.3), this condition is not needed under a suitable smallness assumption on the averaging measure.

The optimality of assumption (2.20) without smallness assumptions needs further clarification.

- The results derived in § 3.2 need  $\lambda_n \in \gamma\mathbb{Z}$  to be a sequence of integers. But the unique continuation property, Corollary 3.1, could have been obtained directly from [13, Corollary 2.3.5]. This result still holds in the case general case where  $(\lambda_n)_n$  satisfies (3.1) and assuming that the values  $\varsigma(\zeta_k)\lambda_n \neq \varsigma(\zeta_l)\lambda_m$  for  $k \neq l$  or  $n \neq m$ .

In addition, results similar to corollaries 3.2 and 3.3 could have been obtained from [19]. More precisely, assume that the sequence  $(\lambda_n)_n$  satisfies the Ingham gap condition (3.1), and that  $\varsigma(\zeta_k)\lambda_n \neq \varsigma(\zeta_l)\lambda_m$  for  $k \neq l$  or  $n \neq m$ . Let us now consider the increasing sequence  $(\Lambda_n)_n$  such that  $\{\Lambda_n, n \in \mathbb{Z}\} = \{\varsigma(\zeta_k)\lambda_m, m \in \mathbb{Z}, k \in \{0, \dots, K\}\}$ . Then for every  $n \in \mathbb{Z}$ , we have  $\Lambda_{n+K+1} - \Lambda_n \geq \gamma \min\{|\varsigma(\zeta_k)|, k \in \{0, \dots, K\}\}$ . Thus [19, Theorem 4] applies and leads to a weighted averaged Ingham inequality valid for every

$$T > \frac{K+1}{\gamma \min\{|\varsigma(\zeta_0)|, \dots, |\varsigma(\zeta_K)|\}}.$$

Notice that this minimal time is greater than  $\sum_{k=0}^K \frac{1}{\gamma|\varsigma(\zeta_k)|}$ , the one obtained in corollaries 3.2 and 3.3, but under stronger assumptions on the sequence  $(\lambda_n)_n$ .

In addition, the results given in [13] and [19] ensure simultaneous observability. Thus it would be interesting to see how the assumption given in these two works could be weakened in order to only ensure averaged observability.

In the proof of Theorem 3.1, we strongly need that the sequence  $(\lambda_n)_n$  satisfies (3.3) and, even for  $\lambda_n = n + \varepsilon(n)$  with  $\varepsilon(n) = o(1)$ , the technique of proof fails. It would be worth exploring whether some improvements could be obtained with a perturbation argument, combined with the ideas of [7] and in particular with Ulrich's result [34].

The analysis of all these examples could contribute to achieve sharp results for the averaged controllability of finitely many string equations (1.1).

- Let us conclude this paper with a general remark linking averaged controllability and simultaneous controllability. The aim is to find controls independent of the parameter performing well for all values of the parameters. With this goal, a first and natural choice was to control the average of the parameter dependent outputs. Of course, the best we could expect is a control, independent of the values of the unknown parameters, steering all parameter dependent trajectories to a common fixed target, i.e. looking to simultaneous controllability. But this is unfeasible in general.

There exists a natural link between the control of the average and the stronger notion of simultaneous control. This link can be made through penalisation and optimal control.

More precisely, for every  $\kappa \geq 0$ , let us consider the following optimal control problem:

$$\begin{array}{l} \min \quad J_\kappa(u) := \frac{1}{2} \|u\|_{L^2([0,T],U)}^2 + \kappa \int_{\mathbb{R}} \|y_\zeta(T) - y^f\|_X^2 \, d\eta_\zeta \\ \left| \begin{array}{l} \int_{\mathbb{R}} y_\zeta(T) \, d\eta_\zeta = y^f, \\ \dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u, \quad y_\zeta(0) = y_\zeta^i. \end{array} \right. \end{array}$$

Notice that for  $\kappa = 0$  this leads the averaged control of minimal norm as we considered here. But, as  $\kappa$  increases, the control, other than ensuring the averaged controllability property, also forces the reduction of the variance of the output.

Of course, under the property of averaged controllability, the minimiser  $u_\kappa$  exists for every  $\kappa > 0$ .

It can also be proved that, if, in addition,  $J_\kappa(u_\kappa)$  is uniformly bounded, then up to a subsequence,  $(u_\kappa)_\kappa$  is weakly convergent to a control simultaneous control  $u_\infty$  solution of:

$$\left| \begin{array}{l} y_\zeta(T) = y^f \quad (\zeta \in \mathbb{R} \quad \eta\text{-a.e.}), \\ \dot{y}_\zeta = A_\zeta y_\zeta + B_\zeta u, \quad y_\zeta(0) = y_\zeta^i. \end{array} \right.$$

This issue is analysed in [28], where this idea is discussed in detail in the finite-dimensional control context.

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