

Numerical approximation of null controls for the heat equation through transmutation

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Abstract

The numerical approximation of exact or trajectory controls for the wave equation is known to be, since the pionnering work of Glowinski-Lions in the nineties, a delicate issue, because of the anomalous behavior of the high frequency spurious numerical waves. Various efficient remedies have been developed and analyzed in the last decade to filter out these high frequency components: Fourier filtering, Tychonoff regularization, mixed finite element methods, multi-grid strategies, etc. Recently convergence rates results have also been obtained.

This work is devoted to analyze this issue for the heat equation, which is the opposite paradigm because of its strong dissipativity and smoothing properties. The existing analytical results guarantee that, at least in some simple situations as in the finite-difference scheme in $1 - d$, the null or trajectory controls for numerical approximation schemes converge. This is due to the intrinsic high frequency damping of the heat equation that is inherited by its numerical approximation schemes. But when developing numerical simulations the topic appears to be much more subtle and difficult. In fact, efficiently computing the null control for a numerical approximation scheme of the heat equation is a difficult problem in itself. The difficulty is strongly related to the ill-posedness of the *backward heat problem*.

The controls of minimal L^2 -norm are characterized as minima of quadratic functionals on the solutions of the adjoint heat equation, or its numerical versions. These functionals are shown to be coercive in very large spaces of solutions, sufficient to guarantee the L^2 character of controls, but very far from being identifiable as energy spaces for the adjoint system. This very weak coercivity of the functionals under consideration makes the approximation problem to be exponentially ill-posed and the functional framework to be far from being well adapted to standard techniques in numerical analysis. In practice, the controls of minimal L^2 -norm exhibit a singular highly oscillatory behavior near the final controllability time, which can not be captured numerically. Standard techniques such as Tychonoff regularization or quasi-reversibility methods allow to slightly smooth the singularities but reduce significantly the quality of the approximation.

In this article we develop some more involved and less standard approaches which turn out to be more efficient. We first discuss the advantages of using controls with

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compact support with respect to the time variable or the effect of adding numerical dissipative singular terms.

But the main contribution of this paper is to develop the numerical version of the so-called transmutation method that allows writing the control of a heat process in terms of the corresponding control of the associated wave process, by means of a "time convolution" with a one-dimensional *controlled fundamental heat solution*. This method, although it can be proved to converge, is also subtle in its computational implementation. Indeed, in one hand, it requires using convergent numerical schemes for the control of the wave equation, a problem that, as mentioned above, is delicate in itself. But it also needs computing an accurate approximation of a controlled fundamental heat solution, an issue that requires its own analysis and significant numerical and computational new developments.

These methods are thoroughly illustrated and discussed along the paper, accompanied by some numerical experiments in one space dimension that show the subtlety of the issue. These experiments allow comparing the efficiency of the various methods. This is done in the case where the control is distributed in some subdomain of the domain where the heat process evolves but similar results and numerical experiments could be derived for other cases such as the one in which the control acts on the boundary.

The techniques we employ here can also be adapted to the multi-dimensional case.

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1 Introduction - Problem statement

Let Ω be a bounded open subset of \mathbb{R}^N , $N \geq 1$, and ω be a non empty open set strictly included in Ω .

The heat equation evolving in Ω with controllers supported in ω is well known to be null or trajectory controllable. To be more precise, for any value of the control time $T > 0$, initial state $y_0 \in L^2(\Omega)$, and a final state y_T which is the value at time $t = T$ of a solution of the heat equation, there exists a control function $v \in L^2((0, T) \times \omega)$ such that the corresponding solution $y \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ of

$$\begin{cases} y' - c\Delta y = v\mathcal{X}_\omega & \text{in } Q_T = (0, T) \times \Omega, \\ y = 0 & \text{on } \Sigma_T = (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega \end{cases} \quad (1)$$

satisfies $y(T, \cdot) = y_T$ in Ω .

This problem of trajectory control is in fact equivalent to the null control one in which the final state y_T is assumed to be the trivial one: $y_T \equiv 0$.

Here and in what follows \mathcal{X}_ω denotes the characteristic function of the subset ω , so that the control function v is active only in ω .

The constant $c > 0$ is the diffusivity coefficient which, of course, by a change of variables in time, can be normalized to $c = 1$ by changing the length of the control time $[0, T]$ or the space domain under consideration. But, in order to better visualize the unstabilities of the numerical control process under consideration, we often assume c to be small to diminish the effect of diffusion.

There is an extensive literature in this subject. The reader is referred to [11, 24] and to the more recent survey article [41].

Similar results hold for the wave equation but under suitable geometric restrictions on the support of the controls (the so-called Geometric Control Condition (GCC)) and the control time $T > 0$ (see [41]). In the case of the wave equation, due to its time-reversibility, both initial and final states can be taken to be equally smooth, of finite energy. Thus, the problems of trajectory control, exact control (when the initial datum and target are arbitrary in the finite energy space) and null control (when the final target is the zero state) are equivalent. But, of course, this is not the case for the heat equation. Because of the smoothing effect, the smoothness of the final target is a necessary condition for controllability and therefore exact controllability does not hold. But, as mentioned above, because of the linearity of the system, the null and trajectory controllability properties are completely equivalent. We refer to [13] for a careful analysis of the cost of control in the various situations (by cost of control we mean the norm of the control needed in terms of the initial and final data to be controlled). Of course, in the case of the heat equation, the geometric conditions on the control set and the lower bound of the time of control (which are due to the finite velocity of propagation for the solutions of the wave equation) do not apply and the null control property holds for all time $T > 0$ and any open non-empty subset ω .

In the sequel, we denote by $\mathcal{C}(T, y_0, y_T)$ the non empty class of controls v which drive the solution of (1) from y_0 at time 0 to y_T at time $t = T$. This set is constituted by infinitely many controls since they can be taken to be, for example, supported in any time subinterval of the form $[\tau, T]$ for any $0 < \tau < T$.

This work is devoted to design and analyze performant numerical schemes allowing to efficiently compute some distinguished elements of the class of admissible controls $\mathcal{C}(T, y_0, y_T)$.

Since the seminal contribution [15], it is well-known and admitted that this is a delicate issue for the wave equation: this is due to the high frequency spurious numerical modes that appear at the discrete level and can not be exactly controlled uniformly with respect to the discretization parameters, typically h and Δt , the space and time mesh sizes respectively. Thus, standard convergent finite element based schemes may produce exponentially divergent sequences $\{v_{h,\Delta t}\}_{h,\Delta t}$ of discrete controls: for such schemes, *exact controllability and numerical approximation processes do not commute*. This specific question has been intensively studied in the last decade and solved efficiently, at least for the constant coefficient case, building uniformly controllable schemes leading to bounded and convergent discrete sequences of controls with respect to h and Δt . These schemes are built by filtering out the high frequency spurious components using, among others, Tychonoff regularization, mixed finite element methods or two-grid filtering. We refer to the review article [39].

The same issue for the heat equation appears to be more subtle, numerically ill-posed and sensitive, which is at a first glance a bit surprising if we take into consideration the regularizing effect of the heat operator and the fact that the null controllability property holds for any T strictly positive and any non-empty open subdomain ω .

To better explain the difficulties we encounter when developing numerical simulations, let us consider, within $\mathcal{C}(T, y_0, y_T)$, the control of minimal L^2 -norm - the so-called HUM control - solution of :

$$\min_{v \in \mathcal{C}(T, y_0, y_T)} J(v) = \int_0^T \int_{\omega} v^2(t, x) dx dt. \quad (2)$$

Through convex duality and following [6], the minimization of J may be replaced by the *a priori* simpler minimization problem of the corresponding conjugate function J^* defined by

$$J^*(\phi_T) = \frac{1}{2} \int_0^T \int_{\omega} \phi^2(t, x) dx dt - \int_{\Omega} y_T(x) \phi_T(x) dx + \int_{\Omega} y_0(x) \phi(0, x) dx$$

with respect to ϕ_T (over a suitable Hilbert space \mathcal{H} that will be made precise below), value at time T of the adjoint state ϕ , solution of

$$\begin{cases} \phi' + c\Delta\phi = 0 & \text{in } Q_T, \\ \phi = 0 & \text{on } \Sigma_T, \\ \phi(T, \cdot) = \phi_T & \text{in } \Omega. \end{cases} \quad (3)$$

The minimal L^2 -norm control is given by $v(t, x) = \phi(t, x)\mathcal{X}_{\omega}$, where ϕ is the solution of (3) corresponding to the minimizer ϕ_T of J^* within the class \mathcal{H} of data ϕ_T such that the corresponding solution of (3) is such that $\phi \in L^2((0, T) \times \omega)$. More precisely, \mathcal{H} is defined as the completion of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|\phi_T\|_{\mathcal{H}} = \left(\int_0^T \int_{\omega} \phi^2(t, x) dx dt \right)^{1/2}.$$

The coercivity of the convex function J^* in this space \mathcal{H} (in the simplest case in which the target $y_T \equiv 0$ is the trivial equilibrium state) is related to the so-called *observability inequality*

$$C(T, \omega) \|\phi(0, \cdot)\|_{L^2(\Omega)}^2 \leq \int_0^T \int_{\omega} \phi(t, x)^2 dx dt, \quad \forall \phi_T \in L^2(\Omega), \quad (4)$$

proved in [24] by Fourier decomposition and Carleman inequalities for packets of eigenfunctions and in [14] by parabolic Carleman inequalities.

Despite of this, the ill-posedness of problem (2), that we shall describe later in detail, is due to the fact that the space \mathcal{H} is very large. In fact, due to the regularizing effect of the heat equation, any initial (at time $t = T$) datum ϕ_T of the adjoint heat equation with finite order singularities away from the control set ω , belongs to \mathcal{H} . Therefore, the minimizer ϕ_T of J^* , for a given pair (y_0, y_T) , may be very singular and therefore difficult to capture/represent numerically with accuracy and robustness: in other terms, the space \mathcal{H} has poor numerical approximation properties.

Although, from a theoretical viewpoint, the existence of the control is due to the observability inequality (4) and therefore is independent of how large the space \mathcal{H} is, its effective numerical computation is intimately related to solving the adjoint system (3). This explains why, in practice, the problem is exponentially ill-posed and why it is difficult to determine numerically with efficiency the control of minimal L^2 -norm. We emphasize that this phenomenon is intrinsic to the heat equation and has no link with the possible lack of uniform observability with respect to the discretization parameters as it occurs to the wave equation. Indeed, this exponential ill-posedness occurs even if the numerical scheme used ensures the uniform boundeness by below of the discrete observability constant in the discrete counterpart of (4). This uniform observability inequality is well known to hold, for instance, for the finite-difference discretizations of the $1 - d$ heat equation, [29], but, as we shall see, despite of this, the efficient numerical computation of controls is a very delicate issue.

In this paper we analyze some possible numerical remedies to this exponential ill-posedness of the problem under consideration. We mainly focus in the $1 - d$ case although most of our methods and results can be extended and adapted to the multidimensional case.

This paper is devoted to the problem of null or trajectory controllability in which the target y_T is assumed to be the value at time $t = T$ of another solution of the heat equation with a right hand side term localized in ω . As we mentioned above, the most typical example is the target $y_T \equiv 0$ which corresponds to the trivial trajectory of the heat equation with null initial data and right hand side. The general case in which y_T is non trivial can also be reduced to the particular one $y_T \equiv 0$, by simply subtracting the corresponding solution leading to y_T , to the solution to be controlled. Note that, accordingly, the final targets y_T to be reached exactly are intrinsically smooth because they are final values of solutions of the heat equation at time $t = T$. Thus, in some sense, the final targets y_T under consideration inherit the smoothing properties of the heat kernel during the time interval $[0, T]$.

Of course, numerically, the exact condition $y(T) = y_T$ is impossible to be achieved. This is way, often, in numerical experiments, this condition is relaxed to an approximate controllability one of the form $\|y(T) - y_T\|_{L^2(\Omega)} \leq \epsilon$. When ϵ is not too small the instabilities above, whose understanding and master are the main goal of this paper, are not detected since they are mainly due to the control of the high frequency components of the system. We emphasize however, that they are unavoidable when addressing truly the null or trajectory control problems since the problem is intrinsically ill-posed. But, to detect this ill-posedness phenomena, numerical simulations have to be very carefully developed paying attention, in particular, to taking sufficiently small values of the ϵ parameter. Since our numerical experiments are performed using double arithmetic precision, for which the roundoff unit error is of order $\epsilon_1 = 10^{-16}$, we assume that the numerical control is a relevant approximation of the continuous one if the corresponding numerical solution $y_{h,\Delta t}$ satisfies the inequality

$$\frac{\|y_{h,\Delta t}(T, \cdot) - y_{T,h,\Delta t}(\cdot)\|_{L^2(\Omega)}}{\|y_{T,h,\Delta t}(\cdot)\|_{L^2(\Omega)}} \leq \epsilon_2$$

with ε_2 of order 10^{-8} , which is the value used in [15]. Here and in the sequel $y_{T,h,\Delta t}$ stands for a numerical approximation with mesh sizes h and Δt of the continuous target y_T .

When taking greater values of ε_2 , which is often done in practice, the problem is much more easily handled numerically and computationally, but then it rather corresponds to the problem of approximate controllability and not to the null controllability one we are considering here.

The paper is organized as follows. In Section 2, we highlight the ill-posedness of problem (2) with some simple $1-d$ analytical computations. In particular, we observe that the control of minimal L^2 -norm exhibits an infinite order oscillatory behavior near the controllability time T . This is intimately related to the hugeness of the Hilbert space \mathcal{H} . Then, in Section 3, we analyze several perturbed or regularized versions of problem (2). In Section 3.1 we first introduce regularization methods (commonly used in inverse problem theory) which consist in augmenting J^* with terms like $\epsilon \|\phi_T\|_{H_0^s(\Omega)}^2$, $\epsilon, s > 0$. This functional is intimately related to relaxing the null controllability problem to the approximate controllability one. We observe a very low rate of convergence (as $\epsilon \rightarrow 0$) of the family of minimizers $\{v_\epsilon\}_{\epsilon > 0}$ of J_ϵ towards the minimizer of J . In Section 3.2 we analyze two singular perturbations: the first one consists in adding the hyperbolic term ϵu_{tt} while the second one adds the higher order vanishing viscosity term $-\epsilon u_{txx}$. Then, in Section 3.3, we observe an improvement in the stability properties if the functional (2) is replaced by a weighted version so that the corresponding control vanishes on $\omega \times (T - \delta, T)$ for some $\delta > 0$. Then, in Section 4, we consider the null controls obtained by the transmutation method (see [31]). This method allows expressing the control of parabolic problems in terms of those of the corresponding hyperbolic one through a convolution with a $1-d$ controlled fundamental solution of the heat equation. The latter one is determined explicitly in terms of a power series expansion, following [23]. In Section 5 we summarize some of the main conclusions of this paper, and discuss some open problems and perspectives.

2 Ill-posedness for the L^2 -norm

In this section, we characterize the control of minimal L^2 -norm and exhibit, by the way of simple analytical computations in one space dimension, the ill-posedness of the problem.

2.1 Analytical expression of the control operator

We take $\Omega = (0, 1)$. By convex duality, the problem of control to trajectories is reduced to the minimization of the conjugate function J^* defined by

$$J^*(\phi_T) = \frac{1}{2} \int_0^T \int_\omega \phi^2(t, x) dx dt - \int_\Omega y_T(x) \phi_T(x) dx$$

over $\phi_T \in \mathcal{H}$ solution at time $t = T$ of the adjoint backward system

$$\begin{cases} \phi' + c\phi_{xx} = 0 & \text{in } Q_T, \\ \phi = 0 & \text{on } \Sigma_T, \\ \phi(T, \cdot) = \phi_T & \text{in } \Omega. \end{cases} \quad (5)$$

Here, for simplicity, we have taken the initial datum to be controlled to be $y_0 \equiv 0$. Note also that the final target y_T is assumed to be the value at time $t = T$ of a solution of the heat equation (controlled or not).

The optimality condition associated with the minimization of J^* ensures that the control of minimal L^2 -norm is then given by $v = \phi \mathcal{X}_\omega$ on Q_T , where ϕ is the solution of (5) associated to the minimizer ϕ_T of J^* .

Let us now compute in Fourier series the map that associates to ϕ_T the datum at time $t = T$ of the adjoint system, the value of the controlled solution y at time $t = T$. We denote this operator by $\Lambda : \mathcal{H} \rightarrow L^2(\Omega)$, $\Lambda \phi_T := y(T)$.

Finding the control of minimal norm amounts to invert the operator Λ . In this section we describe the intrinsic ill-posedness of this problem.

Given ϕ_T , expanded in Fourier series as $\phi_T(x) = \sum_{k \geq 1} a_k \sin(k\pi x)$ with $\{a_k\}_k \in l^2(\mathbb{N})$, the adjoint solution of (5) is of the form (denoting by $\lambda_k = k^2\pi^2$ the eigenvalues of the Dirichlet laplacian in $1 - d$ in the interval $(0, 1)$, for all $k > 0$)

$$\phi(x, t) = \sum_{k \geq 1} a_k e^{c\lambda_k(t-T)} \sin(k\pi x), \quad (t, x) \in Q_T.$$

The controlled state y is then solution of

$$\begin{cases} y' - cy_{xx} = \sum_{k \geq 1} a_k e^{c\lambda_k(t-T)} \sin(k\pi x) \mathcal{X}_\omega(x) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

The solution y can also be expanded in Fourier series as $y(x, t) = \sum_{p \geq 1} b_p(t) \sin(p\pi x)$, where b_p solves the equation

$$b'_p(t) + c\lambda_p b_p(t) = \sum_{k \geq 1} a_k e^{c\lambda_k(t-T)} c_{k,p}(\omega), \quad b_p(0) = 0, \quad (6)$$

and is explicitly given by

$$b_p(t) = \sum_{k \geq 1} a_k c_{k,p}(\omega) g_{k,p}(t), \quad \forall p \geq 1$$

with ¹

$$g_{k,p}(t) = e^{-c(\lambda_k T + \lambda_p t)} \frac{(e^{c(\lambda_k + \lambda_p)t} - 1)}{c(\lambda_p + \lambda_k)},$$

and

$$c_{k,p}(\omega) = 2 \int_{\omega} \sin(k\pi x) \sin(p\pi x) dx.$$

The value of the solution y at time $t = T$ is then given by:

$$y(T, x) = \sum_{k,p \geq 1} a_k c_{k,p}(\omega) g_{k,p}(T) \sin(p\pi x).$$

¹For $\omega = (a, b) \subset \Omega = (0, 1)$ simple computations lead to

$$c_{p,p}(\omega) = (b - a) + \frac{\cos(p\pi a) \sin(p\pi a) - \cos(p\pi b) \sin(p\pi b)}{p\pi}, \quad p > 0$$

and to

$$c_{k,p}(\omega) = \frac{1}{\pi(k^2 - p^2)} \left\{ (k-p) \left(\sin((k+p)\pi a) - \sin((k+p)\pi b) \right) + (k+p) \left(\sin((k-p)\pi b) - \sin((k-p)\pi a) \right) \right\}, \quad p, k > 0, p \neq k.$$

Then, the operator Λ mapping the Fourier coefficients $\{a_k\}_{(k \geq 1)}$ of ϕ_T into $\{b_p\}_{(p \geq 1)}$, those of y_T , is given by

$$b_p = \sum_{k \geq 1} c_{p,k}(\omega) g_{p,k}(T) a_k, \quad p \geq 1 \quad (7)$$

with

$$g_{p,k}(T) = e^{-c(\lambda_k T + \lambda_p T)} \frac{(e^{c(\lambda_k + \lambda_p)T} - 1)}{c(\lambda_p + \lambda_k)} = \frac{1 - e^{-c(\lambda_k + \lambda_p)T}}{c(\lambda_k + \lambda_p)}. \quad (8)$$

This yields the Fourier representation of the map Λ in terms of the weights $c_{p,k}(\omega)$ and $g_{p,k}(T)$. More precisely, Λ can be identified with the infinite matrix $(\Lambda_{k,p})_{k,p \geq 1}$ with

$$\Lambda_{k,p} = c_{k,p}(\omega) g_{k,p}(T).$$

Let us now analyze the projection of the map Λ over the first N Fourier modes: $\Lambda_N = P_N \Lambda P_N$, P_N being the projection operator over the first N Fourier modes. More precisely, setting $\phi_T^N = P_N(\phi_T) = \sum_{k=1}^N a_k \sin(k\pi x)$, $\Lambda_N : L^2(\Omega) \rightarrow L^2(\Omega)$ is defined by $\Lambda_N(\phi_T) := P_N(y_T) = P_N \Lambda P_N \phi_T$. Then Λ_N can be identified with the symmetric matrix $\mathbf{\Lambda}_N = (\Lambda_{k,p})_{1 \leq k,p \leq N}$ of dimension $N \times N$ with $\Lambda_{p,k}$ as above.

The ill-posedness of the control problem, which is equivalent to the inversion of the operator Λ , can be illustrated by the behavior of the conditioning number of $\mathbf{\Lambda}_N$ with respect to N . The conditioning number is defined by $\text{cond}(\mathbf{\Lambda}_N) = \|\mathbf{\Lambda}_N\|_2 \|\mathbf{\Lambda}_N^{-1}\|_2$ where $\|\cdot\|_2$ denotes the Euclidean norm.

For $T = 1$ and different choices of ω , some values of $\text{cond}(\mathbf{\Lambda}_N)$ are reported on Table 1. We obtain, for any $\omega \neq \Omega$, an exponential behavior with respect to N : for instance, $\omega = (0.2, 0.8)$ leads to $\text{cond}(\mathbf{\Lambda}_N) \approx e^{1.95} e^{0.49N}$. Then, the problem is *exponentially ill-posed* (we refer to [20]). We also observe that the variation with respect to ω is important : for $\omega = (0.5, 0.8)$, we have $\text{cond}(\mathbf{\Lambda}_N) \approx e^{1.11} e^{1.15N}$. On the other hand, the variation with respect to T remains low (see Table 2) in agreement with the relation (8).

	$N = 10$	$N = 20$	$N = 40$	$N = 80$
$\omega = (0.2, 0.8)$	9.05×10^2	1.65×10^5	1.66×10^9	6.96×10^{16}
$\omega = (0.5, 0.8)$	3.57×10^5	3.81×10^{10}	7.31×10^{18}	$\geq 10^{20}$
$\omega = (0.7, 0.8)$	1.82×10^7	2.40×10^{14}	$\geq 10^{20}$	$\geq 10^{20}$
$\omega = (0, 1)$	8.61×10^1	3.44×10^2	1.33×10^3	5.51×10^3

Table 1: Conditioning number of $\mathbf{\Lambda}_N$ vs. N for various $\omega \subset \Omega$ and $\omega = \Omega = (0, 1)$: $T = 1$.

	$N = 10$	$N = 20$	$N = 40$	$N = 80$
$T = 1/10$	3.42×10^2	5.23×10^4	4.47×10^8	1.73×10^{16}
$T = 1$	9.05×10^2	1.65×10^5	1.66×10^9	6.96×10^{16}
$T = 10$	1.04×10^3	1.90×10^5	1.91×10^9	7.97×10^{16}

Table 2: Conditioning number of $\mathbf{\Lambda}_N$ vs. N and T for $\omega = (0.2, 0.8)$.

Remark 1 *The case $\omega = \Omega$ is very particular: we simply have $c_{p,k}(\omega) = \delta_{pk}$ so that the matrix $\mathbf{\Lambda}_N$ is diagonal and*

$$(\Lambda_N)_{pk} = \frac{1 - e^{-2c\lambda_p T}}{2c\lambda_p} \delta_{pk}, \quad \text{cond}(\mathbf{\Lambda}_N) = \frac{\lambda_N}{\lambda_1} \frac{1 - e^{-2c\lambda_1 T}}{1 - e^{-2c\lambda_N T}}.$$

In that case, we obtain that $\text{cond}(\mathbf{\Lambda}_N)$ is of order N^2 . Table 2 confirms this fact. The control problem is therefore well-posed when the controls act on the whole domain. This corresponds to the fact that, as it is well known, $\mathcal{H} = H^{-1}(\Omega)$ in this very case. In other words, the following observability inequality holds when $\omega = \Omega$:

$$C(T)\|\phi_T\|_{H^{-1}(\Omega)}^2 \leq \int_0^T \int_{\omega} \phi(t, x)^2 dx dt. \quad (9)$$

As we have mentioned above, in contrast with (9), as soon as ω is strictly included in Ω , the space \mathcal{H} blows-up to contain many data ϕ_T that are very singular away from ω .

Remark 2 If we reproduce these simple computations in the framework of the exact controllability of the wave equation, when the GCC is satisfied so that the observability inequalities hold in the energy space, the corresponding matrices $\mathbf{\Lambda}_N$ do not exhibit this exponential behavior. Thus, in the context of the wave equation, singularities may only appear after a numerical approximation scheme is applied, in which case the presence of high frequency spurious numerical solutions may make the corresponding discrete matrix $\Lambda_{h, \Delta t}$ to be exponentially ill-posed as the mesh-size parameters h and Δt tend to zero (see [32]). But, in the context of the heat equation, the ill-posedness holds at the continuous level. We therefore insist on the fact that the situation is definitely different from that arising in the numerical approximation of the controls for wave equations and, in some sense, as we shall see, more profound and complex.

Remark 3 The phenomenon we observe here has strong analogies with those arising in inverse problems, known to be severely ill-posed, especially when the heat operator is involved. Among the huge literature on that topic, we refer for instance to [1, 16, 17] and the references therein.

One of the most classical ones, the so-called backward heat problem, consists in determining the initial temperature $u(x, 0)$, given the temperature at any time T , u being the solution of $u_t - \Delta u = 0$ on $\mathbb{R}^+ \times \mathbb{R}$: the ill-posedness of this problem is related to the high instability of the solution u with respect to the data $u(T, x)$. Many approaches have been developed to address this problem, including the method of quasi-reversibility due to Lattès-Lions [22] (see also [26, 27]), Tychonoff regularization as well as moment theory. Our control problem is intimately related to this classical ill-posed problem.

More closely related to our control problem, we can also consider the following sidewise Cauchy problem for the heat equation : given f and g , to find u solution of

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0, T) \times (-L, L), \\ u(t, 0) = f(t), u_x(t, 0) = g(t) & \text{in } (0, T). \end{cases} \quad (10)$$

If f and g are real-analytic functions, then, following Cauchy-Kovalesky's Theorem, the solution of (10) may be expressed as follows :

$$u(t, x) = \sum_{n \geq 0} \frac{x^{2n}}{(2n)!} f^{(n)}(t) - \sum_{n \geq 0} \frac{x^{2n+1}}{(2n+1)!} g^{(n)}(t). \quad (11)$$

Both series converge uniformly for bounded t , provided that f and g satisfy $|f^{(n)}(t)| \leq ML^{-1}(2n)!$ and $|g^{(n)}(t)| \leq PL^{-1}(2n)!$ for all $n > 0$ and some constants M and P . Uniqueness follows as well. However, the problems of determining the values of the solution u at the right-hand side boundary $x = 1$ or the corresponding initial datum $u(x, 0)$ are ill-posed problems.

In this subsection we have illustrated the ill-posedness of the control operator Λ and its similarities with some other classical ill-posed problems in the theory of heat processes. However, this fact is compatible with the property of null or trajectory controllability that is well-known to hold.

This is due to the fact that, in the problem of null or trajectory controllability, the final target y_T is assumed to be the value at time $t = T$ of another solution of the heat equation, controlled or not. Assume for instance that y_T is the value of a solution $z = z(t, x)$ of the heat equation with initial data in $L^2(\Omega)$ and Fourier coefficients $\{d_k\}_{k \geq 1}$. In that case y_T is of the form

$$y_T(x) = \sum_{k \geq 1} d_k e^{-\lambda_k T} \sin(k\pi x), \quad x \in \Omega.$$

thus, its Fourier coefficients $\{d_k e^{-\lambda_k T}\}_{k \geq 1}$ are affected by a exponentially small weight factor.

The exponential decay of the Fourier coefficients of y_T compensates the ill-posedness of the inverse of the operator Λ thus making the null and trajectory controllability possible as the existing theory shows. Note however that the corresponding minima ϕ_T of the functional J^* do not belong to $L^2(\Omega)$. This compensation has a unique effect the fact that the corresponding solution ϕ of the adjoint heat equation belongs to $L^2(\omega \times (0, T))$ which, as we have mentioned above, is compatible with the possible infinity order singularities of ϕ_T away from ω .

2.2 Singular behavior of the minimal L^2 -norm control: boundary layer near $t = T$

From the previous explicit computations, we may compute the HUM control v by inverting the matrix $\mathbf{\Lambda}_N$. We choose as a target the trajectory $y_T(x) = e^{-c\lambda_1 T} \sin(\pi x)$. Table 3 depicts the L^2 -norm of the function ϕ_T^N and of the control $\phi^N \mathcal{X}_\omega$ for increasing values of N : as expected, the function ϕ_T^N is not uniformly bounded with respect to N in L^2 but in the larger Hilbert space \mathcal{H} . For $N = 80$, Figure 1 depicts the function ϕ_T^N on Ω and on the subset ω . Figure 2 depicts the $L^2(\Omega)$ -norm of the corresponding control with respect to t . Due to the dissipative property of the heat equation, the control mainly acts at the end of the time interval. In agreement with Table 3, the control blows up at the controllability time T (as $N \rightarrow \infty$). We also observe some oscillations (increasing as $N \rightarrow \infty$) of the control near $t = T$. These oscillations are related to the behavior of the adjoint solution ϕ near the boundary of ω (see Figure 3). Figure 4 depicts the evolution of the corresponding adjoint and controlled solution $\phi^N \mathcal{X}_\omega$ and y^N in Q_T and clearly exhibits the contrast of regularity of these two functions near T . Finally, we give on Figure 5 the distribution of the Fourier coefficients $\{a_k\}_{1 \leq k \leq N}$ of the function ϕ_T^N highlighting the high frequency contributions. For N greater approximatively than 120, $\mathbf{\Lambda}_N$ is so badly conditioned that the linear inversion produces wrong approximations of ϕ_T^N (with in particular a loss of symmetry in space).

	$N = 10$	$N = 20$	$N = 40$	$N = 80$
$\ \phi_T^N\ _{L^2(\Omega)}$	4.27	3.22×10^1	1.68×10^3	5.38×10^6
$\ \phi^N \mathcal{X}_\omega\ _{L^2(Q_T)}$	4.194×10^{-1}	4.410×10^{-1}	4.526×10^{-1}	4.586×10^{-1}

Table 3: L^2 -norm of ϕ_T^N and of the control vs. N for $\omega = (0.2, 0.8)$, $T = 1$ and $c = 1/10$.

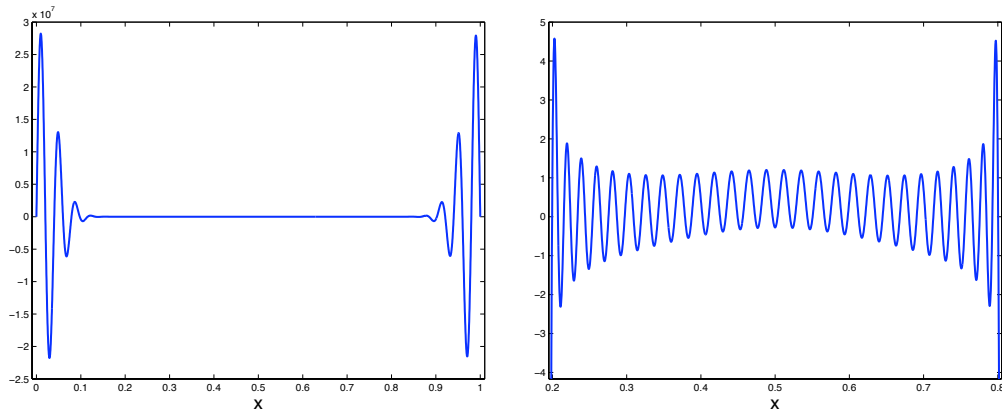


Figure 1: $T = 1, \omega = (0.2, 0.8) : \phi_T^N$ for $N = 80$ on Ω (Left) and on ω (Right).

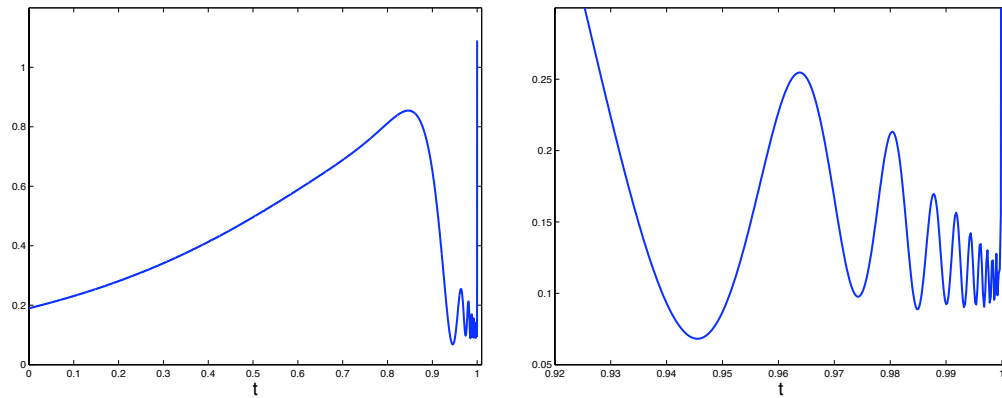


Figure 2: $T = 1, \omega = (0.2, 0.8) : \|\phi^N(\cdot, x)\mathcal{X}_\omega(x)\|_{L^2(\Omega)}$ for $N = 80$ on $[0, T]$ (Left) and on $[0.92T, T]$ (Right).

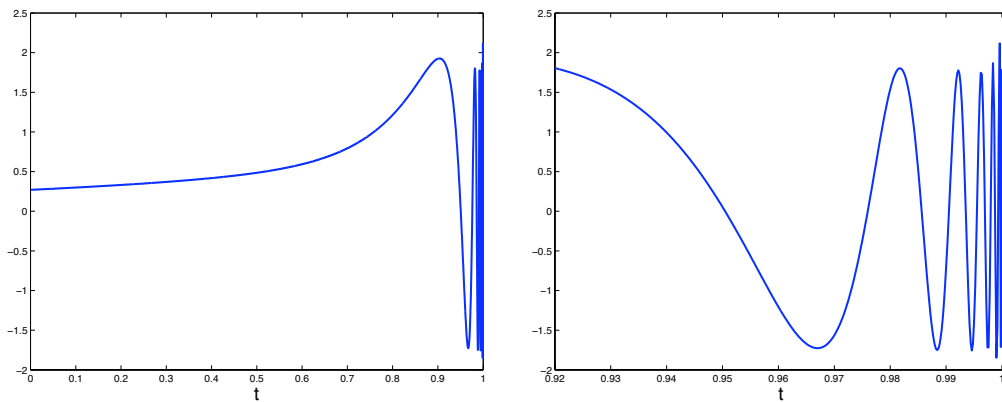


Figure 3: $T = 1, \omega = (0.2, 0.8) : \phi^N(\cdot, 0.8)$ for $N = 80$ on $[0, T]$ (Left) and on $[0.92T, T]$ (Right).

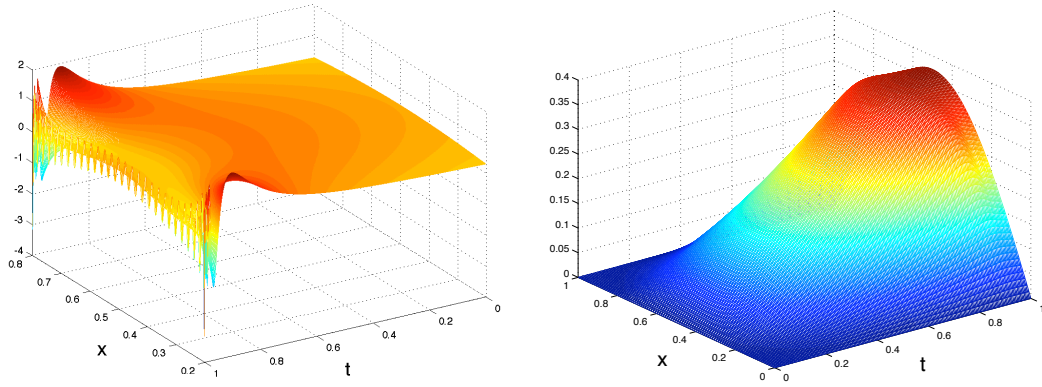


Figure 4: $T = 1$, $\omega = (0.2, 0.8)$, $N = 80$: **Left**: Evolution of the adjoint solution ϕ^N in $(0, T) \times \omega$; **Right**: Evolution of the controlled solution y^N in Q_T .

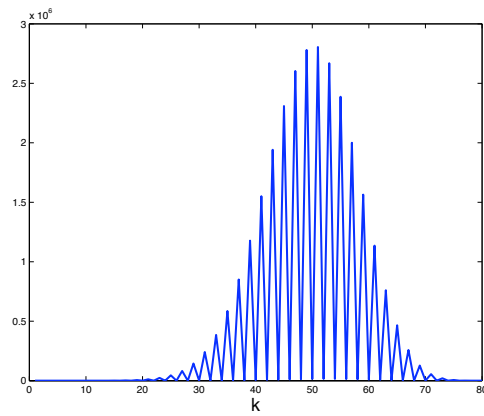


Figure 5: $T = 1$, $\omega = (0.2, 0.8)$, $N = 80$: Distribution of the Fourier coefficients $\{a_k\}_{1 \leq k \leq N}$ of ϕ_T^N .

Remark 4 We emphasize that the null controllability problem in which $y_0 \neq 0$ and $y_T \equiv 0$ produces exactly the same qualitative behavior, i.e. exponentially ill-posedness and high oscillations of the control L^2 -norm near T . The quantitative difference is a lower L^2 -norm of the corresponding null-control, due to the natural dissipation of the system.

Actually, the problem we have considered so far in which $y_0 \equiv 0$ and $y_T(x) = e^{-c\lambda_1 T} \sin(\pi x)$ is equivalent to the null controllability one with $y_0(x) = \sin(\pi x)$ and $y_T \equiv 0$. This can be seen by simply truncating to the controlled solution the solution $y = e^{-c\lambda_1 t} \sin(\pi x)$ of the heat equation with initial datum $\sin(\pi x)$ and final datum $e^{-c\lambda_1 T} \sin(\pi x)$. In particular the control is the same. Thus, the ill-posed character of the problem that we have made explicit in these experiments applies to the problem of null controllability as well.

2.3 Numerical Approximation

We now address the problem of efficiently computing the control. For, we substitute the heat equation by a semi-discrete numerical approximation scheme on a uniform mesh Ω_h , with mesh-size h , an a finite-dimensional state vector $\{Y_h(t)\}_{(h>0)}$, solution of a system of differential equations of the form

$$\begin{cases} M_h Y_h'(t) + K_h Y_h(t) = \mathcal{X}_\omega v_h(t) & \text{in } (0, T) \times \Omega_h, \\ Y_h = 0 & \text{on } (0, T) \times \partial\Omega_h, \\ Y_h(0, \cdot) = Y_{h0}(\cdot) & \text{in } \Omega_h, \end{cases} \quad (12)$$

where M_h and K_h - the mass and stiffness matrices - denote an approximation of the identity Id and $-c\partial_{xx}^2$ respectively. This can be done, for instance, using a centered finite difference approximation. Here and in the sequel $\{v_h(t)\}_{h>0}$ denotes the corresponding control (semi-discrete) approximation.

The explicit knowledge of the spectrum of K_h and M_h allows, as in Section 2.1, to compute the discretized control operator in a closed form (see Appendix 6.1). We denote it by Λ_h .

The problem of the uniform (as $h \rightarrow 0$) null control of semi-discrete approximations of the 1- d heat equation has been previously addressed in a number of articles. For instance, in [29] (see [40] for more details) the standard finite difference approximation is considered and it has been shown that the observability inequalities and the null controllability properties hold uniformly with respect to the mesh-size parameter h . In particular, there exists a positive constant C_1 , independent of h , such that

$$C_1 \|\phi_h(0)\|_{L^2(\Omega)}^2 \leq \int_0^T \int_\omega \phi_h^2(t, x) dx dt, \quad \forall \phi_{Th} \in L^2(\Omega) \quad (13)$$

where $\phi_h(t)$ is the piecewise linear continuous interpolation of the vector $\{\Phi_h\}$ over Ω , the solution of the semi-discrete adjoint system:

$$\begin{cases} M_h \phi_h'(t) - K_h \phi_h(t) = 0 & \text{in } (0, T) \times \Omega_h, \\ \phi_h = 0 & \text{on } (0, T) \times \partial\Omega_h, \\ \phi_h(T, \cdot) = \phi_T(\cdot) & \text{in } \Omega_h. \end{cases} \quad (14)$$

This property - which does not hold for the wave equation because of the high frequency spurious oscillations - implies the uniform coercivity of J_h^* , discrete version of J^* , and ensures the uniform boundedness of the controls obtained through minimization.

Table 4 collects the evolution of the numerical solutions for different values of h , with the numerical data of Section 2.3. The convergence of the numerical controls is confirmed. The ratio $\|\phi_h(0, \cdot)\|_{L^2(\Omega)}/\|\phi_h(t, \cdot)\mathcal{X}_\omega\|_{L^2(Q_T)}$ is observed to be uniformly bounded from above with respect to h . This is in contrast with the fact that the upper bound on the ratio $\|\phi_h(T, \cdot)\|_{L^2(\Omega)}/\|\phi_h(t, \cdot)\mathcal{X}_\omega\|_{L^2(Q_T)}$ diverges as $h \rightarrow 0$, due to the boundary layer appearing at $t = T$.

These controls are computed using a Conjugate Gradient (CG) algorithm, the stopping criterion being that the residue becomes smaller than 10^{-8} . This permits to ensure $\|y_h(T, \cdot) - y_{Th}\|_{L^2(\Omega)}$ to be of order 10^{-9} . Figure 6 depicts the adjoint solution ϕ_{Th} on Ω and the corresponding L^2 -norm of the control on $[0, T)$. We also observe that the number of iterations to reach convergence increases dramatically with $1/h$, in spite of the uniform observability. Of course, this is related to the ill-posedness of the problem, and the difficulty for a numerical method to capture the singular behavior of the control near T . As discussed in [6], this number of iterations is necessary to recover the high frequency components of ϕ_{Th} . The CG algorithm, starting from 0 reconstructs first the low-frequency components of ϕ_{Th} . This is done in a small number of iterations, explaining the very fast convergence to a fairly good approximation. The remaining iterations are used to compute the high frequency components and this takes many iterations since the HUM operator is badly conditioned and strongly damps the effect of the control on high frequency components.

The evolution of the residue is depicted in Figure 7 for $h = 1/80$ and clearly highlights the lack of robustness. If, for any h fixed, we denote by C_{2h} the constant appearing in the inequality

$$\int_0^T \int_\omega \phi_h^2(t, x) dx dt \leq C_{2h} \|\phi_h(0, \cdot)\|_{L^2(\Omega)}^2, \quad \forall \phi_{Th} \in L^2(\Omega), \quad (15)$$

the conditioning number of the control operator Λ_h is bounded by

$$\text{cond}(\Lambda_h) \leq C_1^{-1} C_{2h} h^{-2},$$

C_1 being the observability constant that is known to be uniform.

But $C_1^{-1} C_{2h}$ blows-up exponentially as $h \rightarrow 0$. This is due to the fact that C_{2h} blows up as $h \rightarrow 0$. The situation is *a fortiori* worst if the scheme under consideration is not uniformly observable, i.e. if $C_{1h}^{-1} \rightarrow \infty$ as $h \rightarrow 0$. But we emphasize that here C_{1h}^{-1} remains bounded while C_{2h} blows up exponentially.

We point out that the situation is normally the opposite one for the wave equation: In the wave context, often, for most numerical schemes, the uniform observability property fails and $C_{1h}^{-1} \rightarrow \infty$ blows up exponentially but the analogue of the direct inequality (15) holds uniformly on h .

h	1/20	1/40	1/80	1/160
Number of iterations	36	218	574	1588
$\ v_h\ _{L^2((0,T) \times \omega)}$	4.05×10^{-1}	4.322×10^{-1}	4.426×10^{-1}	4.492×10^{-1}
$\ y_h(T, \cdot) - y_{Th}\ _{L^2(\Omega)}$	2.11×10^{-9}	1.58×10^{-9}	2.65×10^{-9}	2.35×10^{-9}
$\frac{\ \phi_h(0, x)\ _{L^2(\Omega)}^2}{\ \phi_h \mathcal{X}_\omega\ _{L^2(Q_T)}^2}$	4.072×10^{-1}	4.329×10^{-1}	4.429×10^{-1}	4.439×10^{-1}

Table 4: Semi-discrete scheme $\omega = (0.2, 0.8)$, $\Omega = (0, 1)$ and $T = 1$.

Remark 5 Here we have discussed the 1-d finite-difference semi-discretization scheme for which the property of observability is uniform as $h \rightarrow 0$. But, in the two-dimensional case,

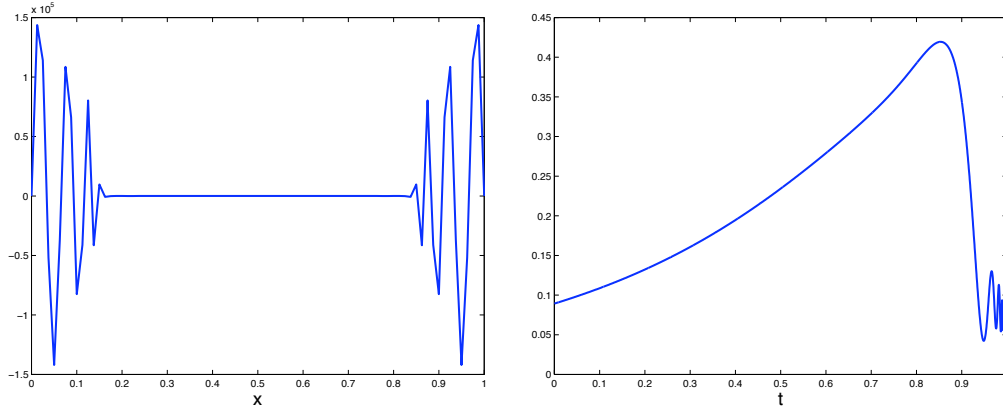


Figure 6: Semi-discrete scheme : $h = 1/80$. **Left:** Adjoint state ϕ_{T_h} at $t = T$ on Ω . **Right:** L^2 - norm of the control v_h on $[0, T)$.

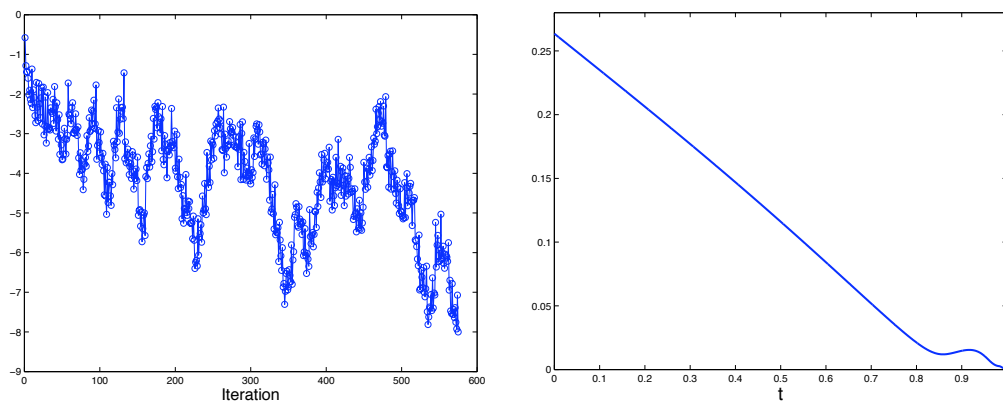


Figure 7: Semi-discrete scheme : $h = 1/80$. **Left:** Evolution of the residue w.r.t. the number of iterations of the GC algorithm. **Right:** Evolution of $\|y_h(\cdot, t) - y_{T_h}\|_{L^2(\Omega)}$ w.r.t time t .

the observability inequality fails to be uniform even for the 5-point finite-difference semi-discrete scheme. In fact, as explained in [40], this is due to the fact that there are some high-frequency numerical eigenmodes (concentrated, for instance, in the diagonal when Ω is a square) that do not fulfill the classical property of unique continuation. Thus, the numerical scheme generates high frequency solutions which are insensitive to the action of the controls, when located away from the diagonal. Note however that the corresponding eigenvalue is of the order of $\lambda \sim ch^{-2}$ so that the energy at time $t = T$ of these uncontrolled modes is of the order of $\exp(-cTh^{-2})$.

In [3], by means of discrete Carleman inequalities, the authors obtain weak uniform observability inequalities which are compatible with this pathology by adding reminder terms of the form $e^{-ch^{-2}} \|\phi_{Th}\|_{L^2(\Omega)}^2$ which vanish asymptotically as $h \rightarrow 0$.

Let us also mention some other closely related articles. In [21] the approximate controllability is derived using semi-group arguments, introducing a vanishing term of the form $h^\beta \|\phi_{Th}\|_{L^2(\Omega)}$, for some $\beta > 0$. We also mention [10, 37] where weak observability inequalities are derived for the time semi-discrete heat equation, using an appropriate filtering of the data.

Despite of this, the search of uniform controllable schemes for the 2-D null controllability of the heat equation remains an open topic, to a large extent.

3 Perturbations of the control problem

As described in the previous section, the very weak coercivity of the functional J^* (in the L^2 -sense) renders ill-posed the control problem and the corresponding trajectory control exhibits a singular behavior that is difficult to capture numerically. This is intimately related to the very bad conditioning of the functionals to be minimized.

In order to overcome this intrinsic ill-posedness phenomenon, several approaches may be introduced. The first one consist in introducing a well-posed regularizing perturbation at the continuous level, either in the state equation or in the functional to be minimized, easier to solve, and whose solution is expected to be *close* to the original one. A second approach consists on doing that directly at the discretized level, defining other numerical schemes or functionals, leading to a lower conditioning number of the discrete control operator. We analyze now how these two approaches perform.

3.1 Regularization method

A simple way to improve the regularity of the minimizer ϕ_T of J^* consists in augmenting the cost J^* as follows

$$J_\epsilon^*(\phi_T) = J^*(\phi_T) + \frac{\epsilon}{2} \|\phi_T\|_{H_0^s(\Omega)}^2$$

for any prescribed $\epsilon > 0$ and well-chosen $s \in \mathbb{N}$.

The corresponding dual function is simply $J_\epsilon(v) = J(v) + \epsilon/2 \|y(T, \cdot) - y_T\|_{H^{-s}(\Omega)}^2$.

The minimizer $\phi_{T,\epsilon}$ of J_ϵ^* satisfies $(\Lambda + \epsilon Id)\phi_{T,\epsilon} = y_T$ for $s = 0$ and $(\Lambda - \epsilon\Delta)\phi_{T,\epsilon} = y_T$ for $s = 1$ respectively, so that the relation between the Fourier coefficients $\{a_{k,\epsilon}\}_{(k \geq 1)}$ and $\{b_p\}_{(p \geq 1)}$ of $\phi_{T,\epsilon}$ and y_T respectively is then

$$b_p = \sum_{k \geq 1} \left(c_{p,k}(\omega) g_{p,k}(T) + \epsilon (k\pi)^{2s} \delta_{p,k} \right) a_{k,\epsilon}, \quad s = 0, 1. \quad (16)$$

The regularization term is reflected in the Fourier representation (16) by the weights $\epsilon(k\pi)^{2s}\delta_{p,k}$ that make $a_{k,\epsilon}$ to decay faster as $k \rightarrow \infty$.

The case $s = 1$ corresponds to the Tychonoff regularization (see [20] for the proof of the equivalence) widely used for ill-posed problems and analyzed in the context of the controllability of the heat equation in [20].

Although this regularization has been introduced at the continuous level, the same can be done for finite-dimensional projections and numerical approximation schemes. In particular, when considering finite-dimensional projections to the first N Fourier modes, in the same manner that we replaced J^* by J_N^* , we can substitute J_ϵ^* by $J_{N,\epsilon}^*$.

Tables 5 and 6 provide some numerical values of the evolution with respect to ϵ obtained with $N = 80$. In particular they give the values of the gap $\|v^N - v_\epsilon^N\|_{L^2((0,T)\times\omega)}$ and the conditioning number of the corresponding operator. The added terms ensure a regularization of the control and the well-posedness of the problem but a very slow rate of convergence towards the heat control. For $s = 1$ and $N = 80$, we observe the behavior $\|v^N - v_\epsilon^N\|_{L^2((0,T)\times\omega)} \approx O(\epsilon^{0.259})$ but for large values of N , we expect a logarithmic convergence with respect to ϵ , which is the typical behavior for exponentially ill-posed problems (see [9]). The case $s = 0$, corresponding to a weaker regularization gives a slightly better rate ($\|v^N - v_\epsilon^N\|_{L^2((0,T)\times\omega)} \approx O(\epsilon^{0.295})$) but a slightly worst conditioning number.

We also refer to the section 7 of [13] where logarithmic estimates are proved when using a penalization method to impose the final condition $y(T) = y_T$. As the penalty parameter tends to infinity the corresponding controls yield approximate ones, but convergence is logarithmically slow. The approximate controllability approach consists in minimizing the functional J over the class of controls which drive the solution of the heat equation to an δ neighborhood of the target:

$$\|y(T, \cdot) - y_T\|_{L^2(\Omega)} \leq \delta.$$

This approximate problem is solved in [6] (see also [15]) using an iterative splitting method: as expected, when δ goes to zero, so that the problem approximates the null or trajectory controllability one, the solution exhibits an oscillatory behavior.

ϵ	10^{-1}	10^{-3}	10^{-5}	10^{-7}	10^{-9}
$\ v^N - v_\epsilon^N\ _{L^2((0,T)\times\omega)}^2$	1.15×10^{-1}	4.72×10^{-2}	2.15×10^{-2}	1.17×10^{-2}	6.89×10^{-3}
$\ \phi_{T,\epsilon}^N\ _{L^2(\Omega)}$	5.47×10^{-1}	2.52×10^0	1.42×10^1	9.20×10^1	6.66×10^2
$\ v_\epsilon^N\ _{L^2((0,T)\times\omega)}$	2.23×10^{-1}	3.85×10^{-1}	4.28×10^{-1}	4.43×10^{-1}	4.49×10^{-1}
$cond(\Lambda_{N,\epsilon})$	5.44×10^0	5.87×10^2	7.46×10^4	7.45×10^6	7.18×10^8

Table 5: $N = 80$, $\omega = (0.2, 0.8)$, $s = 0$: $\|v^N - v_\epsilon^N\|_{L^2((0,T)\times\omega)} \approx O(\epsilon^{0.295})$.

ϵ	10^{-1}	10^{-3}	10^{-5}	10^{-7}	10^{-9}
$\ v^N - v_\epsilon^N\ _{L^2((0,T)\times\omega)}^2$	2.86×10^{-1}	9.97×10^{-2}	5.83×10^{-2}	2.99×10^{-2}	1.70×10^{-2}
$\ \phi_{T,\epsilon}^N\ _{L^2(\Omega)}$	1.90×10^{-1}	6.70×10^{-1}	1.74×10^0	6.50×10^0	2.79×10^1
$\ v_\epsilon^N\ _{L^2((0,T)\times\omega)}$	2.86×10^{-2}	3.36×10^{-1}	3.78×10^{-1}	4.19×10^{-1}	4.37×10^{-1}
$cond(\Lambda_{N,\epsilon})$	4.58×10^3	5.93×10^2	1.50×10^2	3.88×10^3	2.01×10^5

Table 6: $N = 80$, $\omega = (0.2, 0.8)$, $s = 1$: $\|v^N - v_\epsilon^N\|_{L^2((0,T)\times\omega)} \approx O(\epsilon^{0.259})$.

In practice, when trying to obtain the minimizer of J^* , ϵ is taken as being a decreasing parameter of N , so that as $N \rightarrow \infty$, $\epsilon \rightarrow 0$. In this way the control of the perturbed

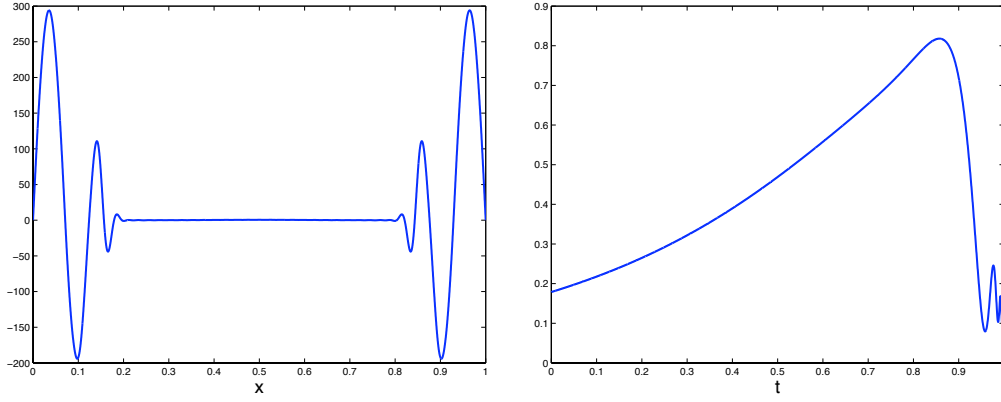


Figure 8: L^2 regularization ($s = 0$) for $\epsilon = 10^{-7}$ and $N = 80$, $T = 1$ and $\omega = (0.2, 0.8)$. **Left:** Adjoint state $\phi_{T,\epsilon}$ at time $t = T$. **Right:** L^2 -norm of the control vs. time t .

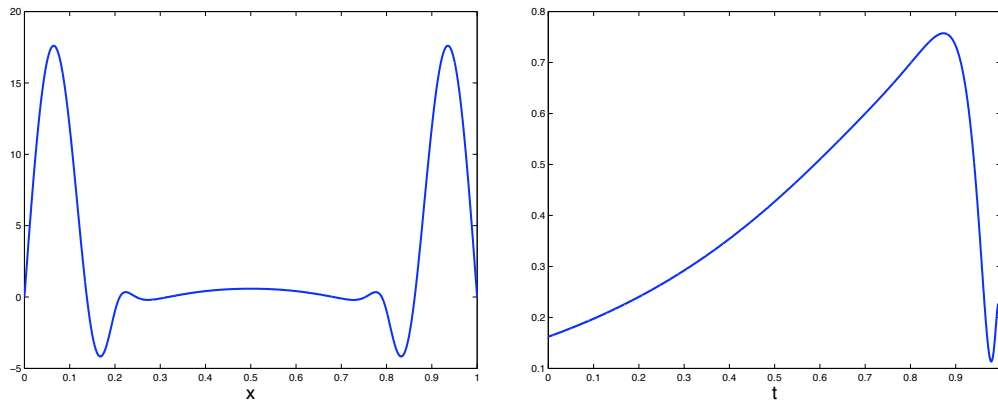


Figure 9: Tychonoff regularization ($s = 1$) for $\epsilon = 10^{-7}$ and $N = 80$, $T = 1$, $\omega = (0.2, 0.8)$. **Left:** Adjoint state $\phi_{T,\epsilon}^N$ at $t = T$. **Right:** L^2 -norm of the control vs. time t .

regularized system is close to the control v of system (1): in other terms, the perturbation vanishes when $N \rightarrow \infty$. At the level of numerical discretization schemes the role of N is played by $1/h$, h being the mesh-size parameter. For the case $s = 1$, the choice $\epsilon = N^{-4}$ provides a good compromise between the value of the conditioning number of the corresponding perturbed matrix $\mathbf{\Lambda}_{N,\epsilon}$ and the rate of convergence of the control: we obtain $\text{cond}(\mathbf{\Lambda}_{N,\epsilon}) \approx e^{-5.97} N^{3.5}$ and $\|v - v_\epsilon^N\|_{L^2(0,T) \times \omega} \approx O(N^{-0.305})$.

N	20	40	80	160	320
$\text{cond}(\mathbf{\Lambda}_{N,\epsilon})$	1.38×10^2	1.26×10^3	1.27×10^4	1.40×10^5	1.58×10^6
$\ \phi_{T,\epsilon}^N\ _{L^2(\Omega)}^2$	4.15×10^0	1.87×10^1	9.94×10^1	5.90×10^2	4.01×10^3
$\ v_\epsilon^N\ _{L^2((0,T) \times \omega)}$	3.942×10^{-1}	4.166×10^{-1}	4.298×10^{-1}	4.382×10^{-1}	4.438×10^{-1}

Table 7: Tychonoff regularization ($s = 1$) for $\epsilon = N^{-4}$, $\omega = (0.2, 0.8)$ and $\Omega = (0, 1)$.

3.2 Singular perturbations

Regularization can also be performed without changing the functionals J and J^* but by modifying the heat equation with the aim to restore the stability of the corresponding backward system. This approach is considered in detail in [22] in the framework of optimal control theory by introducing the so-called quasi-reversibility methods and in [26, 27] in the context of controllability.

For instance, the regular perturbation $\partial_t - \partial_{xx} - \epsilon \partial_{xxxx}$ of the heat operator, for any $\epsilon > 0$ may be considered. But there are some variants. For instance, in [30], the null controllability of the following damped, singularly perturbed wave equation is considered:

$$\begin{cases} \epsilon y_{\epsilon,tt} + y_{\epsilon,t} - c y_{\epsilon,xx} = v_\epsilon \mathcal{X}_\omega, & \text{in } Q_T, \\ y_\epsilon = 0, & \text{on } \Sigma_T, \\ (y_\epsilon(0, \cdot), y'_\epsilon(0, \cdot)) = (y_0, y_1), & \text{in } \Omega \end{cases} \quad (17)$$

for any $\epsilon > 0$ (the elliptic situation where ϵ is replaced by $-\epsilon$ is studied in [22], Chapter 3). In [30] it is proved that, for any $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\omega)$ and $T > 2\sqrt{c^{-1}\epsilon}$, system (17) is uniformly controllable with respect to ϵ with $v_\epsilon \in L^2(Q_T)$ and that the control of minimal L^2 -norm for (17) converges in $L^2((0, T) \times \omega)$ as $\epsilon \rightarrow 0$, towards the control v of minimal L^2 -norm for the heat equation (1).

The functional $J_\epsilon^* : L^2(\Omega) \times H^{-1}(\Omega) \rightarrow \mathbb{R}$ to be minimized for the control of (17) is

$$J_\epsilon^*(\phi_{T\epsilon}, \phi'_{T\epsilon}) = \frac{1}{2} \int_0^T \int_\omega \phi_\epsilon^2(t, x) dx dt + \int_\Omega (y_0 + \epsilon y_1) \phi_\epsilon(0) dx - \epsilon \langle y_0, \phi'_\epsilon(0) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}$$

where ϕ_ϵ is solution of

$$\begin{cases} \epsilon \phi_{\epsilon,tt} - \phi_{\epsilon,t} - c \phi_{\epsilon,xx} = 0 & \text{in } Q_T, \\ \phi_\epsilon = 0 & \text{on } \Sigma_T, \\ (\phi(T, \cdot), \phi'(T, \cdot)) = (\phi_{T\epsilon}, \phi'_{T\epsilon}) & \text{in } \Omega. \end{cases}$$

In this case the functionals J_ϵ^* are uniformly coercive since the following observability inequality

$$\epsilon \|\phi_\epsilon(0)\|_{L^2(\Omega)}^2 + \|\epsilon \phi_{\epsilon,t}(0) - \phi_\epsilon(0)\|_{H^{-1}(\Omega)}^2 \leq C(T) \int_0^T \int_\omega |\phi_\epsilon|^2 dx dt,$$

holds for some constant $C(T)$, independent of ϵ and any $T > 2\sqrt{\epsilon c^{-1}}$ (see Theorem 1.3 in [30]).

Although, at a theoretical level, the functionals J_ϵ^* are uniformly coercive, we found that, for ϵ small enough, the minimization of J_ϵ^* using a CG algorithm is particularly sensitive to the numerical approximation of (17) and to the choice of the norms for the descent direction.

Following section (2.1), we can also compute the finite-dimensional approximation $\Lambda_{2N,\epsilon}$ of the control application Λ_ϵ expanding the solution ϕ_ϵ in Fourier series. We have

$$\phi_\epsilon(t, x) = \sum_{k \geq 1} \left(c_1^k e^{-\frac{1+\mu_k}{2\epsilon}(T-t)} + c_2^k e^{-\frac{1-\mu_k}{2\epsilon}(T-t)} \right) \sin(k\pi x), \quad \mu_k = \sqrt{1 - 4c(k\pi)^2\epsilon}.$$

We then truncate the series to the first N terms and take $\epsilon < (4c(N\pi)^2)^{-1}$ so that $\mu_k \in \mathbb{R}$ for all $k \geq 1$. The tedious computation of the $2N$ order matrix $\Lambda_{2N,\epsilon}$ is developed in Appendix 6.2. For $N = 80$ (which requires $\epsilon < 3.96 \times 10^{-5}$ to ensure that the eigenvalues under consideration are real), $\epsilon = 10^{-6}$, $(y_0, y_1) = (0, 0)$ and the target $(y_T, y'_T) = (1, -c\pi^2)e^{-c\pi^2 T} \sin(\pi x)$, Figure 10 depicts the minimizer $(\phi_{T\epsilon}^N, \phi_{T\epsilon}'^N)$ in Ω of J_ϵ^* . The corresponding function $t \rightarrow \|v_\epsilon^N\|_{L^2(\omega)}$ is illustrated in Figure 11, to be compared with Figure 2-Left. Note that the addition of the hyperbolic term has the effect of increasing slightly the value of $\|v_\epsilon(t, \cdot)\|_{L^2(\omega)}$ for t small. From Table 8, we check the convergence of the control v_ϵ^N as $\epsilon \rightarrow 0$. The convergence is faster than for the Tychonoff type regularization in Section 3.1: in particular, we observe that the addition of the hyperbolic term ϵy_{tt} does not modify the regularity of v_ϵ^N near T , nor the high conditioning number $\text{cond}(\Lambda_{2N,\epsilon})$.

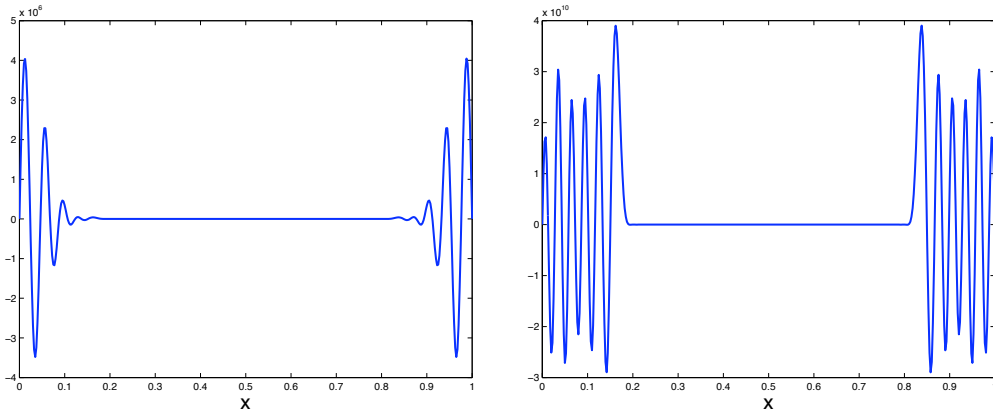


Figure 10: Singular hyperbolic perturbation: $N = 80$, $\epsilon = 10^{-6}$, initial condition $\phi_{T\epsilon}^N$ (Left) and $\phi_{T\epsilon}'^N$ (Right) of ϕ_ϵ in Ω .

ϵ	10^{-6}	10^{-7}	10^{-8}
$\ v^N - v_\epsilon^N\ _{L^2((0,T) \times \omega)}$	2.02×10^{-2}	1.94×10^{-2}	1.74×10^{-2}
$\ v_\epsilon^N\ _{L^2((0,T) \times \omega)}$	4.77×10^{-1}	4.65×10^{-1}	4.61×10^{-1}
$\text{cond}(\Lambda_{2N,\epsilon})$	2.68×10^{18}	3.98×10^{19}	7.48×10^{20}

Table 8: Singular hyperbolic perturbation: $N = 80$, $\omega = (0.2, 0.8)$. Behavior of the control v_ϵ and of the conditioning number to be compared with $\text{cond}(\Lambda_N) = 6.96 \times 10^{16}$, see Table 1.

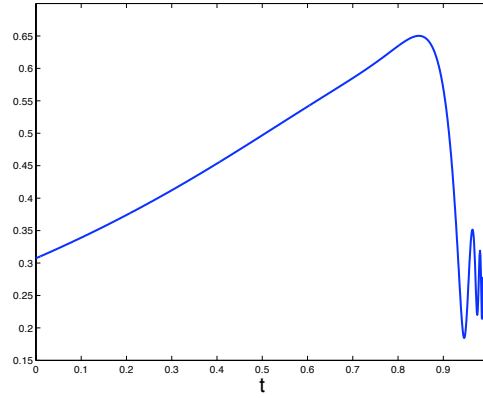


Figure 11: Singular hyperbolic perturbation: $N = 80$, $\epsilon = 10^{-6}$. $L^2(\Omega)$ -norm of the control v_ϵ^N vs. $t \in [0, T]$.

This method has been used before in the context of ill-posed parabolic problems. Indeed, in [36] a similar idea is used to approximate numerically an ill-posed inverse heat problem. There the author first introduces a regularizing second time derivative term and then reverses the space and time variables. This leads to a well-posed hyperbolic problem whose solution is *close* to the desired one.

Another, *a priori* simpler (at the computational level), perturbation of the heat equation is the following one

$$y_{\epsilon,t} - \epsilon y_{\epsilon,txx} - cy_{\epsilon,xx} = v_\epsilon \mathcal{X}_\omega \quad \text{in } Q_T. \quad (18)$$

At the continuous level, the corresponding spectrum is

$$\left\{ \frac{c(k\pi)^2}{1 + \epsilon(k\pi)^2} \right\}_{k>0}$$

that possesses the accumulation point $c\epsilon^{-1}$ (as $k \rightarrow \infty$). This is due to the fact that the generator of the corresponding semigroup is bounded as can be easily seen by writing the operator $\partial_t - \epsilon \partial_{txx}^3 - c \partial_{xx}^2$ in the form $(I - \epsilon \partial_{xx}^2) \partial_t - c \partial_{xx}^2$. Therefore, the generator of the semigroup is given by $c(I - \epsilon \partial_{xx}^2)^{-1} \partial_{xx}^2$. As a consequence of this spectral accumulation, (18) is not null controllable when $\epsilon > 0$.

However, at the numerical level, when performing a finite-difference discretization with mesh-size h , and by letting ϵ to go to zero with $h = 1/N$, one can avoid the spectral concentration phenomenon above to get a uniform controllable scheme. More precisely, it suffices to take $\epsilon < h$. A value of ϵ smaller than h^2 would have no influence on the corresponding discrete spectrum. Therefore, we take $\epsilon = h^\alpha$ with $\alpha \in (1, 2]$. We observe in practice that the resulting scheme, consistent with the initial heat equation, allows a faster convergence of the CG algorithm.

Figure 12 depicts the evolution of the residue for $h = 1/80$ and $\epsilon = h^\alpha$ with $\alpha = 1.75, 2, 0$. System (17) is here fully-discretized with an implicit Euler scheme and $\Delta t = h/2$. For $\epsilon = h^{1.75}$ the convergence is obtained after only 172 iterations to be compared with the 622 iterations corresponding to $\epsilon = 0$. On the other hand, the perturbation slightly reduces the convergence rate of the control with respect to h (see Table 9).

The full-discretization also permits us to appreciate the influence of the time discretization on the CG algorithm. Similarly to the wave equation (see [2]), we observe that this

additional approximation slightly increases the number of iterations. For instance, in the previous example, for $h = 1/80$ and $\Delta t = h/2$, we obtain 622 iterations to be compared with the 574 iterations of the semi-discrete case (see Table 4).

We have also observed that the use of the perturbed operator $(\partial_t - c\partial_{xx} - \epsilon\partial_x^4)$ -as discussed in [22] and [26, 27]- has no influence on the number of iterations. The interest at the numerical level of (18) is to provide a new scheme, convergent towards the continuous heat equation and for which, the CG algorithm is better behaved. In view of the oscillations in the residue observed Figure 12, the conditioning number has still an exponential behavior, but with a smaller rate.

Finally, we underline that the perturbation methods considered here are more involved but provide, for any ϵ , a trajectory control for the heat equation. This is in contrast with the regularization approach of Section 3.1 which leads to approximate controllability.

In the context of the wave equation the use of singular vanishing terms to derive uniformly controllable schemes has been discussed in [2, 32, 39].

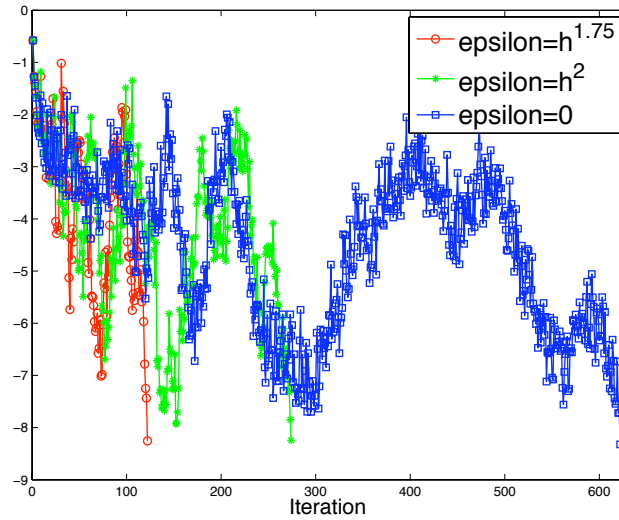


Figure 12: Evolution of the residue for $\epsilon = h^\alpha$ with $\alpha = 1.75, 2$ and 0 , $h = 1/80$, $\Delta t = h/2$, $T = 1$ and $\omega = (0.2, 0.8)$.

h	1/20	1/40	1/80	1/160
$\epsilon = 0$	$4.286 \times 10^{-1}(19)$	$4.822 \times 10^{-1}(129)$	$4.828 \times 10^{-1}(622)$	$4.720 \times 10^{-1}(1456)$
$\epsilon = h^2$	$5.614 \times 10^{-1}(13)$	$5.658 \times 10^{-1}(77)$	$5.094 \times 10^{-1}(274)$	$4.788 \times 10^{-1}(496)$
$\epsilon = h^{1.75}$	$7.696 \times 10^{-1}(13)$	$7.304 \times 10^{-1}(82)$	$5.46 \times 10^{-1}(112)$	$4.894 \times 10^{-1}(209)$

Table 9: Evolution of the $L^2((0, T) \times \omega)$ -norm of the control vs. h for $\epsilon = h^\alpha$ with $\alpha = 1.75, 2$ and 0 , $h = 1/80$, $\Delta t = h/2$, $T = 1$ and $\omega = (0.2, 0.8)$, and, in parenthesis, the number of CG iterations.

3.3 Controls with compact support in time

As observed in Section 2.1, the control of minimal L^2 -norm for the heat equation exhibits a highly oscillatory behavior near the controllability time T , and therefore it is very difficult to capture with robustness by numerical means.

In this section we discuss the efficiency of using time-dependent weights to smooth out the behavior of the control near $t = T$. We thus perturb the L^2 -norm and consider the new problem

$$\min_{v \in \mathcal{C}(T, y_0, y_T)} J_1(v) = \int_0^T \int_{\omega} \rho^{-1}(t) v^2(t, x) dx dt \quad (19)$$

for some strictly positive and smooth function ρ^{-1} . When ρ^{-1} is close to one, this new problem may be seen as a (regular) perturbation of the original problem (2). Moreover, convex duality leads to the following expression of the conjugate function

$$J_1^*(\phi_T) = \frac{1}{2} \int_0^T \int_{\omega} \rho(t) \phi^2(t, x) dx dt - \int_{\Omega} y_T(x) \phi_T(x) dx + \int_{\Omega} y_0(x) \phi(0, x) dx$$

where ϕ solves (3). At last, the unique control which minimizes J_1 is then given by

$$v(t, x) = \rho(t) \phi(t, x) \mathcal{X}_{\omega}(x), \quad (t, x) \in Q_T,$$

where ϕ solves (3) with ϕ_T the minimizer of J_1^* .

An *a priori* simple way to eliminate oscillations near the controllability time is to impose ρ to vanish in the interval $[T - \delta, T]$ for any $\delta > 0$ small enough. A $C^1([0, T])$ example of function fulfilling these conditions is given by

$$\rho(t) = \mathcal{X}_{0 \leq t \leq t_1} + \frac{(t - t_2)^2 (2t - 3t_1 + t_2)}{(t_2 - t_1)^3} \mathcal{X}_{t_1 \leq t \leq t_2}, \quad t \in [0, T] \quad (20)$$

for $0 < t_1 < t_2 = T - \delta < T$. Minimizing J_1^* one obtains controls that vanish in the time interval $[T - \delta, T]$.

Approximations of these minimizers and controls can be computed using a Fourier analysis or a numerical approximation scheme. In particular, the analogue of relation (7) is now

$$b_p = \sum_{k \geq 1} a_k c_{p,k}(\omega) \int_0^T \rho(s) e^{-c(\lambda_p + \lambda_k)(T-s)} ds, \quad p \geq 1. \quad (21)$$

This allows us to construct the corresponding matrix $\mathbf{\Lambda}_{N,\rho}$ of order N . For any N fixed, the conditioning number of $\mathbf{\Lambda}_{N,\rho}$ is larger than the conditioning number $\text{cond}(\mathbf{\Lambda}_N)$. The introduction of the ρ function increases the amplitude of the coefficients on the diagonal. However, when we use a CG algorithm - which never requires the explicit computation of any matrix - we observe a much better behavior with respect to finite-difference space semi-discretizations. This is not a contradiction, the conditioning number being simply an upper measure of the efficiency of the CG algorithm.

Tables 10 and 11 collect some numerical values obtained with the semi-discrete finite difference scheme for $(t_1, t_2) = (0.98, 0.99)$ and $(t_1, t_2) = (0.8, 0.9)$ respectively. We observe that the number of iterations remains bounded with respect to h , with a significant reduction for $(t_1, t_2) = (0.8, 0.9)$. On the other hand, we check that the corresponding control which drives the state to $e^{c\lambda_1 \delta} y_T$ at time $t = t_2 = T - \delta$ has a greater L^2 -norm. From $t = T - \delta$ to $t = T$, the control vanishes, so that the state passes, by diffusion, from $e^{c\lambda_1 \delta} y_T$ to y_T . We recall that $y_T(x) = e^{-c\lambda_1 T} \sin(\pi x)$ involves only the first mode. Tables 10 and 11 also

indicate that the adjoint solution $\phi_h(T, \cdot)$ is uniformly bounded in $L^2(\Omega)$. Figures 13 and 14 depict the corresponding adjoint solution ϕ_{Th} , minimum of J_1^* and the control v_h which presents much less oscillations in time.

The weight function has the effect to filter out the high frequency components: Figure 16 depicts the distribution of the Fourier coefficients of ϕ_T for these two choices of ρ , to be compared with Figure 5. We refer to [12] where compact support controls (in time) are obtained numerically writing the optimality condition of a cost function defined by $J_2(v) = \|\rho_0 y\|_{L^2(Q_T)}^2 + \|\rho v\|_{L^2((0,T)\times\omega)}^2$ for some suitable weight functions ρ_0 and ρ .

h	1/20	1/40	1/80	1/160
Number of iterations	54	805	478	306
$\ v_h\ _{L^2((0,T)\times\omega)}$	4.172×10^{-1}	4.412×10^{-1}	4.472×10^{-1}	4.532×10^{-1}
$\ y_h(\cdot, T) - y_{Th}\ _{L^2(\Omega)}$	1.25×10^{-9}	1.48×10^{-9}	1.77×10^{-9}	1.61×10^{-9}
$\frac{\ \phi_h(0, x)\ _{L^2(\Omega)}^2}{\ \rho(t)\phi_h(t, x)\ _{L^2((0,T)\times\omega)}^2}$	4.19×10^{-1}	4.42×10^{-1}	4.48×10^{-1}	4.54×10^{-1}
$\frac{\ \phi(T, x)\ _{L^2(\Omega)}^2}{\ \rho(t)\phi_h(t, x)\ _{L^2((0,T)\times\omega)}^2}$	7.15×10^5	5.60×10^{12}	1.17×10^{11}	4.75×10^{10}

Table 10: Semi-discrete scheme: $\omega = (0.2, 0.8)$, $\Omega = (0, 1)$, $T = 1$ and $(t_1, t_2) = (0.98, 0.99)$.

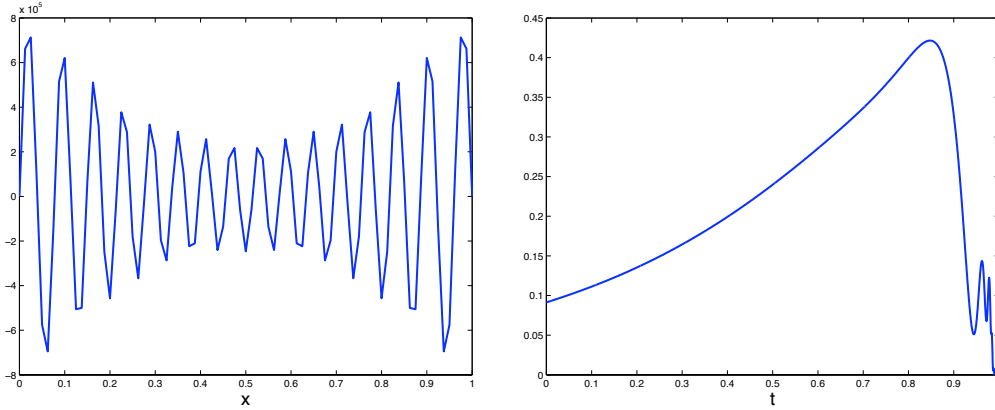


Figure 13: Semi-discrete scheme: $h = 1/80$ and $(r_1, r_2) = (0.98, 0.99)$. **Left:** Adjoint state ϕ_{Th} at time $t = T$. **Right:** L^2 -norm of the control v_h vs. time t .

Remark 6 One may replace the function ρ given by (20) by the exponential $\rho(t) = e^{-\frac{a}{T-t}}$, for any real $a > 0$, which is smooth and vanishes at time T . In [13], the following global estimate, using Carleman techniques, is proved

$$\int_{Q_T} e^{-\frac{A(1+T)}{T-t}} \phi^2(t, x) dx dt \leq e^{C(1+1/T)} \int_0^T \int_{\omega} \phi^2(t, x) dx dt$$

for any solution ϕ of (3) and some constant A and C that depend only on Ω and ω . This observability inequality, stronger than (4), is certainly related to the fact that the compact function ρ improves the stability of the control problem. But this issue needs further analysis.

h	1/20	1/40	1/80	1/160
Number of iterations	56	25	22	19
$\ v_h\ _{L^2((0,T)\times\omega)}$	5.158×10^{-1}	5.236×10^{-1}	5.47×10^{-1}	5.546×10^{-1}
$\ y_h(\cdot, T) - y_{Th}\ _{L^2(\Omega)}$	1.54×10^{-9}	2.40×10^{-9}	2.05×10^{-9}	1.37×10^{-9}
$\frac{\ \phi_h(0,x)\ _{L^2(\Omega)}^2}{\ \rho(t)\phi_h(t,x)\ _{L^2((0,T)\times\omega)}^2}$	5.295×10^{-1}	5.459×10^{-1}	5.558×10^{-1}	5.663×10^{-1}
$\frac{\ \phi_h(T,x)\ _{L^2(\Omega)}^2}{\ \rho(t)\phi_h(t,x)\ _{L^2((0,T)\times\omega)}^2}$	2.48×10^{10}	7.20×10^8	1.49×10^9	1.89×10^9

Table 11: Semi-discrete scheme: $\omega = (0.2, 0.8)$, $\Omega = (0, 1)$, $T = 1$ and $(t_1, t_2) = (0.8, 0.9)$.

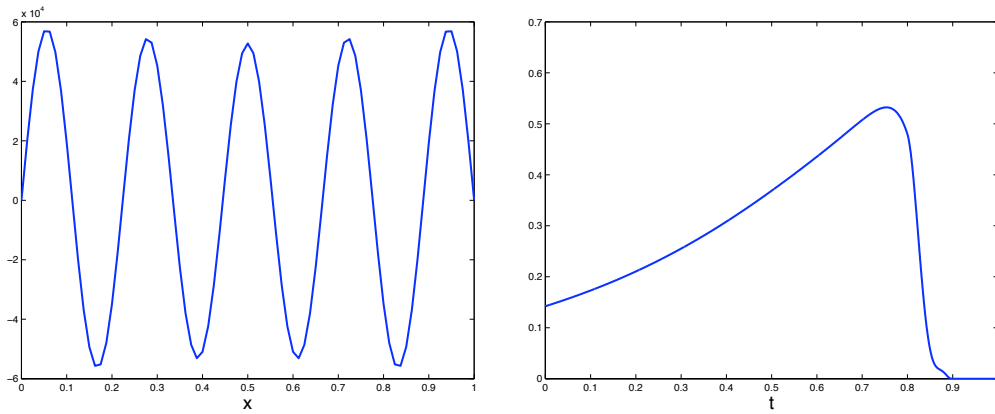


Figure 14: Semi-discrete scheme with $h = 1/80$ and $(t_1, t_2) = (0.8, 0.9)$ - **Left**: Adjoint state ϕ_{Th} at time $t = T$. **Right**: L^2 -norm of the control v_h vs. time t .

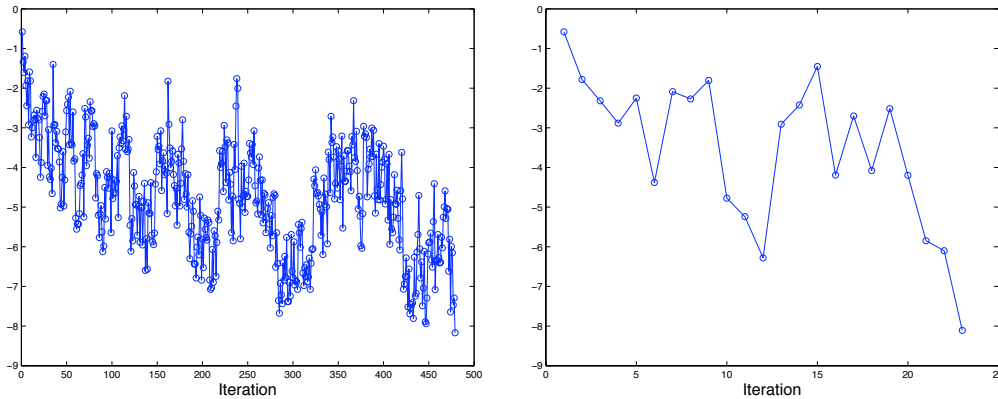


Figure 15: Semi-discrete scheme with $h = 1/80$. Evolution of the residue vs. the number of iterations for $(t_1, t_2) = (0.98, 0.99)$ (**Left**) and $(t_1, t_2) = (0.8, 0.9)$ (**Right**).

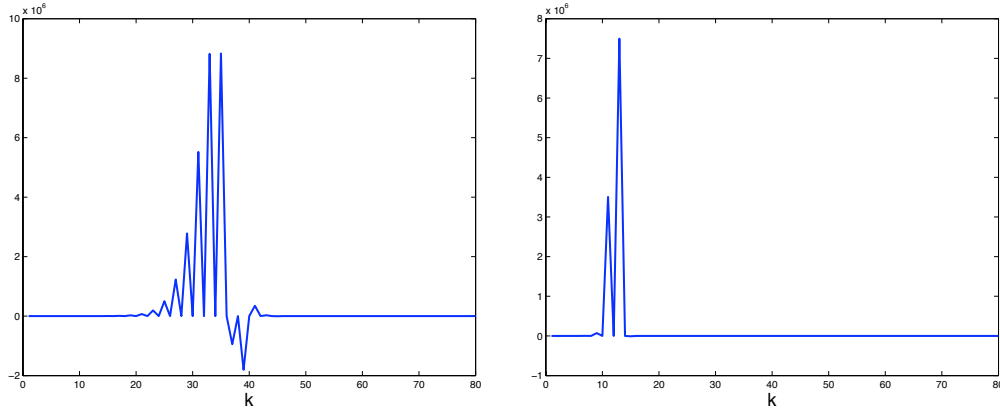


Figure 16: $T = 1$, $\omega = (0.2, 0.8)$, $N = 80$. Distribution of the Fourier coefficients a_k of ϕ_T^N , $k \in [1, N]$ for $(t_1, t_2) = (0.98, 0.99)$ (**Left**) and $(t_1, t_2) = (0.8, 0.9)$ (**Right**).

Remark 7 *Many other perturbation and regularization mechanisms, at the continuous or numerical level, can be implemented. For instance, we may replace the L^2 -norm by the H_0^1 -one in the definition of J . The corresponding control solves then the following equation*

$$\begin{cases} v - v'' = \phi \mathcal{X}_\omega & \text{in } Q_T, \\ v(\cdot, 0) = v(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

This has the effect of smoothing the control near $t = T$ but does not reduce the instability of the problem.

One could also use different approximations such as the C^1 -finite elements, which permit to better reproduce the smoothness of solutions of the heat and the bi-grid strategy (efficient for the wave equation, see [18]) which consists in projecting the descent direction on a coarser spatial grid, at each iteration of the CG algorithm. But we did not observe any essential improvement on the behavior of the algorithms.

4 The control transmutation method

In this section, we search for a control function in $C(T, y_0, 0)$ for (1) driving an initial datum y_0 in $L^2(\Omega)$ to rest at time $t = T$. As mentioned above, this problem is completely equivalent to that of driving an initial datum in $L^2(\Omega)$ to a final target y_T which is the value at time $t = T$ of a solution of the heat equation (controlled or not).

Here we do not consider the control of minimal L^2 -norm since, as we have seen in previous sections, it oscillates tremendously at $t = T$ thus making its efficient computation very difficult. We rather use the so-called *control transmutation method* ([31]) which permits to obtain efficiently a null control for the heat equation, through a time-convolution with the control of the corresponding wave equation. This method, originally introduced in the context of PDE, can also be applied to time semi-discrete systems. Actually it can be easily extended to an abstract semigroup setting.

With this in mind, in this section we develop the transmutation method at the computational level. More precisely, we use it to give an explicit representation formula for the controls of the $1 - d$ heat equation and its finite-difference space discretizations, and

to develop an efficient method to compute numerical approximations of controls. At this level, the rich literature on efficient numerical methods for computing controls of the wave equation plays a key role.

To fix ideas, we perform the change of variables $(\tilde{t}, \tilde{x}) = (ct, x)$ so that the system (1) becomes

$$\begin{cases} \tilde{y}' - \Delta \tilde{y} = c^{-1} \tilde{v} \mathcal{X}_\omega, & \text{in } Q_{\tilde{T}}, \\ \tilde{y} = 0, & \text{on } \Sigma_{\tilde{T}}, \\ \tilde{y}(0, \cdot) = y_0, & \text{in } \Omega \end{cases}$$

with $\tilde{y}(\tilde{t}, \tilde{x}) = y(t, x)$, $\tilde{v}(\tilde{t}, \tilde{x}) = v(t, x)$ and $\tilde{T} = cT$. This normalizes the diffusivity constant c to be $c = 1$ but, its influence is seen in the length of the time interval. In the next section, we drop the tilde for notational simplicity.

4.1 Principle of the method

The goal being to write the control for the heat equation, driving an initial datum y_0 to the zero state at time $t = T$, we first address the control problem for the wave equation: To find a control $f \in L^2([0, L] \times \Omega)$ such that the solution w of the wave equation

$$\begin{cases} w_{ss} - \Delta w = f \mathcal{X}_\omega & (s, x) \in (0, L) \times \Omega, \\ w = 0 & (s, x) \in (0, L) \times \partial\Omega, \\ (w(0, \cdot), w'(0, \cdot)) = (y_0, 0) & x \in \Omega \end{cases} \quad (22)$$

be driven at time $s = L$ to zero: $(w(L), w'(L)) = (0, 0)$ in Ω .

Note that s plays the role of time in this wave equation. But in the application we have in mind it is a pseudo-time parameter, that will be integrated on, to get the dynamics of the controlled heat equation in the real time t .

When the GCC is satisfied, such a control exists for all initial data of final energy in $H_0^1(\Omega) \times L^2(\Omega)$.

In the one-dimensional case, this condition holds if $L \geq L(\omega)$ is large enough, the minimal time of control $L(\omega)$ being twice the length of the maximal segment included in $\Omega \setminus \omega$.

We then extend the solution (w, f) to $(-L, L) \times (0, T)$ as follows:

$$\underline{w}(s, x) = w(s, x) = \underline{w}(-s, x), \quad (s, x) \in (0, L) \times \Omega,$$

and similarly

$$\underline{f}(s, x) = f(s, x) = \underline{f}(-s, x), \quad (s, x) \in (0, L) \times \Omega$$

so that the support of \underline{w} is included in $-L \leq t \leq L$. Then $\underline{w} \in C([-L, L]; H_0^1(\Omega)) \cap C^1([-L, L]; L^2(\Omega))$ and $\underline{f} \in L^2((-L, L) \times \Omega)$.

Secondly, consider the fundamental controlled solution $H \in C^0([0, T], \mathcal{M}(-L, L))$ of the one-dimensional heat equation, satisfying

$$\begin{cases} \partial_t H - \partial_s^2 H = 0 & \text{in } \mathcal{D}'((0, T) \times (-L, L)), \\ H(t = 0) = \delta, \quad H(t = T) = 0. \end{cases} \quad (23)$$

Here and in what follows $\mathcal{M}(-L, L)$ denotes the space of Radon measures on $(-L, L)$. The kernel $H \in L^2(]0, T[\times]-L, L[)$ is extended to $\underline{H} \in L^2(\mathbb{R}^2)$ by $\underline{H} = H$ in $]0, T[\times]-L, L[$ and $\underline{H} = 0$ elsewhere.

Then, the transmutation formulas

$$y(t, x) = \int_{\mathbb{R}} \underline{H}(t, s) \underline{w}(s, x) ds, \quad v(t, x) = \mathcal{X}_{\omega}(x) \int_{\mathbb{R}} \underline{H}(t, s) \underline{f}(s, x) ds,$$

which can be regarded as the analogue of Kannai's formula for the heat kernel at a control theoretical level, define the functions $y \in L^2(\mathbb{R}; H_0^1(\Omega))$ and $v \in L^2(\mathbb{R} \times \Omega)$ such that

$$\begin{cases} y_t - \Delta y = v \mathcal{X}_{\omega} & (t, x) \in (0, T) \times \Omega, \\ y = 0 & (t, x) \in (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0, \quad y(T, \cdot) = 0 & x \in \Omega. \end{cases}$$

Therefore, $v \mathcal{X}_{\omega}$ is a null control for y , solution of the heat equation in Ω with the initial state y_0 , so that v belongs to $\mathcal{C}(T, y_0, 0)$.

Note that this construction requires the initial datum y_0 to be controlled to belong to $H_0^1(\Omega)$ but this is not a restriction since, because of the regularizing effect of the heat equation, by simply letting it evolve freely, solutions starting from $L^2(\Omega)$ enter instantaneously in $H_0^1(\Omega)$.

Observe that the formulas above transform the pseudo-time s for the wave equation into the true time t for the heat equation, the controlled heat kernel \underline{H} playing the role of a convolution kernel. Note also that the time variable s in (22) plays the role of the space variable in (23), explaining the term *transmutation*.

Using the symmetries, we have for all $(t, x) \in (0, T) \times \Omega$

$$y(t, x) = 2 \int_0^L H(t, s) w(s, x) ds, \quad v(t, x) = 2 \mathcal{X}_{\omega}(x) \int_0^L H(t, s) f(s, x) ds. \quad (24)$$

Summarizing, the control transmutation method reduces the null controllability for (1) to the determination of a control for the corresponding linear wave equation plus a fundamental controlled solution for the heat equation. The first point, although sensitive in practice, is now well-understood and efficiently solved numerically, at least for linear homogeneous wave equations on simple geometries (see [39]). In the next section, we discuss how a solution of system (23) can be efficiently computed.

Note also that the transmutation method not only applies to the heat equation but that can be extended to a general class of parabolic like problems and, in particular, to the semi-discrete heat equation in which case the problem of null controllability is reduced to that of the controllability of the corresponding semi-discrete wave equation.

It is important to observe that this method yields controls that differ significantly from those obtained by minimizing functionals of the form J^* , which are solutions of the adjoint heat equation. Here, the controls of the wave equation of minimal L^2 -norm are solutions of the wave equation in the variables (x, s) , since the wave operator is self-adjoint. Thus, the controls v that this transmutation method provides are restrictions to ω of solutions of the heat equation and, therefore, do not coincide with those obtained minimizing functionals of the form J^* .

4.2 Determination of a fundamental controlled solution for the heat equation

This section is devoted to derive explicitly a solution H of (23) following [19] (we also refer to [23] and [8], section 2.5.3) in the one dimensional case.

Let δ be any parameter in $(0, T)$. In the time interval $(0, \delta)$, the function H is taken to be the Gaussian, fundamental solution of the heat equation:

$$H(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad (t, x) \in (0, \delta) \times \mathbb{R}.$$

Therefore, it remains to join $H(\delta, x)$ to the 0 state at time T .

For any $a > 0$ and any $\alpha \geq 1$, we consider the *bump* function

$$h(s) = \exp\left(-\frac{a}{((s-\delta)(T-s))^\alpha}\right), \quad s \in (\delta, T) \quad (25)$$

and then the function

$$p(t) = \frac{1}{\sqrt{4\pi t}} \begin{cases} 1 & t \in (0, \delta) \\ \frac{\int_t^T h(s) ds}{\int_\delta^T h(s) ds} & t \in (\delta, T) \end{cases} \quad (26)$$

so that $p(T) = 0$. Note that $h \in C_c^\infty([\delta, T])$ and $p \in C^\infty([0, T])$. Functions h and p are both Gevrey functions of order $1 + 1/\alpha \in (1, 2]$ ²: this property implies that the series

$$H(t, x) = \sum_{k \geq 0} p^{(k)}(t) \frac{x^{2k}}{(2k)!} \quad (27)$$

converges.

Moreover, (27) defines a solution of the heat equation and satisfies, by definition of p , the null-controllability condition $H(T, x) = 0$ for all $x \in \mathbb{R}$ and $\lim_{t \rightarrow 0^+} H(t, x) = \delta_{x=0}$. In other terms, the restriction to $(0, T) \times (-L, L)$ of H given by (27) is a solution of the controlled problem (23). Remark that, by construction, the function $x \rightarrow H(t, x)$ is odd so that $\partial_x H(t, x) = 0$ at $x = 0$. From this parametrization, the control v for the system (1) takes the following expression

$$\begin{aligned} v(t, x) &= 2 \int_0^L \sum_{k \geq 0} p^{(k)}(t) \mathcal{X}_\omega(x) \frac{s^{2k}}{(2k)!} f(s, x) ds \\ &= 2 \mathcal{X}_\omega(x) \sum_{k \geq 0} p^{(k)}(t) \int_0^L \frac{s^{2k}}{(2k)!} f(s, x) ds \end{aligned}$$

with $f \in L^2((0, L) \times \Omega)$ a control for (22).

On $(0, \delta)$, the control v and the controlled solution y are simply

$$(y(t, x), v(t, x)) = \frac{2}{\sqrt{4\pi t}} \int_0^L e^{-\frac{s^2}{4t}} \left(w(s, x), f(s, x) \mathcal{X}_\omega(x) \right) ds \quad (t, x) \in (0, \delta) \times \Omega.$$

The function H is not unique and depends on the choice of the function h and on the parameters δ, a and α .

Note that the value L is constrained by the size of the set $\Omega \setminus \omega$, due to the finite velocity of propagation of the wave equation and to the GCC.

This explicit representation formulas for the solutions of the heat equation has been used in [23] in the context of approximate controllability.

²A function $z \in C^\infty([0, T]; \mathbb{R})$ is Gevrey of order $s \in [1, \infty[$ if there exist $M > 0$ and $R > 0$ such that $|z^{(m)}(t)| \leq M(m!)^s R^{-m}$, for all $t \in [0, T]$ and $m \in \mathbb{N}$ (see [8], p. 86 and [23]).

Numerically, the difficulty is to perform a robust evaluation of the series (27), and, in particular, a correct evaluation of the (highly oscillating) derivatives of the function p . Figure 17 depicts the fundamental solution for two values of δ obtained with the series (27) truncated to the first forty terms and illustrates the influence of the parameter δ . In view of the definition of p given by (26), we observe that the L^2 -norm of H increases with δ .

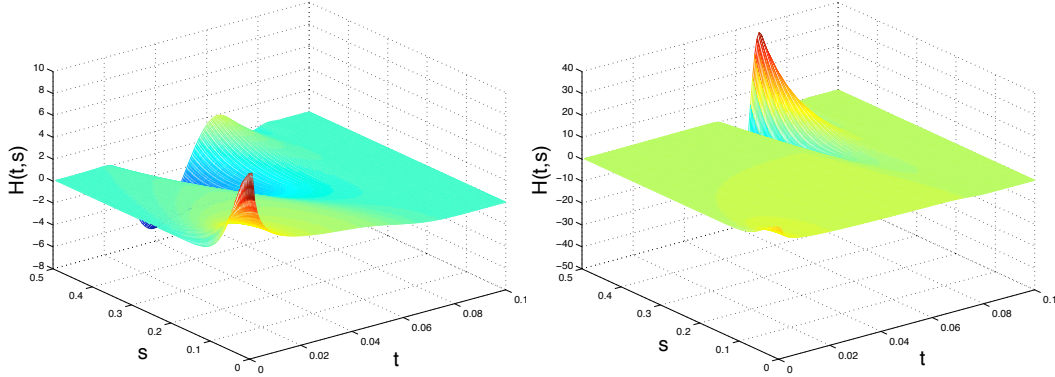


Figure 17: $L = 0.5$, $T = 0.1$. Fundamental solution H on $(0, T) \times (0, L)$ for $(a, \alpha, \delta) = (10^{-2}, 1, T/5)$ (**Left**) and $(a, \alpha, \delta) = (10^{-2}, 1, T/2)$ (**Right**) .

4.3 Numerical experiments

Once the controlled fundamental heat solution above is computed, we *simply* have to control the wave system (22) and then to apply the transmutation representation to get the control of the heat equation.

We consider once again $\omega = (0.2, 0.8)$, $T = 1$ and $c = 0.1$ and assume that the initial condition to be controlled is $y_0 = \sin(\pi x)$. In this case (ω, Ω, L) satisfy the GCC if and only if $L \geq 0.4$. We take $L = 0.5$ and H given by Figures 17. Figure 18 first depicts the controlled wave solution w of (22) and its control of minimal L^2 -norm $f\mathcal{X}_\omega$ on $(0, L) \times \Omega$. The corresponding controlled heat solution y and its control v are given on Figure 19. From the choice of the function p which defines H , the control v vanishes at time T (the support in time of H and y coincide).

As a consequence, the control we obtain is not of minimal L^2 -norm. The function $t \rightarrow \|v(t, \cdot)\|_{L^2(\omega)}$ is given Figure 20-Left and $\|v\|_{L^2((0, T) \times \omega)} \approx 8.74 \times 10^{-1}$ which is greater than the minimal L^2 -norm control which is approximately 2.43×10^{-1} . But, as mentioned above, this is in agreement with the structure of the control itself, which is the restriction to ω of a solution of the heat equation, while the controls of minimal norm are obtained as solutions of the adjoint heat equation.

To validate the numerical efficiency of the method we evaluate the solution y associated with this control v corresponding to several initial data y_0 . We first consider the case $y_0(x) = \sin(\pi x)$. We do it with $h = \Delta t = 1/100$ and the scheme (of order one in time and two in space) used in Section 2.3. We obtained $\|y_h(T, \cdot)\|_{L^2(\Omega)} \approx 4.17 \times 10^{-7}$. This value is small enough and shows that the transmutation method is indeed an efficient way of computing the null controls for the heat equation.

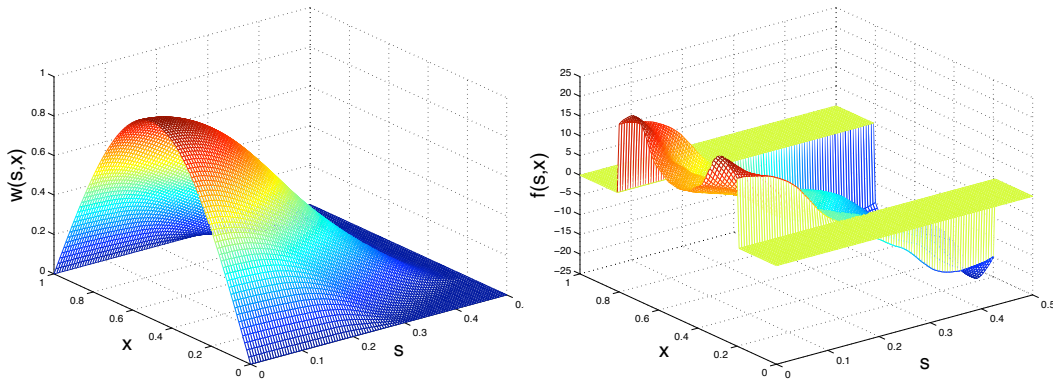


Figure 18: Wave equation: $y_0(x) = \sin(\pi x), L = 0.5, \omega = (0.2, 0.8)$. Controlled wave solution w (Left) and the corresponding control f on $(0, L) \times \Omega$ (Right).

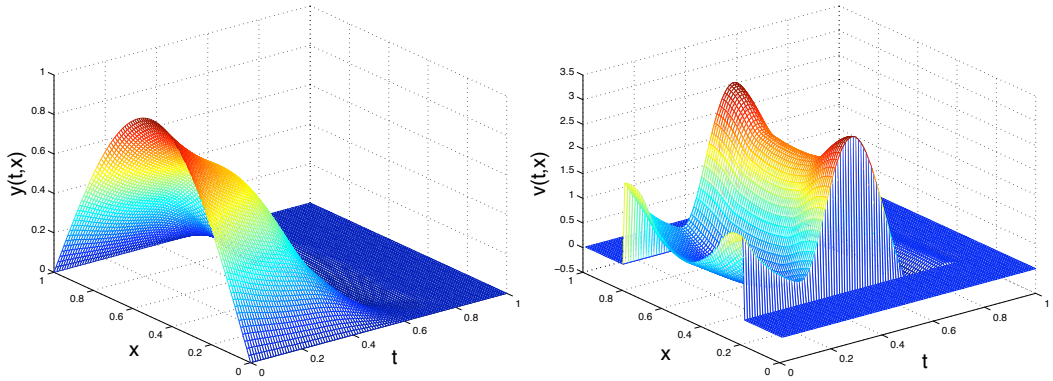


Figure 19: $y_0(x) = \sin(\pi x), T = 1, c = 1/10$ and $(\delta, \alpha) = (T/5, 1)$. Controlled heat solution y (Left) and corresponding transmuted control v on $(0, T) \times \Omega$ (Right).

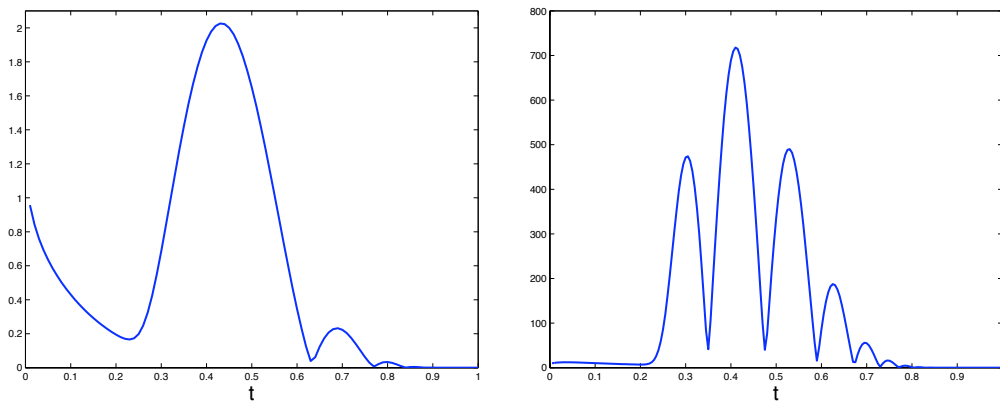


Figure 20: $L^2(\omega)$ -norm of the control v vs time $t \in [0, T]$ for $(y_0(x), T, c) = (\sin(\pi x), 1, 1/10)$ (Left) and $(y_0(x), T, c) = (\sin(3\pi x), 1, 1/5)$ (Right).

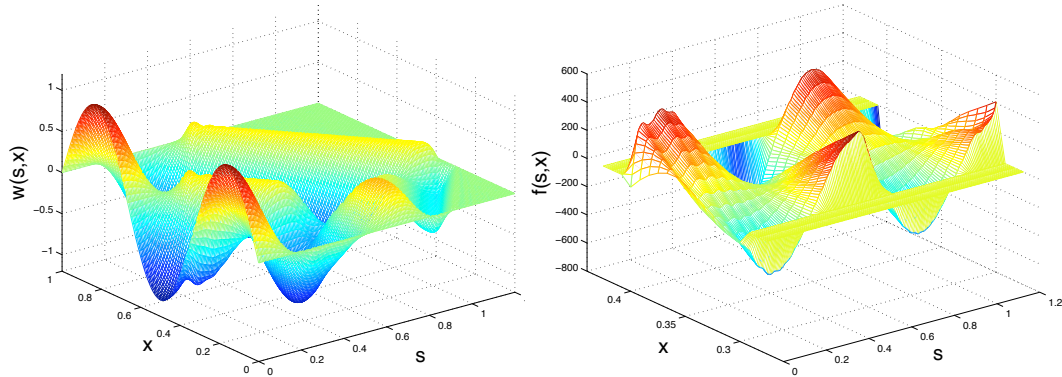


Figure 21: $y_0(x) = \sin(3\pi x)$ and $L = 1.2 - \omega = (0.3, 0.4)$. Controlled wave solution w on $(0, L) \times \Omega$ (**Left**) and corresponding HUM control f on $(0, L) \times (0.25, 0.45) \subset (0, L) \times \Omega$ (**Right**).

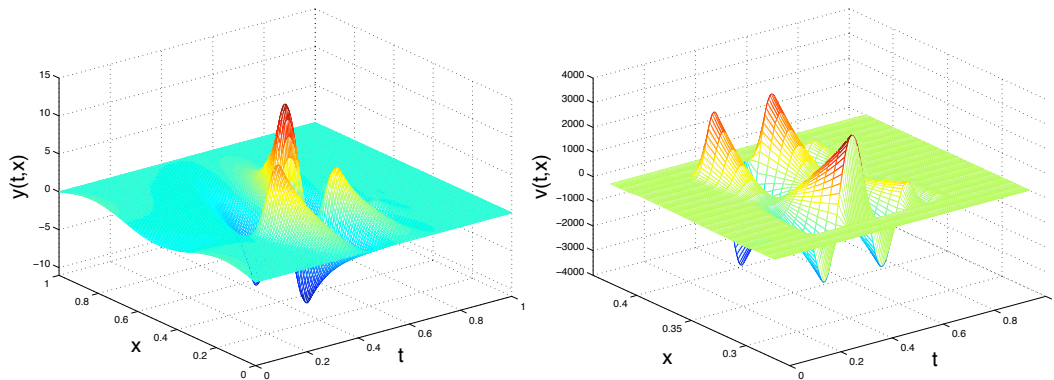


Figure 22: $(y_0(x), T, c, \delta, \alpha) = (\sin(3\pi x), 1, 1/20, T/5, 1)$ and $\omega = (0.3, 0.4)$. Controlled heat solution y on $(0, T) \times \Omega$ (**Left**) and corresponding transmuted control v on $(0, T) \times (0.25, 0.45) \subset (0, T) \times \Omega$ (**Right**).

We now take $\omega = (0.3, 0.4)$, $y_0(x) = \sin(3\pi x)$ and $T = 1$. In order to compensate the faster dissipation of $\sin(3\pi x)$ (with respect to $\sin(\pi x)$), we take a smaller diffusivity coefficient c equal to $1/20$. The choice of ω requires $L \geq 1.2$. Due to the smaller size of ω , this second example is much more singular: the conditioning number of the control operator is so large that the CG algorithm fails to converge to the minimum of J^* , as soon as the discretization parameters h and Δt are small enough. Thus, the control of minimal L^2 -norm can not even be computed. Figure 21, on the right, depicts the wave control f on $(0, L) \times (0.26, 0.44)$ and on the left the corresponding wave solution w on $(0, L) \times \Omega$. The computation of the fundamental heat solution H , taking once again $\delta = T/5$, leads to the controlled heat solution y given on Figure 22-Left. The control has a very oscillatory nature (see Figure 22 (Right)) in the region $(0.2, 0.8) \subset (0, T)$ and its size is of the order of $\|v\|_{L^2((0,T) \times \omega)} \approx 1.41 \times 10^2$. These oscillations are related to the structure of the fundamental solution H . Finally, this control used with a finite element approximation of (1) provides the following estimation for the solution at time T : $\|y_h(T, \cdot)\|_{L^2(\Omega)} \approx 1.07 \times 10^{-6}$. According to the singularity of the problem (due to the smallness of the control set ω) and to the numerical errors in the integration of Kannai type formula (24), this value appears to be very acceptable and one can consider the control as being a very accurate approximation of an actual null control.

Remark 8 *In view of the formula,*

$$v(t, x) = \frac{2}{\sqrt{4\pi t}} \int_0^L e^{-\frac{s^2}{4t}} f(s, x) \mathcal{X}_\omega(x) ds, \quad (t, x) \in (0, \delta) \times \Omega$$

we could ensure that $v(0, \cdot) = 0$ in Ω , by choosing a compact support wave control f for w . More precisely, let $A \in (0, L)$ and assume that $f(s, x) = 0$ in $(0, A) \times \Omega$. Then,

$$v(t, x) = \frac{2}{\sqrt{4\pi t}} \int_A^L e^{-\frac{s^2}{4t}} f(s, x) \mathcal{X}_\omega(x) ds \rightarrow 0 \quad \text{as } t \rightarrow 0, \quad \text{a.e. } x \in \Omega.$$

The control obtained through the transmutation method is not unique since the fundamental solution H is not unique. It may be interesting to minimize the norm $\|v\|_{L^2(Q_T)}$ of the control with respect to the parameters $(\alpha, \delta) \in [1, \infty) \times (0, T)$. Kannai's formula (24) implies that $\|v\|_{L^2(Q_T)} \leq 2\|f\|_{L^2((0,L) \times \Omega)} \|H\|_{L^2((0,T) \times (0,L))}$ so that, *in a first approximation*, the minimization can be performed on the $1-d$ controlled heat kernel H . As observed on Figure 17, a smaller value of δ leads to a smoother function H : this reduces the time period on which the dynamics is governed by the Gaussian. Thus, for the first example with $y_0(x) = \sin(\pi x)$, $\delta = T/5$ leads to $\|v\|_{L^2(Q_T)} \approx 8.7 \times 10^{-1}$ while $\delta = T/10$ gives $\|v\|_{L^2(Q_T)} \approx 7.9 \times 10^{-1}$. On the other hand, a large value of α has the effect of concentrating the support of the function $t \rightarrow H(t, \cdot)$ on the center of the time interval and increasing the amplitude of the control. However, we can replace, at least in a formal way (see [23] where the authors compute boundary controls using divergent series and Gevrey functions of order greater than two), the function h defined in (25) by

$$h(s) = \exp\left(-\frac{a}{(s-\delta)^{\alpha_1}(T-s)^{\alpha_2}}\right). \quad (28)$$

Then, first, take α_1 greater than one, which reduces the values of H near $t = 0$ (as just mentioned) but permits to take a small value of δ and, secondly, take α_2 smaller than one which will allow an active control v near $t = T$. Figure 23-Left depicts the fundamental heat

solution at $s = L = 0.5$ obtained with $\delta = T/10$ and $(\alpha_1, \alpha_2) = (1.1, 0.7)$. The $L^2(\Omega)$ -norm of the corresponding control is given on Figure 23-Right and, as expected, we observe a reduction of the cost: $\|v\|_{L^2(Q_T)} \approx 5.67 \times 10^{-1}$. Remark that the oscillations near T , due to the ones in H , are reminiscent of those obtained for the minimal L^2 -norm control (see Figure 2).

As mentioned above, however, and as it is clear in the plots of the controls obtained through the transmutation method, their shape differs significantly from that of the controls of minimal norm obtained by minimizing functionals of the form J^* . This is particularly true when t approaches the final time $t = T$.

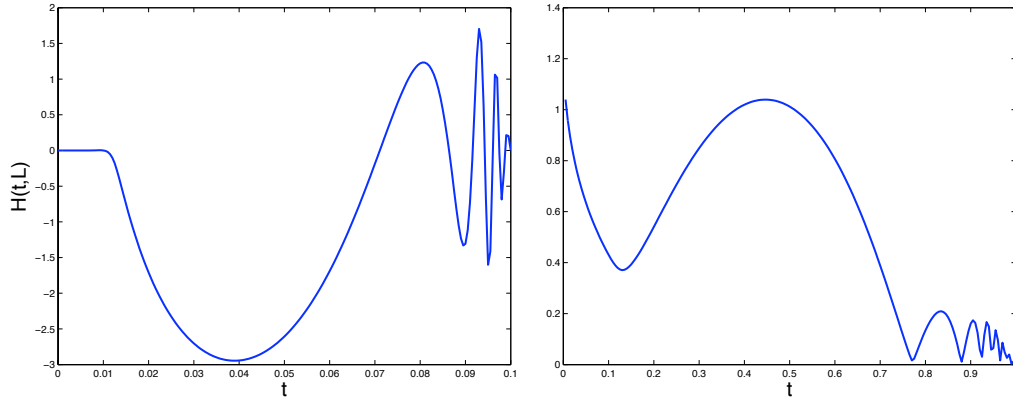


Figure 23: Data: $(y_0(x), c) = (\sin(\pi x), 1/10)$. The heat fundamental solution $H(t, L)$ vs. time $t \in [0, T]$ (**Left**) and $L^2(\Omega)$ -norm of corresponding control v (**Right**).

5 Concluding remarks and perspectives

In this work, we have developed and discussed some numerical methods and experiments related to the distributed null or trajectory controllability of the one dimensional heat equation. This paper is, to our best knowledge, the very first one addressing these issues. Up to now, the existing literature was mainly devoted to the approximate controllability problem.

The high frequency oscillations of the L^2 -controls of minimal norm with respect to the time variable illustrate how rich and complex the control theory for parabolic system may be. It provides a fascinating example of the so-called severally ill-posed problems that render ineffective standard numerical methods. In particular we have illustrated the failure of gradient type methods -widely used for the control of wave processes - even in the framework of convergent and uniformly controllable finite difference schemes.

Figure 7-Left highlights clearly the lack of robustness of the approximation. In order to improve the robustness, similarly as when dealing with PDE's with singularities, it would be very interesting and useful to exhibit singular basis functions for the heat equation that, added to the usual finite polynomial element basis, would allow us to represent better the singularities of the adjoint solutions near T (we refer to Figure 3). As we stressed in Section 4, the functions used to construct the fundamental heat solution H (see (26)) could help on doing this.

We have also seen that, as the consequence of the exponential ill-posedness, the standard

regularization based methods - such as the Tychonoff approach - reduce significantly the quality of the approximation. However, it would be interesting to develop a systematic analysis of the method yielding the best compromise between rate of convergence of the approximation and computational cost. There could be useful elements in the existing extensive literature on ill-posed problems for parabolic equation.

At last, we have seen that the transmutation controlled method provides a quite simple and effective way to approximate a null control for the heat equation, smooth and vanishing near T . Based on the use of controls for a wave equation and Kannai formula, this method offers many advantages. In particular, it may also be applied in the context of fully discrete schemes. Moreover, if the support of the control satisfies the GCC, the transmutation control method may be applied in any dimension. The question is more subtle when the control domain does not fulfill the GCC which, on the other hand, is unnecessary in the context of the heat equation. Despite of this, one expects the transmutation method to be useful for building the null control of the heat equation even when GCC is not fulfilled, because its failure may be compensated by the analyticity of the data to be controlled (see [25]).

Many efforts remain to be done to better understand the numerical approximation of controls for parabolic type equations. We close this section with a list of open problems and future research lines. The list is not exhaustive but collects some of the main issues that arised along the paper:

- **Preconditioners for the CG method.** We have shown the inefficiency of the CG algorithm to derive the optimal control because of the very ill-posedness of the problem. It would be interesting to find efficient preconditioners for the CG algorithm to compensate (to some extent) this ill-posedness.
- **Time-depending weights.** As we have seen, the use of time-depending weights allows to derive more efficient numerical algorithms since they allow to reduce the oscillations of the adjoint state near $t = T$. From a theoretical viewpoint their use is fully justified since the null controllability property is guaranteed by the existing observability inequalities. Note however that the space \mathcal{H}_ρ where the corresponding functional J_ρ^* can be minimized depends on this weight ρ . It would be interesting to analyze what are the optimal weights in various respects: Size of the space \mathcal{H}_ρ , conditioning of the functional J_ρ^* , number of iterations of CG algorithms, etc.
- **Choice of the $1 - d$ controlled heat fundamental solution in transmutation.** As we have seen, the transmutation method can be applied with a whole family of fundamental controlled $1 - d$ kernels H . It would be interesting to analyze the dependence of the controls this leads to, depending on the parameters entering in the definition of H .

Note however that the control of minimal L^2 -norm derived by minimizing functionals of the form J^* can not be obtained within the family of controls obtained through the method of transmutation. Indeed, while the first one is the restriction to ω of a solution of the adjoint heat equation, the second one is a solution of the forward heat problem, two facts that can only arise simultaneously in the trivial null function.

- **Transmutation for fully discrete schemes.** As we have seen, following [31], transmutation can be used to represent controls for heat equations in terms of the controls of wave equations both in the continuous and semi-discrete (time-continuous)

case. A complete analysis of the possible use of this method to build convergent controls for fully discrete approximations of the heat equation is still to be done.

- **Convergence rates for the Tychonoff regularization.** As we have seen, Tychonoff regularization provides a way of approximation the controls, regularizing its highly oscillating behavior at $t = T$. However, as our numerical simulations show, its efficiency is low. It would be interesting to make a systematic analysis of the convergence rates one can get. Following the analysis in [34], one can easily show that, whenever the minimizer ϕ_T of J^* that, in principle, belongs to the huge space \mathcal{H} , belongs to $L^2(\Omega)$, then the minima of the functional regularized by means of the Tychonoff method converge with a polynomial rate. However, the *a priori* assumption that ϕ_T belongs to $L^2(\Omega)$ is of very little practical use since it can not be written in terms of the regularity of the data to be controlled. It would be interesting to see if, using the very properties of \mathcal{H} , that make solutions of the adjoint system with data in \mathcal{H} to belong to $L^2(\Omega)$ for all $t < T$, one can get logarithmic convergence rates without the *a priori* assumption that ϕ_T belongs to $L^2(\Omega)$.

The difficulty of the problem of the regularity of the minimizer ϕ_T of J^* can be easily observed. Let us try to see, for instance, if $\phi_T = \sin(\pi x)$ can be the minimizer of the functional J^* associated to the adjoint system (3) so that it yields the control of minimal L^2 -norm for some $y_0 \in L^2(0, 1)$. If that holds, taking into account that the corresponding solution ϕ is of the form $\phi(x, t) = \exp(-c\pi^2(T - t)) \sin(\pi x)$, then

$$\int_0^T \int_{\omega} \exp(-c\pi^2(T - t)) \sin(\pi x) \psi(x, t) dx dt + \int_{\Omega} y_0(x) \psi(x, 0) dx = 0,$$

for all solution ψ of the adjoint system (3). Thus, if y_0 is written in Fourier series as

$$y_0(x) = \sum_{j \geq 1} y_{0,j} \sin(j\pi x),$$

this is equivalent to the fact that the Fourier coefficients of y_0 , $\{y_{0,j}\}$, satisfy

$$\frac{c_{1,j}(\omega) \int_0^T \exp(-c\pi^2(1 + j^2)(T - t)) dt}{2} + \exp(-c\pi^2 j^2 T) \frac{y_{0,j}}{2} = 0,$$

where $c_{1,j}(\omega)$, as above, denotes $c_{1,j}(\omega) = 2 \int_{\omega} \sin(\pi x) \sin(\pi j x) dx$. This means that

$$y_{0,j} = -\exp(c\pi^2 j^2 T) c_{1,j}(\omega) \frac{1 - \exp(-c\pi^2(1 + j^2)T)}{c\pi^2(1 + j^2)}.$$

This Fourier coefficients grow exponentially as $j \rightarrow \infty$. Thus, clearly, the corresponding datum y_0 that it corresponds to does not belong to any Sobolev space of negative order.

This shows how complex is the link between the data to be controlled and the minimizers of the corresponding functionals J^* . This question is also related to the following one.

- **Regularity of the adjoint state.** Above we have discussed the convergence rates of the Tychonoff method under the assumption that ϕ_T belongs to $L^2(\Omega)$. As far as we know there is no example of data to be controlled in the literature for which the datum at time $t = T$ of the adjoint state, ϕ_T , belongs to $L^2(\Omega)$. It would be interesting to

exhibit such examples or, by the contrary, show that the optimal controls are such that ϕ_T develops singularities away from ω as soon as ω is strictly included and does not coincide with Ω . This problem is also related to that of identifying classes of data for which the existing numerical methods are better behaved or, more generally, to develop *ad'hoc* methods for specific classes of data to be controlled.

- **The multi-dimensional case.** This paper is devoted to the $1 - d$ case but most of our methods can be extended to the multidimensional one. Note however that, in principle, the application of the transmutation method requires the GCC to be satisfied, which imposes restrictions on the geometry of the support of the control. In that sense, in the multi-dimensional setting, an interesting new open problem arises: That of developing efficient numerical methods for the control of multi-dimensional heat processes in the absence of GCC. See the next item.
- **Transmutation in the absence of GCC.** As we have mentioned above, it would be interesting to study analytically and also numerically the possible use of the transmutation method to build controls for the heat equation in the absence of the GCC for the wave equation.
- **Optimal support of the control.** It appears also interesting to analyze deeper the ill-posedness of the problem with respect to the distribution of the support of the control (we refer to [33] in that direction). We also mention the unstudied situation where the support depends on the time variable.
- **Boundary control.** It would also be worth to investigate the boundary control problem developing the methods we have presented in this paper. This problem could even be more unstable numerically than the internal control problem we have considered here.
- **More general parabolic problems.** It would be also interesting to address variable coefficients heat equations and parabolic systems.

Acknowledgments: This work was partially done while the first author was visiting the Basque Center for Applied Mathematics (BCAM - Bilbao). He wishes to thank the members of the Center for their kind hospitality.

6 Appendix

6.1 Semi-discrete approximation of Λ_h

In this appendix, we derive the spatial approximation Λ_h of the control operator Λ associated with a semi-discrete approximation of (1). We use the centered finite approximation on a uniform mesh which produces uniform observability properties. The adjoint system (3) is discretized as follows

$$\begin{cases} \Phi'_h(t) - K_h \Phi_h(t) = 0 & \text{in } Q_{Th} \equiv (0, T) \times \Omega_h, \\ \Phi_h = 0 & \text{on } \Sigma_{Th} \equiv (0, T) \times \partial\Omega_h, \\ \Phi_h(T, \cdot) = \Phi_{hT}(\cdot) & \text{in } \Omega_h, \end{cases}$$

where $\{\Phi_h\}_{h>0}$ denotes a vector of dimension $n = \dim(\Omega_h)$ and $K_h \in \mathbb{R}^{n \times n}$ is the stiffness matrix of order n , discretization of $-c\partial_{xx}^2$. Using the spectral decomposition of K_h so that $V_h, D_h \in \mathbb{R}^{n \times n}$ so that $K_h = V_h D_h V_h^{-1}$, we get that $\Phi_h = V_h Q_h$ with

$$Q_h = \{q_j\}_{1 \leq j \leq n}, \quad q_j(t) = q_j(T) \exp(-\lambda_j(T-t)), \quad 1 \leq j \leq n.$$

Similarly, if we denote by $C_h \in \mathbb{R}^{n \times n}$ the diagonal matrix associated with \mathcal{X}_ω ($C_h = Id_n$ if $\omega = \Omega$), the vectorial approximation Y_h of y is solution of

$$\begin{cases} Y_h'(t) + K_h Y_h(t) = C_h \Phi_h(t) & \text{in } Q_{Th}, \\ Y_h = 0 & \text{on } \Sigma_{Th}, \\ Y_h(0, \cdot) = 0 & \text{in } \Omega_h. \end{cases}$$

From the spectral decomposition of K_h , the vector $Z_h(t) = V_h^{-1} Y_h(t)$ is solution of

$$\begin{cases} Z_h'(t) + D_h Z_h(t) = V_h^{-1} C_h V_h \Phi_h(t) & \text{in } Q_{Th}, \\ Z_h = 0 & \text{on } \Sigma_{Th}, \\ z_h(0, \cdot) = 0 & \text{in } \Omega_h, \end{cases}$$

so that each component z_j of Z_h solves the ordinary differential equation :

$$z_j'(t) + \lambda_j z_j(t) = \sum_k B_{jk} q_k(T) \exp(-\lambda_k T) \exp(\lambda_k t),$$

where $B_h = \{B_{jk}\}_{1 \leq j, k \leq n}$ by $B_h = V_h^{-1} C_h V_h \in \mathbb{R}^{n \times n}$ and $\{\lambda_j\}_{1 \leq j \leq n}$ are the eigenvalues of K_h . We then get

$$z_j(t) = C_j \exp(-\lambda_j t) + \sum_{k=1}^n \frac{B_{jk} q_k(T) \exp(-\lambda_k T) \exp(\lambda_k t)}{\lambda_k + \lambda_j}, \quad 1 \leq j \leq n.$$

The constant C_j is fixed from the initial condition at time $t = 0$:

$$0 = z_j(0) = C_j + \sum_k \frac{B_{jk} q_k(T) \exp(-\lambda_k T)}{\lambda_k + \lambda_j}, \quad 1 \leq j \leq n.$$

so that

$$z_j(T) = \sum_{k=1}^n B_{jk} \frac{1 - \exp(-(\lambda_j + \lambda_k)T)}{\lambda_j + \lambda_k} q_k(T) \quad 1 \leq j \leq n.$$

At last, introducing the matrix $M_h \in \mathbb{R}^{n \times n}$ so that $Z_h(T) = M_h Q_h(T)$, we obtain the vectorial equality $Y_h(T) = V_h M_h V_h^{-1} \Phi_h(T)$. The approximation Λ_h of Λ is then

$$\Lambda_h = V_h M_h V_h^{-1}, \quad h > 0.$$

Remark that the matrix V_h is orthogonal so that the computation of V_h^{-1} is straightforward.

The final structure of Λ_h is similar to the spectral projection Λ_N of Λ with $N \sim 1/h$ but with the spectrum associated to the finite-difference laplacian.

6.2 Approximation of the control operator Λ^ϵ associated to the singular damped wave equation $y_{\epsilon,t} - cy_{\epsilon,xx} + \epsilon y_{\epsilon,tt} = 0$

We give in this appendix the expression of the matrix of order $2N$, approximation to the first N modes of the control application $\Lambda^\epsilon : L^2(\Omega) \times H^{-1}(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ associated with system (17) and defined by $\Lambda^\epsilon(\phi_{T\epsilon}, \phi'_{T\epsilon}) := (y_{T\epsilon}, y'_{T\epsilon})$. As in Section 2.1, we use Fourier decomposition. First, we expand in Fourier series the solution ϕ of the adjoint system

$$\begin{cases} -\phi_{\epsilon,t} - c\phi_{\epsilon,xx} + \epsilon\phi_{\epsilon,tt} = 0, & \text{in } Q_T, \\ \phi_\epsilon = 0 & \text{on } \Sigma_T, \\ (\phi_\epsilon(T, \cdot), \phi'_\epsilon(T, \cdot)) = (\phi_{T\epsilon}, \phi'_{T\epsilon}) & \text{in } \Omega. \end{cases}$$

We get $\phi_\epsilon(t, x) = \sum_{k \geq 1} a_k^\epsilon(t) \sin(k\pi x)$ with

$$a_k^\epsilon(t) = \left(f_1(\mu_k) a_k^0 + f_2(\mu_k) a_k^1 \right) e^{-\frac{(1+\mu_k)}{2\epsilon}(T-t)} + \left(f_1(-\mu_k) a_k^0 + f_2(-\mu_k) a_k^1 \right) e^{-\frac{(1-\mu_k)}{2\epsilon}(T-t)}$$

and $f_1(s) = (s-1)/(2s)$, $f_2(s) = \epsilon/s$ and

$$\mu_k = \sqrt{1 - 4ck^2\pi^2\epsilon}, \quad k \geq 1 \quad (29)$$

so that $(\phi_{T0}(x), \phi_{T1}(x)) = \sum_{k \geq 1} (a_k^0, a_k^1) \sin(k\pi x)$. Then, we solve the system

$$\begin{cases} y_{\epsilon,t} - cy_{\epsilon,xx} + \epsilon y_{\epsilon,tt} = \phi_\epsilon \mathcal{X}_\omega & \text{in } Q_T, \\ y_\epsilon = 0 & \text{on } \Sigma_T, \\ (y_\epsilon(0, \cdot), y'_\epsilon(0, \cdot)) = (0, 0) & \text{in } \Omega \end{cases} \quad (30)$$

assuming that $y_\epsilon(t, x) = \sum_{p \geq 1} b_p^\epsilon(t) \sin(p\pi x)$ and obtain

$$b_p^\epsilon(t) = K_p^1 e^{\frac{1}{2\epsilon}(-1+\mu_p)t} + K_p^2 e^{-\frac{1}{2\epsilon}(1+\mu_p)t} + \sum_{k \geq 1} c_{p,k}(\omega) g_{k,p}(t), \quad p \geq 1$$

for some constants $\{K_p^1, K_p^2\}_{(p \geq 1)}$ and

$$\begin{aligned} g_{k,p}(t) &= \frac{2\epsilon(2a_k^1\epsilon - a_k^0 + \mu_k a_k^0)}{\mu_k(4\mu_k + 3 + \mu_k^2 + 4cp^2\pi^2\epsilon)} e^{-\frac{1}{2\epsilon}(1+\mu_k)(T-t)} \\ &\quad - \frac{2\epsilon(2a_k^1\epsilon - a_k^0 - \mu_k a_k^0)}{\mu_k(-4\mu_k + 3 + \mu_k^2 + 4cp^2\pi^2\epsilon)} e^{-\frac{1}{2\epsilon}(1-\mu_k)(T-t)}, \quad \forall k, p \geq 1. \end{aligned}$$

The coefficients $c_{p,k}(\omega)$ are defined in (6). Then, after simple but tedious computations taking into account the initial condition in (30), we obtain that the matrices of order $2N$ which link the coefficients $\{a_k^0, a_k^1\}_{1 \leq k \leq N}$ to the coefficients $\{b_p^\epsilon(T), b_p^{\epsilon'}(T)\}_{1 \leq p \leq N}$ are given by

$$\begin{pmatrix} \{b_p^\epsilon(T)\}_{1 \leq p \leq N} \\ \{b_p^{\epsilon'}(T)\}_{1 \leq p \leq N} \end{pmatrix}_{2N \times 1} = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}_{2N \times 2N} \begin{pmatrix} \{a_k^0\}_{1 \leq k \leq N} \\ \{a_k^1\}_{1 \leq k \leq N} \end{pmatrix}_{2N \times 1} \quad (31)$$

where the matrix $A^{ij} \in \mathbb{R}^{N \times N}$, $1 \leq i, j \leq 2$ are defined as follows, for all $1 \leq k, p \leq N$:

$$\begin{aligned} A_{pk}^{11} &= f_1(\mu_k) \left(B_{kp}^{11} e^{-\frac{-2+\mu_p-\mu_k}{2\epsilon}T} + B_{kp}^{21} e^{-\frac{-2-\mu_p-\mu_k}{2\epsilon}T} + c_{p,k}(\omega) A_{kp}^1 \right) \\ &\quad + f_1(-\mu_k) \left(B_{kp}^{12} e^{-\frac{-2+\mu_p+\mu_k}{2\epsilon}T} + B_{kp}^{22} e^{-\frac{-2-\mu_p+\mu_k}{2\epsilon}T} + c_{p,k}(\omega) A_{kp}^2 \right), \end{aligned}$$

$$\begin{aligned}
A_{pk}^{12} &= f_2(\mu_k) \left(B_{kp}^{11} e^{-\frac{-2+\mu_p-\mu_k}{2\epsilon} T} + B_{kp}^{21} e^{-\frac{-2-\mu_p-\mu_k}{2\epsilon} T} + c_{p,k}(\omega) A_{kp}^1 \right) \\
&\quad + f_2(-\mu_k) \left(B_{kp}^{12} e^{-\frac{-2+\mu_p+\mu_k}{2\epsilon} T} + B_{kp}^{22} e^{-\frac{-2-\mu_p+\mu_k}{2\epsilon} T} + c_{p,k}(\omega) A_{kp}^2 \right), \\
A_{pk}^{21} &= f_1(\mu_k) \left(\frac{(-1+\mu_p)}{2\epsilon} B_{kp}^{11} e^{-\frac{-2+\mu_p-\mu_k}{2\epsilon} T} - \frac{(1+\mu_p)}{2\epsilon} B_{kp}^{21} e^{-\frac{-2-\mu_p-\mu_k}{2\epsilon} T} + \frac{(1+\mu_k)}{2\epsilon} c_{p,k}(\omega) A_{kp}^1 \right) \\
&\quad + f_1(-\mu_k) \left(\frac{(-1+\mu_p)}{2\epsilon} B_{kp}^{12} e^{-\frac{-2+\mu_p+\mu_k}{2\epsilon} T} - \frac{(1+\mu_p)}{2\epsilon} B_{kp}^{22} e^{-\frac{-2-\mu_p+\mu_k}{2\epsilon} T} + c_{p,k}(\omega) \frac{(1-\mu_k)}{2\epsilon} A_{kp}^2 \right), \\
A_{pk}^{22} &= f_2(\mu_k) \left(\frac{(-1+\mu_p)}{2\epsilon} B_{kp}^{11} e^{-\frac{-2+\mu_p-\mu_k}{2\epsilon} T} - \frac{(1+\mu_p)}{2\epsilon} B_{kp}^{21} e^{-\frac{-2-\mu_p-\mu_k}{2\epsilon} T} + \frac{(1+\mu_k)}{2\epsilon} c_{p,k}(\omega) A_{kp}^1 \right) \\
&\quad + f_2(-\mu_k) \left(\frac{(-1+\mu_p)}{2\epsilon} B_{kp}^{12} e^{-\frac{-2+\mu_p+\mu_k}{2\epsilon} T} - \frac{(1+\mu_p)}{2\epsilon} B_{kp}^{22} e^{-\frac{-2-\mu_p+\mu_k}{2\epsilon} T} + c_{p,k}(\omega) \frac{(1-\mu_k)}{2\epsilon} A_{kp}^2 \right),
\end{aligned}$$

with

$$\left\{ \begin{array}{l}
f_1(\mu_k) = \frac{\mu_k - 1}{2\mu_k}, \quad f_2(\mu_k) = \frac{\epsilon}{\mu_k}, \\
B_{kp}^{11} = -\frac{1}{2\mu_p} c_{p,k}(\omega) A_{kp}^1 (2 + \mu_k + \mu_p), \quad B_{kp}^{12} = -\frac{1}{2\mu_p} c_{p,k}(\omega) A_{kp}^2 (2 + \mu_p - \mu_k), \\
B_{kp}^{21} = \frac{1}{2\mu_p} c_{p,k}(\omega) A_{kp}^1 (2 + \mu_k - \mu_p), \quad B_{kp}^{22} = \frac{1}{2\mu_p} c_{p,k}(\omega) A_{kp}^2 (2 - \mu_k - \mu_p), \\
A_{kp}^1 = \frac{4\epsilon}{4\mu_k + 3 + \mu_k^2 + 4cp^2\pi^2\epsilon}, \quad A_{kp}^2 = \frac{4\epsilon}{-4\mu_k + 3 + \mu_k^2 + 4cp^2\pi^2\epsilon}.
\end{array} \right.$$

For simplicity, we have assumed that $y'_\epsilon(0) = 0$ in (30) but any other choices are possible. The square matrix of order $2N$ defined in (31) is an approximation of the operator Λ^ϵ . In practice, for any N fixed, we take ϵ small enough so that $1 - 4c(N\pi)^2\epsilon \geq 0$ and then $\mu_k \in \mathbb{R}$ for all $1 \leq k \leq N$.

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