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Decay Estimates for some Semilinear Damped Hyperbolic Problems

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Abstract

Let Ω be a bounded open domain in $\boldsymbol{R}^n$, $g: \boldsymbol{R} \to \boldsymbol{R}$ a non-decreasing continuous function such that $g(0) = 0$ and $h \in L^1_{\text{loc}}(\boldsymbol{R}^+; L^2(\Omega))$. Under suitable assumptions on g and h, the rate of decay of the difference of two solutions is studied for some abstract evolution equations of the general form $u'' + Lu + g(u') = h(t, x)$ as $t \to +\infty$. The results, obtained by use of differential inequalities, can be applied to the case of the semilinear wave equation

$$u_{tt} - \Delta u + g(u_t) = h \quad \text{in } \boldsymbol{R}^+ \times \Omega, \qquad u = 0 \text{ on } \boldsymbol{R}^+ \times \partial\Omega.$$

For instance if $g(s) = c\,|s|^{p-1}\,s + d\,|s|^{q-1}\,s$ with $c, d > 0$ and $1 < p \leqq q$, $(n-2)\,q \leqq n+2$, then if $h \in L^\infty(\boldsymbol{R}^+; L^2(\Omega))$, all solutions are bounded in the energy space for $t \geqq 0$ and if u, v are two such solutions, the energy norm of $u(t) - v(t)$ decays like $t^{-1/(p-1)}$ as $t \to +\infty$.

Introduction and notation

Let Ω be a bounded open domain in $\boldsymbol{R}^n$ and $H = L^2(\Omega)$ with norm and inner product respectively denoted by $|\cdot|$ and $(\cdot,\cdot)$. Let V be a real Hilbert space such that $V \subset H$ with dense and continuous imbedding. We denote by $\|\cdot\|$ the norm on V, by $a(\cdot,\cdot)$ the inner product on V and by $L \in \mathcal{L}(V, V')$ the unique operator such that $\langle Lu, v\rangle = a(u, v)$ for all $(u, v) \in V \times V$.

Let $g: \boldsymbol{R} \to \boldsymbol{R}$ be a non-decreasing continuous function such that $g(0) = 0$ and $h \in L^1_{\text{loc}}(\boldsymbol{R}^+; H)$. We consider the nonlinear partial differential equation of evolution

$$\begin{aligned} &u'' + Lu + g(u') = h \quad \text{in } \boldsymbol{R}^+ \times \Omega, \\ &u \in C(\boldsymbol{R}^+; V) \cap C^1(\boldsymbol{R}^+; H). \end{aligned} \tag{0.1}$$

The initial-value problem associated to (0.1) has been studied by J.-L. LIONS, & W. A. STRAUSS [12], L. AMERIO & G. PROUSE [1], H. BREZIS [3] and more recently in [6] and [8].

The problem is essentially well posed in the class $C(R^+; V) \cap C^1(R^+; H)$. More precisely, in [6] it is shown that for any $[u_0, v_0] \in V \times H$, there is a unique solution of some variational formulation of this initial-value problem such that $u(0) = u_0$ and $u'(0) = v_0$ under the assumption that the set $\{v \in V: g(v) \in H\}$ is dense in H.

When in addition g is assumed odd, uniqueness is known for solutions in the sense of distributions on $R^+ \times \Omega$ under the (natural) additional regularity assumption $g(u')\, u' \in L^1_{\mathrm{loc}}(R^+; L^1(\Omega))$, at least when $V = H_0^1(\Omega)$ and $L = -\Delta$.

Without additional hypotheses on g, solutions of (0.1) can be unbounded in V even if $h = h(t): R^+ \to H$ is C^∞ and periodic (resonance phenomenon). If (0.1) has a solution $u \in L^\infty(R^+; V) \cap W^{1,\infty}(R^+; H)$ we say that $h(t, x)$ is *non-resonant*. In this case *any* solution of (0.1) is also bounded in this sense. In particular, any function of the form $h(x) \in H$ is non-resonant, since $u^* = L^{-1} h \in V$ is a solution of the problem.

If g is strictly monotone, it is known that in the non-resonant case the difference of two arbitrary solutions of (0.1) tends to zero weakly in V as $t \to +\infty$ (*cf. e.g.* [6] and [9]). Under some coerciveness and growth assumptions on g, the convergence in V is strong.

Our purpose here is to estimate the rate of decay of such differences under suitable assumptions on g. Such results, in a slightly more restricted framework, are proved in [13] and [14] by a rather complicated method in which mean-values and Stepanov spaces play a basic role. Essentially if $V = H_0^1(\Omega)$, $L = -\Delta$, $g(s) = c\,|s|^{p-1} s$ with $c > 0$, $p > 1$ and $(n-2)\,p \leqq n+2$, it is shown that if h is non-resonant and u, v are two solutions of (0.1), then $\|u(t) - v(t)\| + |u'(t) - v'(t)|$ decays like $t^{-1/(p-1)}$ as $t \to +\infty$. Subsequently in [5] this result was generalized by a similar technique to the case where $g(s) = c\,|s|^{p-1} s + d\,|s|^{q-1} s$ with $c, d > 0$ and $1 < p \leqq q$, $(n-2)\,q \leqq n+2$ under the additional restrictions $q \leqq 2(p+1)$ and $[(n-2)(p+2) - 2n]\,q \leqq (n+2)\,p + 4$. In this paper we present a different method, *based on the construction of suitable Liapunov functionals*. This method enables us to remove this extraneous restriction on p, q.

We treat separately the autonomous and the non-autonomous cases, for which slightly different types of hypotheses and estimates are required. It seems more reasonable to make this distinction in a first approach, although it is in fact possible to include both cases in the same general theorem, *cf.* Remark 3.2. below.

The paper is divided in 4 sections. The first section contains some background material concerning the initial-value problem, boundedness properties on the half-line and some differential inequalities. Section 2 is devoted to the autonomous case and Section 3 to the general case. In Section 4 we give the main examples of applications and some remarks on the possible extensions of the method.

Throughout the text we shall use the standard notation concerning vector-valued Sobolev spaces on an interval of R as well as Stepanov spaces. The L^p norm over Ω will be denoted by $\|\cdot\|_p$. The usual norm on V' will be denoted by $\|\cdot\|_*$. We also use the notation $D(L) := \{v \in V: Lv \in H\}$.

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1. Preliminaries

In this section we recall the main results concerning the existence, uniqueness and regularity of solutions to the initial value problem associated to (0.1) and we establish some lemmas which will be used in Sections 2–3. First we recall the main theorem of [6] concerning existence, uniqueness, stability and regularity of solutions.

Theorem 1.1. *Let* $g \in C(R)$ *be non-decreasing and such that*

$$\{v \in V: g(v) \in H\} \text{ is dense in } H. \tag{1.1}$$

Then for any $[u_0, v_0] \in V \times H$ *and* $h \in L^1_{\mathrm{loc}}(R^+; H)$ *there is a unique solution* $u = u(t, x)$ *of*

$$u \in C(R^+; V) \cap C^1(R^+; H), \tag{1.2}$$

$$u(0) = u_0,\ u'(0) = v_0, \tag{1.3}$$

$$\forall \tau > 0,\ \forall w \in H_0^1(0, \tau; H) \cap L^2(0, \tau; V),$$

$$\int_0^\tau \int_\Omega [j(w) - j(u')]\,dx\,dt \geqq \int_0^\tau [(u', w') - a(u, w)]\,dt + \int_0^\tau \int_\Omega h(w - u')\,dx\,dt$$
$$+ \tfrac{1}{2}\{\|u(\tau)\|^2 + |u'(\tau)|^2\} - \tfrac{1}{2}\{\|u_0\|^2 + |v_0|^2\} \tag{1.4}$$

where $j(s) := \int_0^s g(\sigma)\,d\sigma \quad \forall s \in R$.

In addition, the solution u *has the following properties.*

i) **"Stability".** *If* u_0, v_0, h *are replaced respectively by* $\hat{u}_0, \hat{v}_0, \hat{h}$, *then the corresponding solution* $\hat{u}$ *of* (1.2)–(1.4) *is such that*

$$\forall t \geqq 0,\ \{\|u(t) - \hat{u}(t)\|^2 + |u'(t) - \hat{u}'(t)|^2\}^{1/2} \leqq \{\|u_0 - \hat{u}_0\|^2 + |v_0 - \hat{v}_0|^2\}^{1/2}$$
$$+ \int_0^t |h(s) - \hat{h}(s)|\,ds. \tag{1.5}$$

ii) **"Regularity".** *If in addition we assume* $u_0 \in D(L)$, $v_0 \in V$ *with* $g(v_0) \in H$ *and* $h \in W^{1,1}_{\mathrm{loc}}(R^+; H)$, *then we have*

$$u \in W^{2,\infty}_{\mathrm{loc}}(R^+; H) \cap W^{1,\infty}_{\mathrm{loc}}(R^+; V), \tag{1.6}$$

$$g(u')\,u' \in L^1_{\mathrm{loc}}(R^+; L^1(\Omega)) \ [\textit{in particular } g(u') \in L^1_{\mathrm{loc}}(R^+; L^1(\Omega))]$$

and

$$\int_t^{t+1} \int_\Omega g(u')\, u'\, dx\, ds \leqq E_u(t) + 2^{1/2} \sup_{s\in[t,t+1]} [E_u(s)]^{1/2} \int_t^{t+1} |h(s)|\, ds, \forall\, t \geqq 0 \tag{1.7}$$

with

$$E_u(t) = \tfrac{1}{2}\{\|u(t)\|^2 + |u'(t)|^2\}.$$

Moreover, if V is such that

$$V \cap L^\infty(\Omega) \text{ is dense in } V \tag{1.8}$$

then

$$g(u') \in L^\infty_{\text{loc}}(R^+; V'), \tag{1.9}$$

$$u'' + Lu + g(u') = h(t, x) \quad \text{in} \quad L^\infty_{\text{loc}}(R^+; V'). \tag{1.10}$$

The following general non-resonance is proved also in [6].

Theorem 1.2. *Assume that* (1.8) *holds, that h satisfies*

$$h \in S^1(R^+; H) = \left\{ h \in L^1_{\text{loc}}(R^+; H)/h^* := \sup_{t\geqq 0} \int_t^{t+1} |h(s)|\, ds < +\infty \right\} \tag{1.11}$$

and that g is such that

$$\exists\, \alpha > 0, \exists\, C_1 \geqq 0: g(z)\, z \geqq \alpha\, |z|^2 - C_1 \quad \forall\, z \in R, \tag{1.12}$$

$$\exists\, \gamma > 0, \exists\, C_2 \geqq 0: \|g(v)\|_* \leqq \gamma \int_\Omega g(v)\, v\, dx + C_2 \tag{1.13}$$

(in the sense that if $g(v)\, v \in L^1(\Omega)$ then $g(v) \in V'$ and (1.13) *holds.)*

Assume, in addition that at least one of the two following conditions is fulfilled:

$$2\gamma h^* < 1, \tag{1.14}$$

$$\forall\, \delta > 0, \exists\, \varepsilon > 0 \text{ such that for any measurable set } S \subset [0,1] \text{ with } \operatorname{meas}(S) \leqq \varepsilon, \text{ we have } \sup_{\tau\geqq 0} \int_S |h(t+\tau)|\, dt < \delta. \tag{1.15}$$

Then any solution of (0.1) *in the sense of Theorem* 1.1 *satisfies*

$$E_u(t) := \tfrac{1}{2}\{\|u(t)\|^2 + |u'(t)|^2\} \in L^\infty(0, +\infty). \tag{1.16}$$

Remark 1.3. Some conditions sufficient for h to satisfy properties (1.11) and (1.15) are:

a) h is S^1-almost periodic: $R^+ \to H$. In particular if h is periodic: $R^+ \to H$, condition (1.11) implies (1.15).

b) $h \in S^{1+\varepsilon}(R^+; H)$ for some $\varepsilon > 0$. In particular it is sufficient that $h \in L^\infty(R^+; H)$. •

In the estimates for the rate of decay, it will be convenient to work with solutions in $W^{1,\infty}_{\text{loc}}(R^+; V) \cap W^{2,\infty}_{\text{loc}}(R^+; H)$ and to use Remark 1.4 and Lemma 1.5 below.

Remark 1.4. If h is non-resonant and $\hat{h} \in L^1_{\text{loc}}(R^+; H)$ is such that $h - \hat{h} \in L^1(R^+; H)$, then it follows from the stability inequality (1.5) that $\hat{h}$ is also non-resonant. In particular if $h \in L^1_{\text{loc}}(R^+; H)$ is non-resonant, there exists a sequence $h_n \in C^1(R^+; H)$ with $h_n - h \in L^1(R^+; H)$ for all $n \in N$ and

$$h_n - h \to 0 \quad \text{in } L^1(R^+; H) \quad \text{as } n \to +\infty.$$

Then by approximating also the initial data $[u_0, v_0] \in V \times H$ we can approach the solution $u(t)$ in $C(R^+; V) \cap C^1(R^+; H)$ by a sequence of "regular" solutions u_n in the sense that $u_n \in W^{2,\infty}_{\text{loc}}(R^+; H) \cap W^{1,\infty}_{\text{loc}}(R^+; V)$ for all $n \in N$. This application of the stability property (1.5) is reminiscent of the methods of H. BREZIS [2]. •

Lemma 1.5. *Let $T > 0$ and $w \in W^{1,\infty}(0, T; V) \cap W^{2,\infty}(0, T; H)$.*

Then the functions $\|w(t)\|^2 + |w'(t)|^2$ and $(w(t), w'(t))$ are Lipschitz continuous with

$$\tfrac{1}{2} \frac{d}{dt}(\|w(t)\|^2 + |w'(t)|^2) = \langle Lw(t) + w''(t), w'(t) \rangle \quad \text{a.e. on } [0, T], \tag{1.17}$$

$$\frac{d}{dt}(w(t), w'(t)) = |w'(t)|^2 + \langle w''(t), w(t) \rangle \quad \text{a.e. on } [0, T]. \tag{1.18}$$

In the proofs of Theorems 2.1 and 3.1 we will find some differential inequalities for some modified energy functionals. In the following Lemmas we solve these inequalities and deduce some decay estimates which will be used in the proof of the main results.

Lemma 1.6. *Let $T > 0$, $F \in W^{1,1}(0, T)$, $F \geqq 0$ in $[0, T]$ be such that*

$$F'(t) \leqq -\varrho[F(t)]^a \quad \text{a.e on } [0, T], \tag{1.19}$$

where $\varrho > 0$, $a \geqq 1$ are two constants.

Then if $a > 1$ we have

$$F(t) \leqq k t^{-\alpha} \quad \text{on } [0, T] \tag{1.20}$$

with $\alpha := \dfrac{1}{a-1}$ *and* $k := [\varrho(a-1)]^{-\alpha}$.

If $a = 1$ we have

$$F(t) \leqq F(0)\, e^{-\varrho t} \quad \text{on } [0, T]. \tag{1.21}$$

Lemma 1.7. *Let F and f be two nonnegative functions: $\mathbf{R}^+ \to \mathbf{R}$ such that*

$$F \in W^{1,1}_{\text{loc}}(\mathbf{R}^+), \tag{1.22}$$

$$f \in L^1_{\text{loc}}(\mathbf{R}^+) \quad \textit{with } f^* := \sup_{t \geq 0} \int_t^{t+1} f(s)\, ds < \eta \tag{1.23}$$

$$F'(t) \leqq (f(t) - \eta)\, [F(t)]^a \quad \text{a.e. } \textit{on } [0, T] \tag{1.24}$$

with $a \geqq 1$. Then if $a > 1$ we have

$$F(t) \leqq k t^{-\alpha} \quad \textit{for all } t \geqq T^* := \frac{2f^*}{\eta - f^*} \tag{1.25}$$

with $\alpha := \dfrac{1}{a-1}$ and $k := [\frac{1}{2}(\eta - f^*)(a-1)]^{-\alpha}$.

If $a = 1$ we have

$$F(t) \leqq F(0)\, e^{-[(\eta - f^*)/2]t} \quad \textit{for all } t \geqq T^*. \tag{1.26}$$

Idea of the proofs. The case $a = 1$ is standard. When $a > 1$ we argue as follows: if $F(t_0) = 0$ for some finite t_0, then by Gronwall's Lemma $F(t) \equiv 0$ for $t \leqq t_0$ and the result follows for $t \geqq t_0$. On the other hand if $F(t) > 0$ for $0 \leqq t < T$, then we introduce $G(t) = [F(t)]^{1-a}$ and compute $G'(t)$. The result follows immediately for $t \in [T^*, T[$ by integrating and computing $F(t)$ from $G(t)$. •

2. The autonomous case

Throughout this section we consider the autonomous problem of hyperbolic type

$$u'' + Lu + g(u') = h(x) \quad \text{in } \mathbf{R}^+ \times \Omega,$$

$$u \in C(\mathbf{R}^+; V) \cap C^1(\mathbf{R}^+; H),$$

i.e. equation (0.1) with h independent of t.

A particular solution of this problem is the (unique) equilibrium solution $u^* := L^{-1}(h)$. If $g(s) \neq 0$ for all $s \neq 0$, any solution $u(t)$ of this equation converges to u^* in V and $u'(t)$ converges to zero in H as $t \to +\infty$ (*cf.* [6]–[7]). The main result of this section concerns the rate of decay of $u(t) - u^*$ in the energy space and can be stated as follows.

Theorem 2.1. *Assume that $g \in C(\mathbf{R})$ is a non-decreasing function and that condition (1.8) is satisfied. Then we have the following results:*

i) *Let $\gamma \in [1, +\infty[$ be such that $V \subset L^{\gamma+1}(\Omega)$ with continuous and dense imbedding, and assume that there are constants c, C with $0 < c \leqq C < +\infty$ and $p \in [1, \gamma]$ such that*

$$c\,|z|^p \leqq |g(z)| \quad \forall z \in \mathbf{R}, \tag{2.1}$$

$$|g(z)| \leqq C(|z| + |z|^\gamma) \quad \forall z \in \mathbf{R}. \tag{2.2}$$

Then for any solution of (0.1) we have the following:

a) *If $p = 1$,*

$$\|u(t) - u^*\| + |u'(t)| \leqq M e^{-\delta t} \quad \forall t \geqq 0 \tag{2.3}$$

where M depends on the initial data and δ does not.

b) *If $p > 1$,*

$$\|u(t) - u^*\| + |u'(t)| \leqq M t^{-1/(p-1)} \quad \forall t \geqq 0 \tag{2.4}$$

where M depends on the initial data.

ii) *On the other hand, if $V \subset L^\infty(\Omega)$ with continuous embedding, the same conclusions hold true for any $p \geqq 1$ with (2.2) replaced by*

$$\exists \eta > 0 / |g(z)| \leqq C\,|z| \quad \forall z \in \mathbf{R} : |z| \leqq \eta. \tag{2.2 bis}$$

Proof. If u is replaced by $u - u^*$, it is sufficient to consider the case $h \equiv 0$; then $u^* \equiv 0$.

Let $[u_0, v_0] \in D(L) \times V$ with $g(v_0) \in H$ and let $u = u(t, x)$ be the solution of (0.1) with initial data $[u_0, v_0]$. From Theorem 1.1 the solution u satisfies (1.6) and (1.7).

We introduce

$$\Phi(t) := \tfrac{1}{2}\{\|u(t)\|^2 + |u'(t)|^2\} \tag{2.5}$$

and

$$\Psi(t) := [\Phi(t)]^{(p-1)/2}\,(u(t), u'(t)). \tag{2.6}$$

By (1.6) and Lemma 1.5 we have $\Phi(t) \in W^{1,\infty}_{\text{loc}}(\mathbf{R}^+) \cap L^\infty(\mathbf{R}^+)$ with

$$\Phi'(t) = -\langle g(u'), u'\rangle = -\int_\Omega g(u')\, u'\, dx \leqq -c \int_\Omega |u'|^{p+1}\, dx$$
$$\leqq -c_1\, |u'(t)|^{p+1} \quad \text{a.e. on } \mathbf{R}^+, \tag{2.7}$$

and on the other hand

$$\Psi'(t) = \frac{p-1}{2}\,[\Phi(t)]^{(p-3)/2}\,(u(t), u'(t))\, \Phi'(t)$$
$$+ [\Phi(t)]^{(p-1)/2} \left\{ |u'(t)|^2 - \|u(t)\|^2 - \int_\Omega g(u')\, u\, dx \right\}$$
$$\leqq -C_1 \Phi'(t) + [\Phi(t)]^{(p-1)/2} \left\{ |u'(t)|^2 - \|u(t)\|^2 - \int_\Omega g(u')\, u\, dx \right\} \tag{2.8}$$

with $C_1 = \dfrac{p-1}{2}\,[\Phi(0)]^{(p-1)/2}$ since $\Phi'(t) \leqq 0$ and $\Phi(t) \leqq \Phi(0)$ $\forall t \geqq 0$.

i) In this case we need the following lemma.

Lemma 2.2. *If* (2.2) *is satisfied, then for any* $v \in V$ *we have* $g(v) \in V'$ *and there are constants* $C_2, C_3 > 0$ *(depending only on* C *and* γ*) such that:*

$$\|g(v)\|_* \leqq C_2 \left[\int_\Omega g(v)\, v\, dx\right]^{\gamma/(\gamma+1)} + C_3 \|v\|_{(\gamma+1)/\gamma} \quad \forall v \in V. \tag{2.9}$$

Proof of Lemma 2.2. First, we remark that from (2.2) it follows that

$$|g(z)| \leqq 2C|z| \quad \text{on } \{|z| \leqq 1\}$$

and

$$|g(z)|^{\gamma+1} < 2C|z|^\gamma |g(z)|^\gamma \quad \text{on } \{|z| > 1\}.$$

Then

$$\|g(v)\|_{(\gamma+1)/\gamma} \leqq 2C\|v\|_{(\gamma+1)/\gamma} + (2C)^{1/(\gamma+1)}\left(\int_\Omega g(v)\, v\, dx\right)^{\gamma/(\gamma+1)}. \tag{2.10}$$

On the other hand, from the embedding $V \subset L^{\gamma+1}(\Omega)$ we deduce that $L^{(\gamma+1)/\gamma}(\Omega) \subset V'$ and the estimate (2.9) follows from (2.10). •

By applying Lemma 2.2 we now find

$$\begin{aligned}\left|\int_\Omega g(u')\, u\, dx\right| &= |\langle g(u'), u\rangle| \leqq \|g(u')\|_* \|u\| \\ &\leqq C_2\left(\int_\Omega g(u')\, u'\, dx\right)^{\gamma/(\gamma+1)} \|u\| + C_3 |u'|\, \|u\|\end{aligned} \tag{2.11}$$

and therefore, if we apply the Cauchy-Schwarz inequality to the last term of (2.11), the estimate (2.8) yields

$$\begin{aligned}\Psi'(t) \leqq &-C_1\Phi'(t) + C_4[\Phi(t)]^{(p-1)/2}|u'(t)|^2 - \tfrac{1}{2}[\Phi(t)]^{(p-1)/2}\|u(t)\|^2 \\ &+ C_2[\Phi(t)]^{(p-1)/2}\left(\int_\Omega g(u')\, u'\, dx\right)^{\gamma/(\gamma+1)} \|u(t)\|,\end{aligned} \tag{2.12}$$

where C_4 depends continuously on C_3.

By applying Young's inequality with exponents $(\gamma+1)/\gamma$ and $\gamma+1$ to the last term we have

$$\begin{aligned}\Psi'(t) \leqq &-C_1\Phi'(t) + C_4[\Phi(t)]^{(p-1)/2}|u'(t)|^2 \\ &- \tfrac{1}{4}[\Phi(t)]^{(p-1)/2}\|u(t)\|^2 + C_5 \int_\Omega g(u')\, u'\, dx \\ = &-(C_1 + C_5)\,\Phi'(t) + C_4[\Phi(t)]^{(p-1)/2}|u'(t)|^2 - \tfrac{1}{4}[\Phi(t)]^{(p-1)/2}\|u(t)\|^2.\end{aligned} \tag{2.13}$$

We define

$$\Phi_\varepsilon(t) = (1 + (C_1 + C_5)\,\varepsilon)\,\Phi(t) + \varepsilon\Psi(t) \tag{2.14}$$

and we remark that if $\varepsilon > 0$ is small enough (depending only on $E_u(0)$) then

$$\tfrac{1}{2}\Phi_\varepsilon(t) \leqq \Phi(t) \leqq 2\Phi_\varepsilon(t). \tag{2.15}$$

Now by combining (2.7) and (2.13) we find that

$$\Phi_\varepsilon'(t) \leqq -c_1|u'|^{p+1} - \frac{\varepsilon}{4}[\Phi(t)]^{(p-1)/2}\|u(t)\|^2 + C_4\varepsilon[\Phi(t)]^{(p-1)/2}|u'(t)|^2. \tag{2.16}$$

We now distinguish the cases $p = 1$ and $p > 1$.

a) Case $p = 1$. The result follows from Lemma 1.6 and (2.15) since for ε possibly smaller

$$\Phi_\varepsilon'(t) \leqq -\frac{c_1}{2}|u'|^2 - \frac{\varepsilon}{4}\|u\|^2 \leqq -\frac{\varepsilon}{2}\Phi(t) \leqq -\frac{\varepsilon}{4}\Phi_\varepsilon(t). \tag{2.17}$$

b) Case $p > 1$. We note that from (2.16) the next inequality follows:

$$\Phi_\varepsilon'(t) \leqq -c_1|u'(t)|^{p+1} - \frac{\varepsilon}{2}[\Phi(t)]^{(p+1)/2} + (C_4 + \tfrac{1}{2})\,\varepsilon[\Phi(t)]^{(p-1)/2}|u'(t)|^2. \tag{2.18}$$

Now we have for all $\lambda > 0$

$$[\Phi(t)]^{(p-1)/2}|u'(t)|^2 \leqq (\lambda^{-1}|u'|^2)^{(p+1)/2} + \{\lambda[\Phi(t)]^{(p-1)/2}\}^{(p+1)/(p-1)} \tag{2.19}$$

and by choosing $\lambda > 0$ small enough we deduce from (2.18) and (2.19) that for $\varepsilon > 0$ small we have

$$\Phi_\varepsilon'(t) \leqq -\frac{\varepsilon}{4}[\Phi(t)]^{(p+1)/2}. \tag{2.20}$$

Finally from (2.15) and (2.20) the proof is completed by applying Lemma 1.6.

ii) The case where $V \subset L^\infty(\Omega)$ follows easily from the following lemma.

Lemma 2.3. *If* $z \in V$, *then* $g(z) \in V'$ *and there is a constant* $C > 0$ *such that*

$$\|g(v)\|_* \leqq C\left(|v| + \eta^{-1}\int_\Omega g(v)\, v\, dx\right) \quad \forall v \in V. \tag{2.21}$$

The conclusion now follows by a density argument for any solution of (0.1) (*i.e.* for any $[u_0, v_0] \in V \times H$) and from the fact that the constants that appear in the estimates above depend only on $\Phi(0)$ in a bounded manner. •

Remark 2.4. If, in particular, $V = H_0^1(\Omega)$ and $L = -\Delta$, this proof can be carried out directly for any solution u of (0.1) (for any $[u_0, v_0] \in V \times H$) since the "differentiation of the energy" (2.7) can be justified rigorously also for weak solutions (*cf.* [10]). •

Remark 2.5. Hypothesis (2.1) can be relaxed to

$$c\inf\{|z|, |z|^p\} \leqq |g(z)| \quad \forall z \in R \tag{2.22}$$

for any $p \geqq 1$ (see [16]). •

3. The non-autonomous case

In this section we consider the general case of problem (0.1) and we prove the following result which extends that of M. NAKAO [13] (*cf.* also [7]).

Theorem 3.1. *Assume that* (1.8) *is verified and that* $g \in C(R)$ *is a non-decreasing function satisfying either* i) *or* ii) *below.*

i) *There is a* $\gamma \in [1, +\infty[$ *such that* $V \subset L^{\gamma+1}(\Omega)$ *with continuous embedding, and real constants* c, C, p *with* $0 < c, C < +\infty$ *and* $p \in [1, \gamma]$ *such that*

$$c\,|z_1 - z_2|^p \leqq |g(z_1) - g(z_2)| \qquad \forall\, z_1, z_2 \in R \tag{3.1}$$

$$|g(z_1) - g(z_2)| \leqq C\{1 + [g(z_1)\,z_1 + g(z_2)\,z_2]^{(\gamma-1)/(\gamma+1)}]\}\,|\,|z_1 - z_2| \qquad \forall\, z_1, z_2 \in R. \tag{3.2}$$

ii) $V \subset L^\infty(\Omega)$ *with continuous embedding and there are constants* c, C, *and* p *as above such that* g *satisfies* (3.1) *and*

$$|g(z_1) - g(z_2)| \leqq C(1 + g(z_1)\,z_1 + g(z_2)\,z_2)\,|z_1 - z_2| \qquad \forall\, z_1, z_2 \in R. \tag{3.2 bis}$$

Then if $h \in S^1(R^+; H)$ *is non-resonant and* u, v *are two solutions of* (0.1), *we have the following conclusions*:

a) *If* $p = 1$

$$\|u(t) - v(t)\| + |u'(t) - v'(t)| \leqq Me^{-\delta t} \qquad \forall\, t \geqq 0 \tag{3.3}$$

where M *depends on the initial data and* δ *does not.*

b) *If* $p > 1$

$$\|u(t) - v(t)\| + |u'(t) - v'(t)| \leqq Mt^{-1/(p-1)} \qquad \forall\, t \geqq 0 \tag{3.4}$$

with M *depending on the initial data.*

Proof. We introduce $w = u - v$ and the functionals

$$\Phi(t) := \tfrac{1}{2}\{\|w(t)\|^2 + |w'(t)|^2\}, \tag{3.5}$$

$$\Psi(t) := [\Phi(t)]^{(p-1)/2}\,(w(t), w'(t)). \tag{3.6}$$

As a consequence of Theorem 1.1 and Remark 1.4, we may assume that $h \in C^1(R^+; H)$, $u(0), v(0) \in D(L)$ and $u'(0), v'(0) \in V$ with $g(u'(0)), g(v'(0)) \in H$. In such a case Theorem 1.1 and Lemma 1.5 give $\Phi(t), \Psi(t) \in W^{1,\infty}_{\mathrm{loc}}(R^+)$ with

$$\begin{aligned}\Phi'(t) &= -\langle g(u') - g(v'), w'\rangle = -\int_\Omega (g(u') - g(v'))\,w'\,dx \leqq -c\int_\Omega |w'|^{p+1}\,dx \\ &\leqq -c_1\,|w'(t)|^{\mathrm{PM}}, \quad \text{a.e. in } R^+.\end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned}\Psi'(t) &= \frac{p-1}{2}[\Phi(t)]^{(p-3)/2}\,(w(t), w'(t))\,\Phi'(t) \\ &\quad + [\Phi(t)]^{(p-1)/2}\left\{|w'(t)|^2 - \|w(t)\|^2 - \int_\Omega (g(u') - g(v'))w\,dx\right\} \\ &\leqq -C_1\Phi'(t) + [\Phi(t)]^{(p-1)/2}\left\{|w'(t)|^2 - \|w(t)\|^2 - \int_\Omega (g(u') - g(v'))\,w\,dx\right\}.\end{aligned} \tag{3.8}$$

i) In order to estimate the last term of the right-hand side we write:

$$\begin{aligned}\left|\int_\Omega (g(u') - g(v'))\,w\,dx\right| &\leqq \int_\Omega |[g(u') - g(v')]/w'|\,|w|\,|w'|\,dx \\ &\leqq \int_\Omega |[g(u') - g(v')]/w'|\left[\tau\,|w|^2 + \frac{1}{4\tau}|w'|^2\right]dx\end{aligned} \tag{3.9}$$

and therefore, from (3.2), Hölder's inequality applied with the exponents $(\gamma + 1)/(\gamma - 1)$ and $(\gamma + 1)/2$ and the fact that $V \subset L^{\gamma+1}(\Omega)$ we have

$$\begin{aligned}\left|\int_\Omega (g(u') - g(v'))\,w\,dx\right| &\leqq C\tau\int_\Omega [1 + (g(u')\,u' + g(v')\,v')^{(\gamma-1)/(\gamma+1)}]\,|w|^2\,dx \\ &\quad + \frac{1}{4\tau}\int_\Omega (g(u') - g(v'))\,w'\,dx \\ &\leqq C_1\tau\left\{\|w\|^2 + \left[\int_\Omega (g(u')\,u' + g(v')\,v')\,dx\right]^{(\gamma-1)/(\gamma+1)}\|w\|^2\right\} \\ &\quad + \frac{1}{4\tau}\int_\Omega (g(u') - g(v'))\,w'\,dx.\end{aligned} \tag{3.10}$$

On the other hand the boundedness of the solutions u and v and (1.7) implies that

$$\int_t^{t+1}\int_\Omega [g(u')\,u' + g(v')\,v']\,dx \leqq C(u, v) \qquad \forall\, t \geqq 0. \tag{3.11}$$

Therefore, (3.10) and (3.11) give

$$\left|\int_\Omega (g(u') - g(v'))\,w\,dx\right| \leqq \int_\Omega (g(u') - g(v'))\,w'\,dx + f(t)\,\|w(t)\|^2 \tag{3.12}$$

with

$$f(t) = C_1\tau\left\{\left[\int_\Omega (g(u')\,u' + g(v')\,v')\,dx\right]^{(\gamma-1)/(\gamma+1)} + 1\right\}. \tag{3.13}$$

From (3.8) and (3.13) we find

$$\Psi'(t) \leqq -C_1\Phi'(t) + [\Phi(t)]^{(p-1)/2}$$
$$\times \left\{|w'(t)|^2 - (1-f(t))\,\|w(t)\|^2 + \frac{1}{4\tau}\int_\Omega (g(u')-g(v'))\,w'\,dx\right\}$$
$$= -\left(C_1 + \frac{1}{4\tau}\right)\Phi'(t) + [\Phi(t)]^{(p-1)/2}\{|w'(t)|^2 - (1-f(t))\,\|w(t)\|^2.$$

On the other hand, by choosing τ small enough we have from (3.13)

$$\int_t^{t+1} f(s)\,ds \leqq 2^{-4} \qquad \forall\, t \geqq 0. \tag{3.15}$$

We now define for all $\varepsilon > 0$

$$\Phi_\varepsilon(t) = \left(1 + \left(C_1 + \frac{1}{4\tau}\right)\varepsilon\right)\Phi(t) + \varepsilon\Psi(t). \tag{3.16}$$

It is easily seen that

$$\Phi_\varepsilon'(t) \leqq -\int_\Omega (g(u')-g(v'))\,w'dx - \varepsilon(1-f(t))\,[\Phi(t)]^{(p-1)/2}\,\|w(t)\|^2$$
$$+ \varepsilon[\Phi(t)]^{(p-1)/2}\,|w'(t)|^2 \tag{3.17}$$
$$\leqq -c_1\,|w'|^{p+1} - \varepsilon(1-f(t))\,[\Phi(t)]^{(p-1)/2}\,\|w(t)\|^2 + \varepsilon[\Phi(t)]^{(p-1)/2}\,|w'(t)|^2$$

and for ε small enough

$$\tfrac{1}{2}\,[\Phi_\varepsilon(t)]^{(p+1)/2} \leqq [\Phi(t)]^{(p+1)/2} \leqq 2[\Phi_\varepsilon(t)]^{(p+1)/2} \qquad \forall\, t \geqq 0. \tag{3.18}$$

We now distinguish the cases $p = 1$ and $p > 1$.

a) Case $p = 1$.
The result follows at once from Lemma 1.7, since

$$\Phi_\varepsilon'(t) \leqq -(c_1-\varepsilon)\,|w'|^2 - \varepsilon(1-f(t))\,\|w\|^2$$
$$= -(c_1-\varepsilon)\,|w'|^2 - \varepsilon(1-f(t))[2\Phi(t) - |w'|^2] \tag{3.19}$$

and for ε possibly smaller

$$\Phi_\varepsilon'(t) \leqq -\varepsilon(1-4f(t))\,\Phi_\varepsilon(t) \qquad \forall\, t \geqq 0. \tag{3.20}$$

b) Case $p > 1$.
It follows from (3.17) that

$$\Phi_\varepsilon'(t) \leqq -c_1\,|w'|^{p+1} - 2\varepsilon(1-f(t))\,[\Phi(t)]^{(p+1)/2} + 2\varepsilon[\Phi(t)]^{(p-1)/2}\,|w'(t)|^2. \tag{3.21}$$

Now we have for all $\lambda > 0$

$$[\Phi(t)]^{(p-1)/2}\,|w'(t)|^2 \leqq (\lambda^{-1}|w'|^2)^{(p+1)/2} + \{\lambda[\Phi(t)]^{(p-1)/2}\}^{(p+1)/(p-1)}; \tag{3.22}$$

therefore by choosing $\lambda = 2^{-1}$ we obtain from (3.21)

$$\Phi_\varepsilon'(t) \leqq -(c_1 - \varepsilon C_2)\,|w'|^{p+1} - \varepsilon(1-2f(t))\,[\Phi(t)]^{(p+1)/2} \tag{3.23}$$

for some positive constant C_2.

Finally for ε possibly smaller we obtain

$$\Phi_\varepsilon'(t)) \leqq -\varepsilon\,(\tfrac{1}{2} - 4f(t))\,[\Phi_\varepsilon(t)]^{(p+1)/2}. \tag{3.24}$$

Therefore, from (3.19), (3.24) and (3.15) we get (3.4) by using Lemma 1.7.

ii) If $V \subset L^\infty(\Omega)$, the result can be obtained in a similar way. It is enough to remark that the following estimate follows from (3.2 bis):

$$\left|\int_\Omega (g(u')-g(v'))\,w\,dx\right| \leqq C\tau\,\|w\|^2 \int_\Omega [1 + (g(u')\,u' + g(v')\,v')]\,dx$$
$$+ \frac{1}{4\tau}\int_\Omega (g(u')-g(v'))\,w'\,dx. \tag{3.25}$$

The conclusion now follows by density for any pair of weak solutions of (0.1) from the fact that the constants which appear in the estimates above depend only on $E_u(0)$ and $E_v(0)$ in a bounded manner. •

Remark 3.2. The method of proof of Theorem 3.1 can be used also to recover the results given in Theorem 2.1 in the autonomous case.

Indeed, the proof of Theorem 3.1 shows that it is enough for hypotheses (3.1)–(3.2) (or (3.2 bis)) to be satisfied for $z_1 = u'(t,x)$ and $z_2 = v'(t,x)$, $\forall\,(t,x) \in R^+ \times \Omega$. In the autonomous case $v' \equiv 0$ and our hypotheses are the following:

$$c\,|z|^p \leqq |g(z)| \qquad \forall\, z \in \boldsymbol{R}, \tag{3.26}$$
$$|g(z)| \leqq C\{1 + [g(z)\,z]^{(\gamma-1)/(\gamma+1)}]\}\,|z| \qquad \forall\, z \in \boldsymbol{R}, \tag{3.27}$$
$$|g(z)| \leqq C(1 + g(z)\,z)\,|z| \qquad \forall\, z \in \boldsymbol{R}. \tag{3.27 bis}$$

It is easy to verify that (3.26), (3.27) and (3.27 bis) are respectively equivalent to (2.1), (2.2) and (2.2 bis) of Theorem 2.1. •

Remark 3.3. Hypothesis (3.2) of Theorem 3.1 is easily seen to be equivalent to

$$g \in W^{1,\infty}_{\mathrm{loc}}(\boldsymbol{R}) \text{ with } g'(z) \leqq C\{1 + [g(z)\,z]^{(\gamma-1)/(\gamma+1)}]\} \text{ for almost all } z \in \boldsymbol{R}, \tag{3.28}$$

where C is some non-negative constant. It is satisfied, for example, if the following conditions hold:

$$|g(z_1) - g(z_2)| \leqq C\{1 + |z_1|^{\gamma-1} + |z_2|^{\gamma-1}\}\,|z_1 - z_2| \qquad \forall\, z_1, z_2 \in \boldsymbol{R} \tag{3.29}$$

and

$$|g(z)| \geqq c\,|z|^\gamma - C \qquad \forall\, z \in \boldsymbol{R}. \tag{3.30}$$

Condition (3.2 bis) is easily seen to be equivalent to

$$g \in W^{1,\infty}_{\mathrm{loc}}(\boldsymbol{R}),\ g'(z) \leqq C(1 + g(z)\,z) \quad \text{for almost all } z \in \boldsymbol{R} \text{ and some constant } C \geqq 0. \tag{3.31}$$

For instance, this condition is satisfied if there are constants $K, S \geqq 0$ such that $m(s) := \log(|g(s)|)$ is of class C^1 for $|s| \geqq S$ and $|m'(s)| \leqq K|s|$ for $|s| \geqq S$. •

Remark 3.4. Hypothesis (3.1) can be relaxed to

$$c \inf\{|z_1 - z_2|, |z_1 - z_2|^p\} \leqq |g(z_1) - g(z_2)| \quad \forall z_1, z_2 \in \boldsymbol{R} \tag{3.32}$$

for any $p \geqq 1$ (see [16]). •

4. Examples and additional remarks

Let Ω be a bounded open set in $\boldsymbol{R}^n$. We apply Theorems 2.1 and 3.1 to the following examples of semilinear partial differential equations on $\boldsymbol{R}^+ \times \Omega$.

Example 4.1. Let $V = H_0^1(\Omega)$ and $h = h(t, x) \in L^1_{\text{loc}}(\boldsymbol{R}^+; L^2(\Omega))$. We consider the problem

$$\begin{aligned} u_{tt} - \Delta u + g(u_t) = h \quad &\text{in } \boldsymbol{R}^+ \times \Omega, \\ u = 0 \quad &\text{on } \boldsymbol{R}^+ \times \partial\Omega. \end{aligned} \tag{4.1}$$

It is well known that $V \subset L^p(\Omega)$ with

$$\begin{aligned} &p \in [1, +\infty], \quad \text{if } n = 1, \\ &p \in [1, +\infty[, \quad \text{if } n = 2, \\ &p \in [1, 2n/(n-2)], \quad \text{if } n > 2. \end{aligned} \tag{4.2}$$

We first consider the autonomous case in which $h \equiv h(x)$. The results of Theorem 2.1 hold under the coerciveness assumption (2.1) in the following cases:

a) If $n = 1$, for any $p \geqq 1$ under the additional hypothesis (2.2 bis).

b) If $n = 2$, for any $p \in [1, +\infty[$ under the additional assumption:

$$\exists\, k > 0 \text{ such that } |g(z)| \leqq C\{1 + |z|^k\}\,|z| \quad \forall z \in \boldsymbol{R}. \tag{4.3}$$

c) If $n > 2$, for any $p \in [1, (n+2)/(n-2)]$ under the additional hypothesis

$$|g(z)| \leqq C\{1 + |z|^{4/(n-2)}\}\,|z| \quad \forall z \in \boldsymbol{R}. \tag{4.4}$$

Now let $h = h(t, x) \in S^1(\boldsymbol{R}^+; L^2(\Omega))$ be any non-resonant forcing term for (4.1). The conclusions of Theorem 3.1 hold true for all solutions of (4.1) under the coerciveness assumption (3.1) in the following cases:

a) If $n = 1$, for any $p \geqq 1$ under the additional hypotheses (3.31).

b) If $n = 2$, for any $p \in [1, +\infty[$ under the additional assumption that (3.29)–(3.30) is satisfied for some $\gamma \in [1, +\infty[$.

c) If $n > 2$, for any $p \in [1, (n+2)/(n-2)]$ under the additional hypothesis that (3.29)–(3.30) is satisfied for some $\gamma \in [1, (n+2)/(n-2)]$.

Example 4.2. Let $V = H_0^m(\Omega)$ with $m \geqq 1$ and $h = h(t, x) \in L^1_{\text{loc}}(\boldsymbol{R}^+; L^2(\Omega))$. We consider the problem

$$\begin{aligned} u_{tt} + (-1)^m \Delta^m u + g(u_t) = h \quad &\text{in } \boldsymbol{R}^+ \times \Omega, \\ u = \partial u/\partial \nu = \ldots = \partial^{m-1} u/\partial \nu^{m-1} = 0 \quad &\text{on } \boldsymbol{R}^+ \times \partial\Omega \end{aligned} \tag{4.5}$$

where "$\partial\cdot/\partial\nu$" denotes the derivative in the direction of the outward unit normal ν to Ω.

It is well known that $V \subset L^p(\Omega)$ with

$$\begin{aligned} &p \in [1, +\infty], \quad \text{if } n < 2m, \\ &p \in [1, +\infty[, \quad \text{if } n = 2m, \\ &p \in ([1, 2n/n - 2m)], \quad \text{if } n > 2m. \end{aligned} \tag{4.6}$$

In the autonomous case ($h \equiv h(x)$) the results of Theorem 2.1 hold true under the coerciveness assumption (2.1) in the following cases:

a) If $n < 2m$, for any $p \geqq 1$ under the additional hypothesis (2.2 bis).

b) If $n = 2m$, for any $p \in [1, +\infty[$ under the additional assumption:

$$\exists\, k > 0 \text{ such that } |g(z)| \leqq C\{1 + |z|^k\}\,|z| \quad \forall z \in \boldsymbol{R}. \tag{4.7}$$

c) If $n > 2m$, for any $p \in [1, (n+2m)/(n-2m)]$ under the additional hypothesis

$$|g(z)| \leqq C\{1 + |z|^{4m/(n-2m)}\}\,|z| \quad \forall z \in \boldsymbol{R}. \tag{4.8}$$

Now let $h = h(t, x) \in S^1(\boldsymbol{R}^+; L^2(\Omega))$ be any non-resonant forcing term for (4.1). The conclusions of Theorem 3.1 hold true for all solutions of (4.1) under the coerciveness assumption (3.1) in the following cases:

a) If $n < 2m$, for any $p \geqq 1$ under the additional hypothesis (3.31).

b) If $n = 2m$, for any $p \in [1, +\infty[$ under the additional assumption that (3.29)–(3.30) is satisfied for some $\gamma \in [1, +\infty[$.

c) If $n > 2m$, for any $p \in [1, (n+2m)/(n-2m)]$ under the additional assumption that (3.29)–(3.30) is satisfied for some $\gamma \in [1, (n+2m)/(n-2m)]$.

Remark 4.3. It follows from Theorem 1.2 and Remark 1.3 that for both equations (4.1) and (4.5), either of conditions a) or b) in Remark 1.3 is sufficient for h to be non-resonant if g satisfies the conditions required for the decay estimate. •

Remark 4.4. If $h \equiv 0$ and $[u_0, v_0] \in D(L) \times V$, the decay estimate can be obtained under weaker conditions on the growth of g at infinity, *cf.* M. NAKAO [15]. This kind of property can also be deduced by the method of this paper, *cf.* A. CHABI [4]. •

Remark 4.5. The method is also applicable, when $h \equiv 0$, to obtain decay estimates when a conservative term $f(u)$ is added to the left-hand side of (0.1). These estimates, applied to related equations and some adapted energy functionals, are used in [11] to prove convergence to an equilibrium when the set of equilibria is one dimensional, *cf.* also [16]. Finally, it is quite reasonable to expect that the method is applicable to some types of non-local damping terms. •

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