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EXPONENTIAL DECAY FOR THE SEMILINEAR WAVE EQUATION WITH LOCALLY DISTRIBUTED DAMPING

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0. Introduction

Let Ω be a bounded, open, connected set in $\mathbf{R}^n$ $(n \geq 1)$ having a boundary $\Gamma = \partial\Omega$ of class C^2. Let $f \in C^1(\mathbf{R})$ be a function such that

$$(0.1) \qquad f(s)s \geq 0 \qquad \forall s \in \mathbf{R}$$

and satisfying the following growth condition

$$(0.2) \qquad \begin{cases} |f(x) - f(y)| \leq C(1 + |x|^{p-1} + |y|^{p-1})|x - y| & \forall x, y \in \mathbf{R} \\ \text{for some } C > 0 \quad \text{and } p > 1 \quad \text{with } (n-2)p \leq n. \end{cases}$$

Let $a = a(x) \in L^\infty(\Omega)$ be a nonnegative bounded function such that

$$a \geq a_0 > 0 \quad \text{a.e. in} \quad \omega \tag{0.3}$$

for some non empty open subset ω of Ω and some positive constant $a_0 > 0$.

Let us consider the following semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + f(u) + a(x)u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = 0 & \text{on} \quad \Gamma \times (0, \infty) \\ u(0) = u^0, \quad u_t(0) = u^1. \end{cases} \tag{0.4}$$

Under the conditions above, problem (0.4) is well posed in the space $H_0^1(\Omega) \times L^2(\Omega)$, i.e. for any initial data $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a unique weak solution of (0.4) in the class

$$u \in C([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega)). \tag{0.5}$$

Let us consider the energy

$$E(t) = \frac{1}{2} \int_\Omega (|\nabla u(x,t)|^2 + |u_t(x,t)|^2) dx + \int_\Omega F(u(x,t)) dx \tag{0.6}$$

where

$$F(s) = \int_0^s f(z) dz. \tag{0.7}$$

For every solution of (0.4),(0.5) the following identity holds

$$E(t_2) - E(t_1) = - \int_{t_1}^{t_2} \int_\Omega a(x) |u_t(x,t)|^2 dx dt \tag{0.8}$$

$$\forall t_2 > t_1 \geq 0$$

and therefore the energy is a non increasing function of the time variable t.

The aim of this paper is to give sufficient conditions on the nonlinearity f and the open subset ω (where the damping term is effective) ensuring the uniform exponential decay of the energy, i.e. the existence of some constants $C > 1$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t}E(0), \qquad \forall t \geq 0 \tag{0.9}$$

for every solution of (0.4), (0.5)

The linear case (i.e. $f(s) = \alpha s \quad \forall s \in \mathbf{R}$ with $\alpha \geq 0$) is by now well understood.

On one hand, the results by C. Dafermos [6] and A. Haraux [9] , based on LaSalle invariance principle, show that the energy of every solution goes to zero as t goes to infinity (even when (0.3) is satisfied for some measurable set ω with positive measure and we have a nonlinear monotone damping term).

On the other hand, recent results by C. Bardos, G. Lebeau and J. Rauch [2], [3] show that, when Ω and a are of class C^∞, (0.9) holds if and only if the following "geometric control condition "is satisfied: there exists some $T > 0$ such that every ray of geometric optics intersects the set $\omega \times (0, T)$. The results of [2], [3] are always stated in the framework of the boundary damping, however the result mentioned above is a direct consequence of their proofs (C. Bardos [1]). The canonical example of open subset ω verifying the "geometric control condition " is when ω is a neighbourhood of the boundary (here and in what follows by a neighbourhood of the boundary or of a portion $\Gamma_0 \subset \Gamma$ of the boundary we mean the intersection of Ω with some neighbourhood of those sets in $\mathbf{R}^n$). The proofs of [2] and [3] are based on microlocal analysis and do not extend to the semilinear problem (0.4).

Let us also mention the results of J.L. Lions [17], Chap. VII, that show how multiplier techniques may be used in order to achieve a priori estimates leading to exact controllability results for the undamped linear wave equation with controls on a neighbourhood of the boundary. These estimates, combined with the result by A. Haraux [10] allows proof of the exponential decay for (0.4) when $f = 0$ and ω is a neighbourhood of the boundary, by assuming only a C^2 regularity for the boundary of Ω.

Very little is known of the semilinear problem (0.4). To our knowledge, the only situation where a proof for the exponential decay has been given is when $\omega = \Omega$, i.e. the damping term is effective everywhere in Ω. This is a classical result that may be proved, for instance, constructing a perturbed energy functional

$$E_\varepsilon(t) = E(t) + \varepsilon \int_\Omega u(x,t) u_t(x,t) dx$$

for which differential inequalities leading to the desired exponential decay are easily proved when $\varepsilon > 0$ is small enough (see for instance E. Zuazua [22] where nonlinear damping terms are also considered).

The goal of the present paper is to handle the semilinear problem in the case where the damping is not effective everywhere in Ω, in order to extend the results of the linear case mentioned above.

We have not been able to adapt to problem (0.4) the method above based on the construction of suitable perturbed energy functionals. Therefore, inspired by the work by J. Rauch and M. Taylor [19] we shall look for energy estimates of type

$$E(T) \le C \int_0^T \int_\Omega a(x) |u_t(x,t)|^2 dx dt$$

which, combined with (0.8) and the semigroup property, allows the uniform exponential decay to be concluded.

We shall distinguish two different situations.

The first concerns the case where f is globally Lipschitz, i.e. $f' \in L^\infty(\mathbf{R})$. We can treat this case as a perturbation of the linear problem. However, in order to obtain the desired energy estimates we shall need to absorb some lower order terms. This will be done by using a "compactness-uniqueness argument" that has recently been used in the study of the exact controllability and stabilizability problems for the wave equation and for some plate models (c.f. for instance C. Bardos, G. Lebeau and J. Rauch [2], [3], J.L. Lions [18] and E. Zuazua [21], [23]). This argument reduces the question of absorbing these lower order terms to a uniqueness or unique continuation problem that may be solved by applying recent results by A. Ruiz [20]. However, in order to handle the nonlinear term we shall need to assume the existence of the following limits

$$(0.11) \qquad \lim_{s \to -\infty} f'(s) = f'(-\infty); \qquad \lim_{s \to +\infty} f'(s) = f'(+\infty).$$

Therefore, we shall slightly generalize the results that are by now well known in the linear framework.

We shall then study the case where f is superlinear, i.e.

$$(0.12) \qquad \exists \delta > 0 : f(s)s \geq (2+\delta)F(s) \qquad \forall s \in \mathbf{R}.$$

This situation may not be treated as a perturbation of the linear case. We shall therefore restrict our attention to the particular case where ω is a neighbourhood of the boundary Γ. We shall adapt the multiplier technique developed in J.L. Lions [17, p. 409-419] in order to obtain suitable energy estimates of type (0.10) (the motivation of [17] was to prove exact controllability results for the undamped linear wave equation with controls supported on a neighbourhood of the boundary). Once again a compactness-uniqueness argument will be needed in order to absorb lower order terms. In this case (0.12) will suffice to establish the uniform exponential decay.

The methods of this paper are general and allow us to treat other boundary conditions and also some models of plates.

The paper is organized as follows. In section 1 we shall state and prove the main results in the case where f is globally Lipschitz. The case where

f is superlinear will be treated in section 2. Finally, in section 3 we shall discuss some possible extensions of these results and we shall also briefly mention the links of this problem with that of the boundary stabilization.

1. The case of a globally Lipschitz nonlinearity

Let us consider the following linear conservative wave equation

$$\text{(1.1)} \qquad \begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad \Omega \times (0,\infty) \\ \varphi = 0 & \text{on} \quad \Gamma \times (0,\infty) \\ \varphi(0) = \varphi^0, \quad \varphi_t(0) = \varphi^1. \end{cases}$$

We assume that the open subset $\omega \subset \Omega$ is such that there exists a time $T > 0$ so that:

$$\text{(1.2)} \qquad \begin{aligned} &\exists C > 0 : \|\varphi^0\|^2_{H^1_0(\Omega)} + \|\varphi^1\|^2_{L^2(\Omega)} \le \\ &\qquad \le C\left\{ \int_0^T \int_\omega |\varphi_t(x,t)|^2 dx dt + \|\varphi\|^2_{L^2(\Omega\times(0,T))} \right\} \\ &\qquad \forall \{\varphi^0, \varphi^1\} \in H^1_0(\Omega) \times L^2(\Omega) \end{aligned}$$

and the following unique continuation result holds

$$\text{(1.3)} \qquad \left. \begin{aligned} &w_{tt} - \Delta w + b(x,t) w = 0 \quad \text{in } \Omega \times (0,T) \\ &w = 0 \quad \text{on } \partial\Omega \times (0,T); \quad w = 0 \quad \text{in } \omega \times (0,T) \\ &b \in L^\infty(\omega \times (0,T)); \quad w \in H^1(\Omega \times (0,T)) \end{aligned} \right\} \Rightarrow w \equiv 0.$$

The main result of this section is as follows.

Theorem 1.1.

Let $f \in C^1(\mathbf{R})$ be such that (0.1) is satisfied, $f' \in L^\infty(\mathbf{R})$ and the limits (0.11) exist.

Assume that $a \in L^\infty_+(\Omega)$ verifies (0.3) for some $a_0 > 0$ and some open subset $\omega \subset \Omega$ such that (1.2) and (1.3) hold.

Then, there exist some constants $C > 1, \gamma > 0$ such that the estimate (0.9) holds for every solution of (0.4) with initial data $\{u^0, u^1\} \in H^1_0(\Omega) \times L^2(\Omega)$.

Remark 1.2.

From J.L. Lions [17], Chap. VII, Lemme 2.4, we know that (1.2) is satisfied if Ω is of class C^2, ω is a neighbourhood of the boundary and $T >$ the diameter of Ω. In fact, we know that given any point $x^0 \in \mathbb{R}^n$, (1.2) is satisfied when ω is a neighbourhood of $\overline{\Gamma(x^0)}$ where

$$\Gamma(x^0) = \{x \in \Gamma : (x - x^0) \cdot \nu(x) > 0\} \tag{1.4}$$

and $\nu(x)$ is the unit outward normal at $x \in \Gamma$.

From C. Bardos, G. Lebeau and J. Rauch [2], [3] we know that, when Ω is of class C^∞, (1.2) is satisfied when ω satisfies the "geometric control property "mentioned in the introduction.

On the other hand, the unique continuation principle (1.3) holds when ω is a neighbourhood of the boundary and $T >$ diameter of Ω (cf. A. Ruiz [20]).

Therefore Theorem 1.1 applies, in particular, when ω is a neighbourhood of the whole boundary.

We conjecture that the hypothesis (1.3) is unnecessary since, given any open nonempty subset ω of Ω, the unique continuation principle (1.3) should hold for $T > T(\omega, \Omega)$ with $T(\omega, \Omega)$ large enough. This result is by now well known in some particular cases: i) when b is a real analytic function (1.3) holds for every ω as a consequence of Holmgren's Uniqueness Theorem; ii) when b depens only on the space variable, the arguments of [2] allow proving that (1.2) implies (1.3); iii) when b depends only on the time variable, (1.3) may be proved as in A. Haraux [11] for every ω. The problem seems to be open in the general case.

Proof of Theorem 1.1.

We note that it suffies to prove the following estimate

$$E(T) \leq C_0 \int_0^T \int_\Omega a(x)|u_t(x,t)|^2 dxdt. \tag{1.5}$$

Indeed, from (0.8) and (1.5) we easily deduce

$$E(T) \leq \frac{C_0}{1+C_0} E(0). \tag{1.6}$$

This estimate, combined with the semigroup property, implies (0.9) with

$$\begin{cases} C &= 1 + \frac{1}{C_0} \\ \gamma &= \frac{1}{T} \log(1 + \frac{1}{C_0}). \end{cases} \tag{1.7}$$

In order to prove (1.5) we write the solution $u = u(x,t)$ of (0.4) as

$$u = \varphi + \psi$$

where $\varphi = \varphi(x,t)$ solves (1.1) with initial data

$$\varphi^0 = u^0, \qquad \varphi^1 = u^1$$

and $\psi = \psi(x,t)$ satisfies

$$\begin{cases} \psi_{tt} - \Delta\psi = -f(u) - a(x)u_t & \text{in } \Omega \times (0,\infty) \\ \psi = 0 & \text{on } \Gamma \times (0,\infty) \\ \psi(0) = \psi_t(0) = 0. \end{cases} \tag{1.8}$$

From $(0.3) - (1.2)$ and the non increasing character of the energy we deduce

$$\begin{aligned} (1.9)\ E(T) \leq E(0) &\leq C \left\{ \|u^0\|^2_{H_0^1(\Omega)} + \|u^1\|^2_{L^2(\Omega)} \right\} \leq \\ &\leq C \left\{ \int_0^T \int_\omega |\varphi_t|^2 dxdt + \|\varphi\|^2_{L^2(\Omega\times(0,T))} \right\} \\ &\leq C \left\{ \int_0^T \int_\Omega a(x)|u_t|^2 dxdt + \|u\|^2_{L^2(\Omega\times(0,T))} + \|\psi\|^2_{H^1(\Omega\times(0,T))} \right\} \end{aligned}$$

Standard energy estimates for (1.8) yield

$$(1.10)\ \|\psi\|^2_{H^1(\Omega\times(0,T))} \leq C\|f(u) + a(x)u_t\|^2_{L^1(0,T;L^2(\Omega))}$$

$$\leq C\left\{\int_0^T\int_\Omega a(x)|u_t|^2dxdt + \|u\|^2_{L^2(\Omega\times(0,T))}\right\}.$$

(we have implicitely used the fact that $|f(s)| \leq C|s|$, $F(s)| \leq C|s|^2$ for every $s \in \mathbf{R}$).

Combining (1.9) and (1.10) we obtain

$$E(T) \leq C\left\{\int_0^T\int_\Omega a(x)|u_t|^2dxdt + \|u\|^2_{L^2(\Omega\times(0,T))}\right\}. \tag{1.11}$$

We note that the constant $C > 0$ in (1.11) depends on $\|f'\|_{L^\infty(\mathbf{R})}$ in a bounded manner.

It remains to prove the estimate

$$\|u\|^2_{L^2(\Omega\times(0,T))} \leq C\int_0^T\int_\Omega a(x)|u_t|^2dxdt. \tag{1.12}$$

We argue by contradiction. If (1.12) is not satisfied for some $C > 0$, there exists a sequence of solutions $\{u_n\}$ of (0.4), (0.5) verifying

$$\lim_{n\to\infty}\frac{\|u_n\|^2_{L^2(\Omega\times(0,T))}}{\int_0^T\int_\Omega a(x)|(u_n)_t|^2dxdt} = \infty. \tag{1.13}$$

We set

$$\lambda_n = \|u_n\|_{L^2(\Omega\times(0,T))} \tag{1.14}$$

and

$$v_n(x,t) = \frac{u_n(x,t)}{\lambda_n}. \tag{1.15}$$

The function v_n satisfies

$$\begin{cases} (v_n)_{tt} - \Delta v_n + f_n(v_n) + a(x)(v_n)_t = 0 & \text{in } \Omega\times(0,\infty) \\ v_n = 0 & \text{on } \Gamma\times(0,\infty) \end{cases} \tag{1.16}$$

where

$$f_n(s) = \frac{1}{\lambda_n} f(\lambda_n s) \quad \forall s \in \mathbf{R}, \forall n \in \mathbf{N}. \tag{1.17}$$

On the other hand

$$\|v_n\|_{L^2(\Omega\times(0,T))} = 1 \tag{1.18}$$

$$\int_0^T \int_\Omega a(x) |(v_n)_t|^2 dx dt \to 0. \tag{1.19}$$

Taking into account that

$$\|(f_n)'\|_{L^\infty(\mathbf{R})} = \|f'\|_{L^\infty(\mathbf{R})}$$

and the fact that the estimate (1.11) is therefore uniform on n, we deduce from (1.18)-(1.19) that

$$\{v_n\} \text{ is bounded in } L^\infty(0,\infty; H_0^1(\Omega)) \cap W^{1,\infty}(0,\infty; L^2(\Omega)). \tag{1.20}$$

We extract a subsequence (still denoted by $\{v_n\}$) such that

$$v_n \to v \quad \text{weakly in } H^1(\Omega \times (0,T)) \tag{1.21}$$

$$\begin{cases} v_n \to v & \text{strongly in} \quad L^2(\Omega \times (0,T)) \\ v_n \to v & \text{a.e. in} \quad \Omega \times (0,T). \end{cases} \tag{1.22}$$

From (1.18), (1.22) we deduce that

$$\|v\|_{L^2(\Omega\times(0,T))} = 1 \tag{1.23}$$

and (1.19) implies

$$v_t = 0 \quad \text{a.e. in} \quad \{a > 0\} \times (0,T). \tag{1.24}$$

On the other hand we note that

$$f_n(s) = h_n(s)s = h(\lambda_n s)s \qquad \forall s \in \mathbf{R},\ \forall n \in \mathbf{N}$$

where

$$h(z) = \frac{f(z)}{z}.$$

Remark that $h \in L^\infty(\mathbf{R})$ (h is bounded by the Lipschitz constant of f'), $h \geq 0$. Thus

$$f_n(v_n) = h_n(v_n)v_n$$

with $\{h_n(v_n)\}$ uniformly bounded in $L^\infty(\Omega \times (0,T))$.

Therefore we may extract a subsequence (still denoted by $h_n(v_n)$) such that

$$h_n(v_n) \to p(x,t) \text{ in } L^\infty(\Omega \times (0,T) \text{ weak star}$$

for some $p \in L^\infty_+(\Omega \times (0,T))$.

This allows us to pass to the limit in (1.16) obtaining (from (1.24))

$$(1.25) \qquad \begin{cases} v_{tt} - \Delta v + p(x,t)v = 0 & \text{in} \quad \Omega \times (0,T) \\ v = 0 & \text{on} \quad \Gamma \times (0,T). \end{cases}$$

However, the fact that, in principle, the potential $p(x,t)$ might depend on t does not allow to prove directly (from (1.24)-(1.25)) that $v \equiv 0$ in order to contradict (1.23).

In order to solve this difficulty we distinguish the following three situations:

a) There exists a subsequence of $\{\lambda_n\}$ (still denoted by $\{\lambda_n\}$) such that

$$\lambda_n \to \lambda \in (0,\infty).$$

In this case we easily see that

$$(1.26) \qquad p(x,t)v = \frac{1}{\lambda} f(\lambda v).$$

Then, $w = v_t$ satisfies:

$$w_{tt} - \Delta w + f'(\lambda v)w = 0 \quad \text{in } \Omega \times (0,T) \tag{1.27}$$

with

$$w = 0 \qquad \text{a.e. on } \omega \times (0,T). \tag{1.28}$$

b) We are not in situation (a) and there exists a subsequence $\{\lambda_n\}$ such that

$$\lambda_n \to 0.$$

In this case

$$p(x,t) = f'(0) \qquad \text{a.e. in } \Omega \times (0,T) \tag{1.29}$$

and $w = v_t$ satisfies, in addition to (1.28),

$$w_{tt} - \Delta w + f'(0)w = 0 \qquad \text{in } \Omega \times (0,T). \tag{1.30}$$

c) The sequence $\{\lambda_n\}$ goes to infinity.

In this case we take the derivative of (1.16) with respect to t and deduce that $w_n = (v_n)_t$ satisfies

$$(w_n)_{tt} - \Delta w_n + f'(\lambda_n v_n)w_n + a(x)(w_n)_t = 0 \qquad \text{in} \quad \Omega \times (0,T). \tag{1.31}$$

From (1.21) we know that

$$w_n \rightharpoonup w = v_t \qquad \text{weakly in} \quad L^2(\Omega \times (0,T)). \tag{1.32}$$

On the other hand, $\{f'(\lambda_n v_n)\}$ is uniformly bounded in $L^\infty(\Omega \times (0,T))$ but this does not suffice to pass the limit in (1.31). However,

$$f'(\lambda_n v_n)w_n \rightharpoonup z(x,t) \qquad \text{weakly in} \quad L^2(\Omega \times (0,T))$$

for some subsequence.

Therefore,

$$w_{tt} - \Delta w + z(x,t) = 0 \quad \text{in} \quad \Omega \times (0,T). \tag{1.33}$$

In order to identify the limit $z(x,t)$ we divide the cylinder $Q = \Omega \times (0,T)$ into two subsets

$$Q = Q_1 \cup Q_2 \quad \text{with} \quad Q_1 = \{v \neq 0\},\ Q_2 = \{v = 0\}.$$

From (0.11) and (1.22) we easily deduce, by Lebesgue's Theorem, that

$$f'(\lambda_n v_n) \to f'(-\infty)\chi_{\{v<0\}} + f'(+\infty)\chi_{\{v>0\}} = q(x,t) \tag{1.34}$$

$$\text{strongly in } L^2(Q_1)$$

(we denote by χ_A the characteristic function of A).

Therefore

$$z(x,t) = q(x,t)w(x,t) \quad \text{a.e. in} \quad Q_1. \tag{1.35}$$

On the other hand, the fact that (by definition)

$$v = 0 \quad \text{a.e. in} \quad Q_2$$

and $v \in H^1(\Omega \times (0,T))$, $v_{tt} - \Delta v \in L^2(\Omega \times (0,T))$, imply

$$v_{tt} - \Delta v = 0 \quad \text{a.e. in} \quad Q_2.$$

But clearly, $z \in L^2(\Omega \times (0,T))$ satisfies

$$z = -\frac{d}{dt}(v_{tt} - \Delta v) \quad \text{in} \quad \Omega \times (0,T)$$

and therefore

$$z = 0 \quad \text{a.e. in} \quad Q_2. \tag{1.36}$$

From (1.33), (1.35) and (1.36) we conclude that, in addition to (1.28), w satisfies

$$w_{tt} - \Delta w + \tilde{q}(x,t)w = 0 \quad \text{in} \quad \Omega \times (0,T)$$

with

$$\tilde{q}(x,t) \in L^{\infty}(\Omega \times (0,T)); \quad \tilde{q} = \begin{cases} q & \text{in} \quad Q_1 \\ 0 & \text{in} \quad Q_2. \end{cases}$$

Recapitulating, we see that, in each of these three possible situations (a),(b) and (c), the function $w \in L^2(\Omega \times (0,T))$ satisfies (1.28) and a wave equation of type

$$(1.37) \qquad \begin{cases} w_{tt} - \Delta w + b(x,t)w = 0 & \text{in } \Omega \times (0,T) \\ w = 0 & \text{on } \Gamma \times (0,T) \end{cases}$$

for some potential $b \in L^{\infty}_{+}(\Omega \times (0,T))$.

In order to apply (1.3) we must prove that $w \in H^1(\Omega \times (0,T))$. This can be done by proving (by the same perturbation argument that we have used above) an estimate of type (1.11) for system (1.37). Applying (1.3) we deduce $w \equiv 0$ and therefore $v = v(x)$.

Taking into account that $v = v(x)$ is a stationary solution of (1.25) we deduce that

$$(1.38) \qquad \begin{cases} -\Delta v + p(x,t)v = 0 & \text{in} \quad \Omega, \ \forall t \in (0,T) \\ v \in H^1_0(\Omega) \end{cases}$$

and since $p \geq 0$, $v \equiv 0$. This clearly contradicts (1.23) and the proof of the theorem is now completed.

Remark 1.3.

The hypothesis on the existence of the limits (0.11) is probably unnecessary. However, it has been crucial for us in order to treat the situation (c). We note that if limits (0.11) do not exist, there is no reason to expect the strong convergence of the sequence $\{f'(\lambda_n v_n)\}$ as $\lambda_n \to \infty$ and therefore to obtain an equation of type (1.37) for w (which is essential on the application of a unique continuation principle).

In Theorem 1.1 the hypothesis on the existence of the limits (0.11) may be replaced by the existence of the following one

$$\lim_{|s|\to\infty} \frac{f(s)}{s} = \alpha. \tag{1.39}$$

Under this assumption the situation (c) can be easily treated since in equation (1.25) we have $p(x,t) = \alpha$ a. e. in $\Omega \times (0,T)$.

Remark 1.4.

The only place where the continuity of the derivative of f has been used is when treating the situation (b). Theorem 1.1 remains true if the hypothesis $f \in C^1(\mathbf{R})$ is relaxed to $f \in W^{1,\infty}_{loc}(\mathbf{R})$ and f' is continuous at $s = 0$.

Remark 1.5.

As mentioned in the introduction, Theorem 1.1 extends recent results by C. Bardos, G. Lebeau and J. Rauch [2], [3], G. Chen, S.A. Fulling, F. J. Narcowich and S. Sun [4] and A. Haraux [10] on the exponential decay for linear waves with locally distributed damping.

2. The superlinear case

In this section we study the exponential decay of solutions of (0.4) – (0.5) in the case where the nonlinearity $f \in C^1(\mathbf{R})$, in addition to (0.1), (0.2), satisfies (0.12), i.e. f is superlinear.

As it was pointed out in the introduction, (0.4) may not be treated in this case as a perturbation of the linear problem where $f \equiv 0$.

Therefore we shall restrict our attention to the particular case where ω is a neighbourhood of a subset of the boundary of type $\overline{\Gamma(x^0)}$. The proofs of our estimates are based on multiplier techniques.

The main result of this section concerns the simplest case where ω is a neighbourhood of the whole boundary Γ.

Theorem 2.1.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with boundary of class C^2. Let us assume that $f \in C^1(\mathbf{R})$ satisfies (0.1), (0.2) and (0.12). Let us assume also that $a \in L^\infty_+(\Omega)$ satisfies (0.3) where ω is a neighbourhood of the boundary.

Then there exist some constants $C > 1$, $\gamma > 0$ such that the estimate (0.9) holds for every solution of (0.4), (0.5).

Proof of Theorem 2.1.

As in the proof of Theorem 1.1, we note that it suffices to prove an estimate of type (1.5) to obtain (0.9) with $C, \gamma > 0$ given by (1.7).

In order to prove (1.5) we proceed in several steps.

Step 1. We multiply equation (0.4) by $q(x) \cdot \nabla u(x,t)$ with a vector field $q \in (W^{1,\infty}(\Omega))^n$. (We denote by $\cdot$ the scalar product in $\mathbf{R}^n$).

Following the integrations by parts of Lemme 3.7, Chap. I of J.L. Lions [17] we easily deduce the identity.

$$\text{(2.1)}\quad \left(\int u_t q \cdot \nabla u\right)\Big|_0^T + \frac{1}{2}\int\int (div\, q)[|u_t|^2 - |\nabla u|^2] + \int\int \frac{\partial q_k}{\partial x_j}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_j} - \int\int (div\, q)F(u) + \int\int a u_t q \cdot \nabla u = \frac{1}{2}\int\int_\Sigma (q\cdot\nu)|\frac{\partial u}{\partial \nu}|^2.$$

In (2.1), for brevity, we have omitted the variables x, t of the functions under integral signs and we have used the convention of summation of repeated indexes. On the other hand, the following notations have been used:

(i) $$\int = \int_\Omega dx$$

(ii) $$\int\int = \int_0^T \int_\Omega dx dt$$

(iii) $$\int\int_\Sigma = \int_0^T \int_\Gamma d\Gamma dt$$

(iv) $div\, q = \text{divergence of } q = \dfrac{\partial q_k}{\partial x_k}$

(v) $(\cdot)|_0^T = \cdot(T) - \cdot(0)$

(vi) $\dfrac{\partial \cdot}{\partial \nu} = \text{normal derivative.}$

Applying identity (2.1) with

$$q(x) = x - x^0 = m(x)$$

for some $x^0 \in \mathbf{R}^n$ we deduce

$$\begin{aligned}(2.2)\qquad &\left(\int u_t m \cdot \nabla u\right)\Big|_0^T + \frac{n}{2}\int\int(|u_t|^2 - |\nabla u|^2) + \\ &+ \int\int |\nabla u|^2 - n\int\int F(u) + \int\int a u_t \quad m \cdot \nabla u = \\ &= \frac{1}{2}\int\int_\Sigma (m\cdot\nu)|\frac{\partial u}{\partial \nu}|^2 \le \frac{1}{2}\int\int_{\Sigma(x^0)}(m\cdot\nu)|\frac{\partial u}{\partial \nu}|^2\end{aligned}$$

(we have used the notation $\int\int_{\Sigma(x^0)} = \int_0^T \int_{\Gamma(x^0)} d\Gamma dt$).

We now multiply equation (0.4) by $\xi(x)u$ with $\xi \in W^{1,\infty}(\Omega)$. Integrating by parts we obtain the following identity

$$\begin{aligned}(2.3)\qquad \left(\int \xi u(u_t + \frac{au}{2})\right)\Big|_0^T = &\int\int \xi(|u_t|^2 - |\nabla u|^2) + \\ &+ \int\int u\nabla\xi\cdot\nabla u - \int\int \xi f(u)u.\end{aligned}$$

Applying (2.3) with $\xi = 1$ we deduce

$$(2.4)\qquad \left(\int u(u_t + \frac{a}{2}u)\right)\Big|_0^T = \int\int(|u_t|^2 - |\nabla u|^2) - \int\int f(u)u.$$

Combining (2.2) and (2.4) we obtain

$$\begin{aligned}(2.5)\qquad &\left(\int[u_t m\cdot\nabla u + \alpha u(u_t + \frac{a}{2}u)]\right)\Big|_0^T + \left(\frac{n}{2} - \alpha\right)\int\int |u_t|^2 + \\ &+ \left(1 + \alpha - \frac{n}{2}\right)\int\int |\nabla u|^2 + \alpha\int\int f(u)u - n\int\int F(u) + \\ &+ \int\int a u_t m\cdot\nabla u \le \frac{1}{2}\int\int_{\Sigma(x^0)}(m\cdot\nu)|\frac{\partial u}{\partial\nu}|^2\end{aligned}$$

for any $\alpha \in \mathbf{R}$.

We note that (0.12) implies the existence of some constant $\alpha \in \left(\frac{n-2}{2}, \frac{n}{2}\right)$ for which ($\alpha \in \left(0, \frac{1}{2}\right)$ when $n = 1$)

$$f(s)s \geq \left(\frac{n+\gamma}{\alpha}\right) F(s) \qquad \forall s \in \mathbf{R}$$

for some $\gamma > 0$.

With this choice of α, from (2.5) we deduce, for some $C > 0$, the estimate

$$C \int_0^T E(t)dt \leq \frac{1}{2} \int_{\Sigma(x^0)} (m \cdot \nu) |\frac{\partial u}{\partial \nu}|^2 + |\int\int a u_t m \cdot \nabla u| + \mathcal{X} \tag{2.6}$$

with

$$\mathcal{X} = |\int [u_t m \cdot \nabla u + \alpha u(u_t + \frac{a}{2} u)] \Big|_0^T |. \tag{2.7}$$

We have

$$|\int\int a u_t m \cdot \nabla u| \leq \varepsilon \|m\|^2_{L^\infty(\Omega)} \int\int |\nabla u|^2 + \frac{1}{2\varepsilon} \|a\|_{L^\infty(\Omega)} \int\int a|u_t|^2 \tag{2.8}$$

for any $\varepsilon > 0$.

Combining (2.6) with (2.8) where $\varepsilon > 0$ is taken small enough, we deduce

$$\int_0^T E(t)dt \leq C \left\{ \int_{\Sigma(x^0)} (m \cdot \nu) |\frac{\partial u}{\partial \nu}|^2 + \int\int a|u_t|^2 + \mathcal{X} \right\}. \tag{2.9}$$

Step 2. We now estimate the quantity

$$\int_{\Sigma(x^0)} (m \cdot \nu) |\frac{\partial u}{\partial \nu}|^2$$

in terms of

$$\int\int a|u_t|^2.$$

Following the method of proof of Lemme 2.3, Chap. VII in J.L. Lions [17], we construct a neighbourhood $\hat{\omega}$ of $\overline{\Gamma(x^0)}$ such that

$$\overline{\hat{\omega}} \cap \Omega \subset \omega$$

and a vector field $h \in (W^{1,\infty}(\Omega))^n$ such that

$$h = \nu \quad \text{on} \quad \Gamma(x^0); \quad h \cdot \nu \geq 0 \quad \text{a.e. in} \quad \Gamma \tag{2.10}$$

and

$$h = 0 \quad \text{on} \quad \Omega \setminus \hat{\omega} \tag{2.11}$$

(see Remark 3.2, Chap. I of [17] for the construction of this vector field).

Applying identity (2.1) with $q = h$ we easily deduce the existence of some constant $C > 0$ such that

$$\begin{aligned} \int\int_{\Sigma(x^0)} \left|\frac{\partial u}{\partial \nu}\right|^2 \leq \int_{\Sigma} (h \cdot \nu) \left|\frac{\partial u}{\partial \nu}\right|^2 \leq \\ \leq C \int_0^T \int_{\hat{\omega}} [|u_t|^2 + |\nabla u|^2 + F(u)] dx dt + 2 \left(\int u_t h \cdot \nabla u \right) \Big|_0^T \end{aligned} \tag{2.12}$$

We then construct a function $\eta \in W^{1,\infty}(\Omega)$ satisfying

$$0 \leq \eta \leq 1 \quad \text{a.e. in} \quad \Omega; \quad \eta = 1 \quad \text{a.e. in} \quad \hat{\omega} \tag{2.13}$$

$$\eta = 0 \quad \text{a.e. in} \quad \Omega \setminus \omega \tag{2.14}$$

and

$$\frac{|\nabla \eta|^2}{\eta} \in L^\infty(\omega) \tag{2.15}$$

(see Lemme 2.4, Chap. VII in [17] for the construction of this function).

Applying identity (2.3) with $\xi = \eta$ we deduce

$$\int\int \eta[|\nabla u|^2 + f(u)u] - \int\int u\nabla\eta\cdot\nabla u \leq \leq C\left\{\int_0^T\int_\omega |u_t|^2 dxdt + \mathcal{Y}\right\} \tag{2.16}$$

with

$$Y = |\int[\eta u(u_t + \frac{au}{2})]\Big|_0^T| \tag{2.17}$$

On the other hand

$$|\int\int u\nabla\eta\cdot\nabla u| \leq \varepsilon\int\int \eta|\nabla u|^2 + \frac{1}{2\varepsilon}\int\int\frac{|\nabla\eta|^2}{\eta}|u|^2. \tag{2.18}$$

Combining (2.16) with (2.18) for $\varepsilon \in (0,1)$ we deduce

$$\begin{aligned}\int_0^T\int_\omega [|\nabla u|^2 + F(u)]dxdt &\leq \int\int \eta[|\nabla u|^2 + F(u)] \leq \\ &\leq C\int\int \eta[|\nabla u|^2 + f(u)u] \leq \\ &\leq C\left\{\int_0^T\int_\omega |u_t|^2 dxdt + \int\int |u|^2 + \mathcal{Y}\right\}.\end{aligned} \tag{2.19}$$

From (2.12) and (2.19) we conclude

$$\begin{aligned}&\int\int_{\Sigma(x^0)} |\frac{\partial u}{\partial\nu}|^2 \leq \\ &\leq C\left\{\int\int a|u_t|^2 + \int\int u^2 + |\int(u_t h\cdot\nabla u)\Big|_0^T| + \mathcal{Y}\right\}.\end{aligned} \tag{2.20}$$

Combining (2.9) and (2.20) we get

$$\begin{aligned}&TE(T) \leq \int_0^T E(t)dt \leq \\ &\leq C\left\{\int\int a|u_t|^2 + \int\int u^2 + \mathcal{X} + \mathcal{Y} + |\int(u_t h\cdot\nabla u)\Big|_0^T|\right\}.\end{aligned} \tag{2.21}$$

We now remark that

$$(2.22)\qquad \mathcal{X} + \mathcal{Y} + \left| \int (u_t h \cdot \nabla u) \Big|_0^T \right| \leq C(E(0) + E(T)) = $$
$$= C \left\{ 2E(T) + \int\int a|u_t|^2 \right\}.$$

Combining (2.21) with (2.22) for $T > 0$ large enough, we obtain

$$(2.23)\qquad E(T) \leq C \left\{ \int\int a|u_t|^2 + \int\int |u|^2 \right\}.$$

Step 3. As in the proof of Theorem 1.1 we must get the following estimate

$$(2.24)\qquad \int\int |u|^2 \leq C \int\int a|u_t|^2.$$

We argue by contradiction. If (2.24) is not satisfied for some $C > 0$, there exists a sequence $\{u_n\}$ of solutions of (0.4), (0.5) verifying (1.13). We define $\{\lambda_n\}, \{v_n\}$ by (1.14) and (1.16). The functions $\{v_n\}$ satisfy (1.16) with the nonlinearities f_n given by (1.17). On the other hand the sequence $\{v_n\}$ satisfies also (1.18), (1.19).

We now remark that the constant $C > 0$ on the estimate (2.23) depends on the nonlinearity f but only in terms of the constant δ of hypothesis (0.12). But this constant δ is uniform with respect to the rescaled family of nonlinearities (1.17). Therefore, the constant C on (2.23) is uniform on $\{f_n\}$.

Thus, from (1.18)-(1.19) we deduce that $\{v_n\}$ is bounded in $H^1(\Omega \times (0,T))$.

Therefore, we may extract a subsequence verifying (1.21) and (1.22). The limit $v \in H^1(\Omega \times (0,T))$ will satisfy (1.23) and (1.24).

In order to contradict (1.23) we want to apply again a uniqueness argument showing that $v \equiv 0$. But the question is now simpler than in Theorem 1.1.

We first remark that the sequence $\{\lambda_n\}$ is necessarily bounded. Indeed, assume that there is a subsequence (still denoted by $\{\lambda_n\}$) such that

$$\lambda_n \to \infty. \tag{2.25}$$

From the uniform estimate (2.23) we know that

$$\{F_n(v_n)\} \quad \text{is uniformly bounded in} \quad L^1(\Omega \times (0,T)) \tag{2.26}$$

where

$$F_n(z) = \int_0^z f_n(s)ds = \frac{1}{\lambda_n^2} F(\lambda_n z). \tag{2.27}$$

But from (0.12) one easily deduces that

$$F(s) \geq c|s|^{2+\delta} \qquad \forall |s| \geq 1 \tag{2.28}$$

with $c = \min\{F(1), F(-1)\}$.

Combining (2.26), (2.28) we conclude that

$$\lambda_n^\delta \int_0^T \int_{\{|v_n| \geq \lambda_n^{-1}\}} |v_n|^{2+\delta} + \int_0^T \int_{\{|v_n| \leq \lambda_n^{-1}\}} F(\lambda_n v_n) \leq C$$

that implies

$$\lim_{n\to\infty} \int\int |v_n|^{2+\delta} = 0$$

which contradicts (1.23).

The sequence $\{\lambda_n\}$ being bounded we must only consider the situations (a) and (b) of step 3 of the proof of Theorem 1.1.

In the situation (a) we proceed as follows. Since v_n is bounded in $L^\infty(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$, then it is relatively compact in $L^\infty(0,T;H^{1-\varepsilon}(\Omega))$ for every $\varepsilon > 0$. Taking into account that p satisfies $(n-2)p \leq 2n$, we deduce that the sequence $f_n(v_n)$ converges strongly to $\lambda^{-1}f(\lambda v)$ in $L^\infty(0,T;L^r(\Omega))$ for every $r \in [1, \frac{2n}{p(n-2)})$ (when $n = 1,2$ the convergence holds in $L^\infty(0,T;L^r(\Omega))$ for every $r \geq 1$). Therefore we may pass to the limit in (1.16) and we deduce that $w = v_t$ (where v is the limit of v_n) satisfies (1.27)-(1.28). On the other hand, since $(n-2)p \leq n$, we deduce that $b = f'(\lambda v) \in L^\infty(0,T;L^n(\Omega))$ (when $n = 1,2$, $b \in L^\infty(0,T;L^r(\Omega))$ for every $r \geq 1$).

In the situation (b) we proceed as in the proof of Theorem 1.1 and we deduce that w satisfies (1.28) and (1.30).

In both cases we see that $w \in L^2(\Omega \times (0,T))$ solves an equation of type (1.37) with $b \in L_+^\infty(0,T;L^n(\Omega))$ and verifies (1.28). Applying the unique continuation result of [20] we deduce, if $T >$ diameter of Ω, $w \equiv 0$. Thus, $v = v(x)$ and solves (1.38). This implies $v \equiv 0$ which contradicts (1.23).

The proof of the theorem is now completed.

Remark 2.2.

Let us consider the more general case where ω is a neighbourhood of a subset of the boundary of type $\overline{\Gamma(x^0)}$. The calculations of steps 1 and 2 apply in this case and the estimate (2.23) holds. If we had a unique continuation result applicable to solutions of (1.37) vanishing in ω, the arguments of step 3 would apply and the exponential decay would hold. However, as we have mentioned in Remark 1.2, this unique continuation result does not seem to be proved.

Remark 2.3.

We note that the proof of the estimate (2.24) is in this superlinear case slightly simpler than it was in Theorem 1.1. Indeed, hypothesis (0.12), ensuring the superlinear behaviour of f at ∞, has now played the role of the assumption on the existence of the limits (0.11) of the globally Lipschitz case.

3. Some comments on the extension of the main results

In this section we discuss some possible extensions of the methods and results of sections 1 and 2.

3.1 Equations with potentials

Theorems 1.1 and 1.2 are still valid for semilinear wave equations with potentials of type

$$\text{(3.1)} \quad \begin{cases} u_{tt} - \Delta u + W(x)u + f(u) + a(x)u_t = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Gamma \times (0,\infty) \\ u(0) = u^0 \in H_0^1(\Omega), \quad u_t(0) = u^1 \in L^2(\Omega) \end{cases}$$

provided $W(x) \in L^p(\Omega)$, $W \geq 0$ a.e. in Ω, with

$$\text{(3.2)} \quad \begin{cases} p = 2 & \text{if } n = 1 \\ p > n & \text{if } n \geq 2. \end{cases}$$

The condition $W \geq 0$ may be relaxed to

$$\text{(3.3)} \quad \exists \alpha > 0 : \int_\Omega [|\nabla u|^2 + W|u|^2] dx \geq \alpha \int_\Omega |\nabla u|^2 dx \quad \forall u \in H_0^1(\Omega).$$

The methods of proof are exactly the same. Condition (3.2) ensures compactness of the lower order perturbation Wu.

3.2 Equations with variable coefficients in the principal part

Let us consider a wave equation of type

$$(3.4)\quad \begin{cases} u_{tt} - \partial_j(a_{ij}(x)\partial_i u) + f(u) + a(x)u_t = 0 & \text{in} \quad \Omega \times (0,\infty) \\ u = 0 & \text{on} \quad \Gamma \times (0,\infty) \\ u(0) = u^0 \in H_0^1(\Omega), \quad u_t(0) = u^1 \in L^2(\Omega) \end{cases}$$

with $a_{ij} \in C^\infty(\bar\Omega)$, $a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$, $\forall \xi \in \mathbf{R}^n$, with $\alpha > 0$ and $a_{ij} = a_{ji}$, $\forall i, j \in \{1, ..., n\}$.

The results in [2], [3] apply also in this case provided $f = 0$ and Ω to be smooth.

Theorem 1.1 may be extended to this case. The results in [2],[3] show that the estimate which correpods to (1.2) holds when ω verifies the "geometric control property". The results in [20] show that the unique continuation result which corresponds to (1.3) also holds for equations of type

$$(3.5)\qquad w_{tt} - \partial_j(a_{ij}(x)\partial_i w) + b(x,t)w = 0$$

when ω is a neighbourhood of the boundary.

We may also try to treat the superlinear case by multiplier techniques. Following J. Lagnese [15] and V. Komornik [12] we may prove the estimate (2.23) under the very restrictive condition:

$$(3.6)\qquad a_{ij}\xi_i\xi_j - \frac{(x_k - x_k^0)}{2}\partial_k(a_{ij})\xi_i\xi_j \geq \beta|\xi|^2, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbf{R}^n$$

and some $\beta > 0$.

Therefore, the exponential decay will hold under the additional hypothesis (3.6). The general case remains open.

3.3 Equations with nonlinear damping terms

Let us consider the following wave equation with nonlinear damping:

$$(3.7)\quad \begin{cases} u_{tt} - \Delta u + f(u) + a(x)g(u_t) = 0 & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \Gamma \times (0,\infty) \\ u(0) = u^0 \in H_0^1(\Omega), \quad u_t(0) = u^1 \in L^2(\Omega) \end{cases}$$

where g is globally Lipschitz, i.e. $g' \in L^\infty(\mathbf{R})$, and satisfies

$$(3.8)\qquad \exists c > 0 : g(s)s \geq c|s|^2 \quad \forall s \in \mathbf{R}$$

Theorems 1.1 and 1.2 easily extend to this case.

It would be interesting to see whether, polynomial uniform rates of decay may be obtained when g is, for instance, of the form

$$g(s) = |s|^{p-1}s, \qquad p > 1.$$

3.4 Neumann boundary conditions

Let us consider now the following wave equation with Neumann boundary conditions

$$(3.9)\quad \begin{cases} u_{tt} - \Delta u + u + f(u) + a(x)u_t = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \times (0,\infty) \\ u(0) = u^0 \in H^1(\Omega), \quad u_t(0) = u^1 \in L^2(\Omega). \end{cases}$$

The energy of the system is now given by

$$(3.10)\quad E(t) = \frac{1}{2}\int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2 + |u(x,t)|^2]dx + \int_\Omega F(u(x,t))dx.$$

The result of Theorem 1.1 easily extends to this problem under the same hypotheses. The subset ω needs to satisfy properties of type (1.2) – (1.3). The results in [2], [3] show that, when Ω is C^∞, the corresponding condition (1.2) is satisfied when ω verifies the "geometric control property". On the other hand, we have a unique continuation result of type (1.3) when ω is a neighbourhood of the whole boundary (cf. [20]).

As it was pointed out by D. Cioranescu and P. Donato [5], the multiplier methods of Chap. VII in [17] may be adapted to get exact controllability results for the linear wave equation with Neumann boundary conditions with controls supported on a neighbourhood of the whole boundary. Combining the ideas of section 2 and [5] we may extend Theorem 2.1 to the system (3.9) provided ω is a neighbourhood of the boundary and Ω is of class C^2. The case where ω is a neighbourhood of a subset of the boundary of type $\overline{\Gamma(x^0)}$ is open.

We may also consider systems of type

$$(3.11)\qquad \begin{cases} u_{tt} - \Delta u + f(u) + a(x)u_t = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} + u = 0 & \text{on } \Gamma \times (0,\infty) \\ u(0) = u^0 \in H^1(\Omega) \\ u_t(0) = u^1 \in L^2(\Omega) \end{cases}$$

and obtain analogous results. We note that the energy associated to the system is now

$$(3.12)\qquad E(t) = \frac{1}{2}\int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2]dx + \int_\Omega F(u(x,t))dx + \\ + \frac{1}{2}\int_\Gamma |u(x,t)|^2 d\Gamma.$$

Remark that the additional boundary term on the energy ensures the coerciveness on $H^1(\Omega) \times L^2(\Omega)$. This was the objet of the additional u term in (3.9), i.e. to exclude the existence of non trivial equilibrium solutions.

We may also consider Dirichlet / Neumann mixed boundary conditions

$$(3.13)\qquad \begin{cases} u_{tt} - \Delta u + f(u) + a(x)u_t = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma(x^0) \times (0,\infty) \\ u = 0 & \text{on } \Gamma_*(x^0) \times (0,\infty) \\ u(0) = u^0 \in H^1_{\Gamma_*(x^0)}(\Omega) \\ u_t(0) = u^1 \in L^2(\Omega) \end{cases}$$

with $\Gamma(x^0)$ as in (1.4), $\Gamma_*(x^0) = \Gamma \setminus \Gamma(x^0) \neq \emptyset$ and $H^1_{\Gamma_*(x^0)}(\Omega) = \{u \in H^1(\Omega) | u = 0 \quad \text{on } \Gamma_*(x^0)\}$.

Theorems 1.1 and 2.1 may be easily extended to this problem.

We note that when

$$\overline{\Gamma(x^0)} \cap \overline{\Gamma_*(x^0)} \neq \emptyset$$

the solutions of (3.13) may develop singularities at the interfaces. In this case, solutions are not smooth enough to do the integrations by parts needed to apply multiplier techniques. In this case, following P. Grisvard [7], [8], V. Komornik and E. Zuazua [13], [14] and E. Zuazua [24] we must to restrict our attention to dimensions $n \leq 3$.

3.5 Plate models

Let us consider the simplified semilinear plate model

$$(3.14) \qquad \begin{cases} u_{tt} + \Delta^2 u + f(u) + a(x)u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma \times (0, \infty) \\ u(0) = u^0 \in H_0^2(\Omega) \\ u_t(0) = u^1 \in L^2(\Omega). \end{cases}$$

The methods of sections 1 and 2 allow, once again, to reduce the problem of the exponential decay for (3.14) in both globally Lipschitz and superlinear case, to a unique continuation question. Indeed, if we knew that the unique solution to

$$w_{tt} + \Delta^2 w + b(x,t)w = 0 \quad \text{in} \quad \Omega \times (0,T);\ w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \Gamma \times (0,T)$$

with $b \in L^\infty(\Omega \times (0,T))$; $w \in H^2(\Omega \times (0,T))$, satisfying $w = 0$ in $\omega \times (0,T)$ is $w \equiv 0$, the exponential decay for the corresponding energy would hold. However, this question seems to be a open problem.

Recently, in J. Lagnese [16], Chap. V, the exponential decay for a Von Karman model with dissipative boundary conditions was proved by multiplier techniques. It would by very interesting to see whether the techniques of this paper may be used to establish the exponential decay for Von Karman models with damping localized on a neighbourhood of the boundary.

3.6 Links with boundary stabilization

Recently, in [13], [14], the exponential decay was proved for the following semilinear wave equation with boundary damping

$$
(3.15) \qquad \begin{cases} u_{tt} - \Delta u + f(u) = 0 & \text{in } \Omega \times (0,\infty) \\ \frac{\partial u}{\partial \nu} = -[(x - x^0)\cdot \nu(x)]u_t & \text{on } \Gamma(x^0) \times (0,\infty) \\ u = 0 & \text{on } \Gamma_*(x^0) \times (0,\infty) \\ u(0) = u^0 \in H^1_{\Gamma_*(x^0)}(\Omega) & \\ u_t(0) = u^1 \in L^2(\Omega) & \end{cases}
$$

when $\Omega \subset \mathbf{R}^n$, $n \leq 3$ and f satisfies (0.12).

This result was proved combining multiplier techniques and the idea of constructing ad hoc modified energy functionals mentioned in the introduction.

May be surprisingly, the proof of the exponential decay is more involved when the damping is effective in a neighbourhood of the boundary (an additional compactness-uniqueness argument is needed). It would be very interesting to see if the result of Theorem 2.1 may be achieved by constructing modified energy functionals.

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