On the optimality of some observability inequalities for plate systems with potentials*

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Summary. In this paper, we derive sharp observability inequalities for plate equations with lower order terms. More precisely, for any $T > 0$ and suitable observation domains (satisfying the geometric conditions that the multiplier method imposes), we prove an estimate with an explicit observability constant for plate systems with an arbitrary finite number of components and in any space dimension with lower order bounded potentials. These inequalities are relevant for control theoretical purposes and also in the context of inverse problems. We also prove the optimality of this estimate for plate systems with bounded potentials in even space dimensions $n \geq 2$. This is done by extending a construction due to Meshkov to the bi-Laplacian equation, to build a suitable complex-valued bounded potential $q = q(x)$, with a non-trivial solution u of $\Delta^2 u = qu$ in $\mathbb{R}^2$, with the decay property $|u(x)| + |\nabla u(x)| + |\nabla \Delta u(x)| \leq \exp(-|x|^{4/3})$ for all $x \in \mathbb{R}^2$.

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1 Introduction

1.1 Formulation of the problem

Let $n \geq 1$ and $N \geq 1$ be two integers. Let $T > 0$ be given, Ω be a bounded domain in $\mathbb{R}^n$ with C^4 boundary Γ, and ω be a nonempty open subset of Ω. Put $Q \stackrel{\triangle}{=} (0,T)\times\Omega$ and $\Sigma \stackrel{\triangle}{=} (0,T)\times\Gamma$. For simplicity, we will use the notation $y_j = \frac{\partial y}{\partial x_j}$, where x_j is the jth coordinate of a generic point $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$. Throughout this paper, we will use $C = C(\Omega, \omega)$ and $C^* = C^*(T, \Omega, \omega)$ to denote generic positive constants which may vary from line to line.

Set

$$Y \stackrel{\triangle}{=} \{y \in H^3(\Omega) |\ y|_\Gamma = \Delta y|_\Gamma = 0\}.$$

We consider the following $\mathbb{R}^N$-valued plate system with a potential $a \in L^\infty(0,T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n, \infty]$:

$$\begin{cases} y_{tt} + \Delta^2 y + ay = 0 & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad y_t(0) = y^1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $y = (y_1, \dots, y_N)^\top$, and the initial datum (y^0, y^1) is supposed to belong to $Y^N \times (H_0^1(\Omega))^N$, the state space of the system (1.1). It is easy to show that the system (1.1) admits a unique weak solution $y \in C([0,T]; Y^N) \cap C^1([0,T]; (H_0^1(\Omega))^N)$.

In what follows, we shall denote by $|\cdot|$, $|\cdot|_p$, $\|\cdot\|_p$ and $|||\cdot|||_p$ the (usual) norms on $\mathbb{R}^N$, $W^{1,\infty}(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$, $L^\infty(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$ and $L^\infty(0,T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$, respectively.

We shall study the observability constant $P(T,a)$ of the system (1.1), defined as the smallest (possibly infinite) constant such that the following observability estimate for system (1.1) holds:

$$\begin{aligned} &\|\Delta y^0\|^2_{(H_0^1(\Omega))^N} + \|y^1\|^2_{(H_0^1(\Omega))^N} \\ &\quad \leq P(T,a) \int_0^T \int_\omega (|\nabla y|^2 + |\nabla \Delta y|^2) dt dx, \qquad \forall\, (y^0, y^1) \in Y^N \times (H_0^1(\Omega))^N. \end{aligned} \tag{1.2}$$

This inequality, the *observability inequality*, allows estimating the total energy of solutions in terms of the energy localized in the observation subdomain ω. It is relevant for control problems. In particular, in the linear setting, it is equivalent to the so-called exact controllability problem, i.e., that of driving solutions to rest by means of control forces localized in $\omega \times (0,T)$ (see [8]). This type of inequality, with explicit estimates on the observability constant, is also relevant for the control of semilinear problems (see [11]). Similar inequalities are also useful for solving a variety of inverse problems.

The main purpose of this paper is to analyze the dependence of $P(T,a)$ on the potential a. The main tools to derive this kind of explicit observability estimates are the so-called *Carleman inequalities*. Here we have chosen to work in the space $Y^N \times (H_0^1(\Omega))^N$ in which Carleman inequalities can be applied more naturally. But some other choices of the state space are possible. For example, one may consider similar problems in state spaces of the form $(H_0^1(\Omega))^N \times (H^{-1}(\Omega))^N$ or $(H^2(\Omega)\cap H_0^1(\Omega))^N \times (L^2(\Omega))^N$ where the plate system is also well posed. But the corresponding analysis, in turn, is technically more involved and will be treated elsewhere ([6]).

1.2 Preliminaries on the heat and wave equations

Similar problems have been considered for the heat and wave equations in [3].

Consider the following heat and wave equations (or systems) with potentials:

- The heat equation/system with potential:

$$\begin{cases} z_t - \Delta z + az = 0, & \text{in } Q, \\ z = 0, & \text{on } \Sigma, \\ z(0) = z^0, & \text{in } \Omega. \end{cases} \tag{1.3}$$

- The wave equation/system with potential:

$$\begin{cases} w_{tt} - \Delta w + aw = 0, & \text{in } Q, \\ w = 0, & \text{on } \Sigma, \\ w(0) = w^0, \quad w_t(0) = w^1, & \text{in } \Omega. \end{cases} \tag{1.4}$$

In [3] the observability inequalities for these equations were analyzed in the state spaces $(L^2(\Omega))^N$ and $(L^2(\Omega))^N \times (H^{-1}(\Omega))^N$, respectively.

More precisely, by definition, the heat and wave observability constants $H(T,a)$ and $W(T,a)$ are the smallest (possibly infinite) constants such that the following observability estimates hold:

- The heat equation/system with potential:

$$\|z(T)\|^2_{(L^2(\Omega))^N} \le H(T,a)\int_0^T\int_\omega |z|^2 dtdx, \qquad \forall\, z^0 \in (L^2(\Omega))^N \tag{1.5}$$

  for systems of the form (1.3).
- The wave equation/system with potential:

$$\|w^0\|^2_{(L^2(\Omega))^N} + \|w^1\|^2_{(H^{-1}(\Omega))^N} \le W(T,a)\int_0^T\int_\omega |w|^2 dtdx, \quad \forall\, (w^0,w^1) \in (L^2(\Omega))^N \times (H^{-1}(\Omega))^N \tag{1.6}$$

  for systems of the form (1.4).

For systems (1.3), due to the infinite speed of propagation and its parabolic nature, it is shown that for any $T > 0$ and ω, the observability constant $H(T, a)$ is bounded by (*see* [3, Theorem 2.1])

$$H(T, a) \le \exp\left[C\left(1 + \frac{1}{T} + T\|a\|_p + \|a\|_p^{\frac{1}{3/2 - n/p}}\right)\right]; \tag{1.7}$$

while for system (1.4), due to the finite speed of propagation and its hyperbolic nature, it is shown that for any fixed triple (T, ω, Ω) satisfying suitable geometric assumptions (say, the classical ones arising when applying multiplier methods ([8])), the observability constant $W(T, a)$ is bounded by (see [3, Theorem 2.2 ii)])

$$W(T, a) \le \exp[C^*(1 + \|a\|_p^{\frac{1}{3/2 - n/p}})]. \tag{1.8}$$

In particular the observability constant in (1.7) includes three different terms, i.e., (1.7) can be rewritten as

$$H(T, a) = H_1(T,\, a) H_2(T,\, a) H_3(T,\, a),$$

where

$$\begin{aligned} &H_1(T,\, a) = \exp\left[C\left(1 + \frac{1}{T}\right)\right], \quad H_2(T,\, a) = \exp(CT\|a\|_p), \\ &H_3(T,\, a) = \exp(C\|a\|_p^{\frac{1}{3/2 - n/p}}). \end{aligned} \tag{1.9}$$

As explained in [3, 12], the role that each of these constants plays in the observability inequality is of a different nature: $H_1(T,\, a)$, which blows up exponentially as $T \downarrow 0$, is the observability constant for the special case that $a \equiv 0$; $H_2(T,\, a)$ is the constant which arises naturally when applying Gronwall's inequality to establish the energy estimate for solutions of system (1.3); while $H_3(T,\, a)$ is the one arising when using global Carleman estimates (see [7], [4]) to derive the observability inequality (1.5) by absorbing the undesired lower order terms.

In a similar spirit, by [3, Theorem 2.2 ii)], it is easy to see that one can decompose the right-hand side of (1.8) into two different terms, i.e., (1.8) can be rewritten as

$$W(T,\, a) \le W_1(T,\, a) W_2(T,\, a),$$

where

$$W_1(T,\, a) = e^{C^*}, \quad W_2(T,\, a) = \exp(C^*\|a\|_p^{\frac{1}{3/2 - n/p}}). \tag{1.10}$$

Here, $W_1(T,\, a)$ is the observability constant in (1.6) for the special case that $a \equiv 0$ but it is finite only, for suitable subsets ω and for T large enough (unless $\omega = \Omega$); and $W_2(T,\, a)$ is the one arising when using a global Carleman estimate to derive the observability inequality (1.6) by absorbing the undesired lower order terms. We emphasize again that for this purpose one needs some geometric assumptions on the triple (T, Ω, ω).

In this case the sharp observability constant does not involve the term related with the Gronwall estimate for evolution of energy in time, since it may be bounded above in terms of $W_1(T, a)$ and $W_2(T, a)$.

Based on the construction by Meshkov ([10]), it is shown in [3, Theorems 1.1 and 1.2] that both $H_3(T, a)$ and $W_2(T, a)$ are optimal for systems $(N \geq 2)$ with bounded potentials (i.e., $p = \infty$), in even dimensions $n \geq 2$ for certain ranges of the observability time T. In [3] an extension of Meshkov's construction is also given for odd dimensions, showing that the above estimates are almost sharp in that case.

1.3 The sharp observability constant for plate systems

Plate systems can be viewed as intermediate ones between the heat and the wave systems. Indeed, on the one hand, system (1.1) is time-reversible, which is, typically, a hyperbolic property; on the other hand, similar to the heat system, the solutions of system (1.1) propagate with infinite velocity.

As we shall see, under suitable geometric conditions on the pair (Ω, ω), $P(T, a)$ is finite with the following decomposition:

$$P(T,\, a) = P_1(T,\, a) P_2(T,\, a) P_3(T,\, a). \tag{1.11}$$

Here $P_1(T, a)$ is the observability constant in (1.2) for the special case that $a \equiv 0$; $P_2(T, a)$ is the constant which arises when applying Gronwall's inequality to establish an energy estimate for solutions of system (1.1); while $P_3(T, a)$ is the one arising when using global Carleman estimates to derive the observability inequality (1.2) by absorbing the undesired lower order terms.

More precisely, the first main purpose of this paper is to show that

$$\begin{aligned} &P_1(T,\, a) = \exp\left[C\left(1 + \frac{1}{T}\right)\right], \quad P_2(T,\, a) = \exp(CT |||a|||_p^{\frac{1}{2-n/2p}}), \\ &P_3(T,\, a) = \exp(C\|a\|_p^{\frac{1}{3-5n/2p}}). \end{aligned} \tag{1.12}$$

Note that the power of $|||a|||_p$ in $P_2(T, a)$ is always less than 1 for $p \in [n, \infty]$. This may be achieved as a consequence of a modified energy estimate, because the plate system is second order in time. But this term may not be absorbed by $P_3(T, a)$ when $p > 2n$. In this sense the estimate we get is closer to that on the heat equation $H(T, a)$ since the observability constant contains three different terms. Another important analogy of this estimate with the heat equation is that the observability inequality holds for all $T > 0$. Note however that, for plate systems, the subdomain ω needs to fulfill suitable geometric conditions for the observability to hold.

There are two other important differences between (1.12) and (1.9)–(1.10), i.e.,

1) The power of $\|a\|_p$ in $P_3(T, a)$ is $1/(3-5n/2p)$, but the ones in $H_3(T, a)$ and $W_2(T, a)$ are $1/(3/2-n/p)$. This is due to the different scaling of the various terms arising in the Carleman inequality for plate systems. The main reason for that is that the plate system is fourth order in space, while the heat and wave systems are of order 2.
2) There is only one norm $\|a\|_p$ in (1.9)–(1.10) but one needs to use two different norms, $\|a\|_p$ and $|||a|||_p$, in (1.12). The reason is as follows. For the well-posedness of the heat and wave systems (1.3) and (1.4) in $(L^2(\Omega))^N$ and $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ respectively, it is sufficient to assume the potential a to belong to $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$. But, for system (1.1), one needs a to be more regular, i.e., to be in $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^{N\times N}))$ for establishing its well-posedness in $Y^N \times (H_0^1(\Omega))^N$, as we shall see below. This extra regularity assumption on the potential can be replaced by $a \in W^{1,\infty}(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$. But one always needs to assume that one of the derivatives of the potential a (either in space or time) belong to $L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$.
   Of course, in (1.11), one may replace the right-hand side of $P_3(T, a)$ in (1.12) by $\exp(C|||a|||_p^{\frac{1}{3-5n/2p}})$, through which one ends up with only one norm $|||a|||_p$ in (1.12). But, the observability estimate that one obtains that way fails to be optimal. Actually, the second main purpose of this paper is to show the optimality of $P_3(T, a)$.

The rest of this paper is organized as follows. In Section 2 we recall some preliminary results concerning energy and boundary trace estimates for plate systems, and weighted pointwise estimates for the Schrödinger equation. In Section 3 we state the sharp observability estimate for the plate system. In Section 4 we recall the construction by Meshkov [10], indicating its consequences for the bi-harmonic operator. In Section 5 we prove the optimality of the observability estimate. We conclude in Section 6 indicating some closely related issues and open problems. We refer to [6] for the details of the proofs of the results in this paper and other results in this context.

2 Preliminaries

In this section, we recall some preliminary results.

2.1 Energy estimates for plate systems

Denote the energy of the system (1.1) by

$$E(t) = \frac{1}{2}[|y_t(t,\cdot)|^2_{(H_0^1(\Omega))^N} + |\Delta y(t,\cdot)|^2_{(H_0^1(\Omega))^N}], \tag{2.1}$$

for the solutions y of the system (1.1). Consider also a modified energy of (1.1):

$$\mathcal{E}(t) = \frac{1}{2}[|y_t(t,\cdot)|^2_{(H_0^1(\Omega))^N} + |\Delta y(t,\cdot)|^2_{(H_0^1(\Omega))^N} + |||a|||_p^{\frac{4}{4-n/p}} |y(t,\cdot)|^2_{(H_0^1(\Omega))^N}].$$

It is clear that both energies are equivalent

$$E(t) \leq \mathcal{E}(t) \leq C(1 + |||a|||_p^{\frac{4}{4-n/p}})E(t). \tag{2.2}$$

The following estimates hold for the modified energy:

Lemma 2.1 *i) Let $a \in L^\infty(0,T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n,\infty]$. Then*

$$\mathcal{E}(t) \leq Ce^{CT|||a|||_p^{\frac{2}{4-n/p}}}\mathcal{E}(s), \qquad \forall\, t,s \in [0,T]. \tag{2.3}$$

ii) Let $a \in W^{1,\infty}(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n,\infty]$. Then

$$\mathcal{E}(t) \leq Ce^{CT|a|_p^{\frac{2}{4-n/p}}}\mathcal{E}(s), \qquad \forall\, t,s \in [0,T]. \tag{2.4}$$

Clearly, $2/(4-n/p) < 1$ for any $p \in [n,\infty]$. Therefore, the modified estimate in (2.3) is finer than the usual energy estimate (which gives $E(t) \leq Ce^{CT|||a|||_p}E(s)$ for all $t,s \in [0,T]$). However, the optimality of the estimates above is still to be investigated. The problem of the well- and ill-posedness of wave equations with low regularity coefficients in the principal part has been intensively investigated (see, for instance, [2] and [1]). Obtaining examples of equations with constant coefficients in the principal part, and low regularity zero order potentials for which the energy estimates of the above form are shown to be optimal, seems to be open both in the context of wave and plate models.

Unlike in the state spaces of the form $(H_0^1(\Omega))^N \times (H^{-1}(\Omega))^N$ or $(H^2(\Omega) \cap H_0^1(\Omega))^N \times (L^2(\Omega))^N$, to derive energy estimates in $Y^N \times (H_0^1(\Omega))^N$ further regularity assumptions on the potential a are needed. This is due to the fact that, for deriving energy estimates in this space, one needs to multiply the equation by Δy_t. In this way we get the term $\int_\Omega ay\Delta y_t$ that can not be estimated directly using the terms entering in the energy since the latter only involves the norm of y_t in $H^1(\Omega)$ and not in $H^2(\Omega)$. Thus, we have to integrate by parts:

$$\int_\Omega ay\Delta y_t = -\int_\Omega \nabla(ay)\cdot\nabla y_t dx.$$

Once this is done the integral can be estimated in terms of the energy but at the price of using an L^p estimate on ∇a. A similar argument can be done, after integration in time, to get energy estimates under an L^p assumption on the time derivative a_t of the potential.

2.2 Boundary trace estimates for plate systems

For proving the optimality of the observability estimates which we shall derive in Section 3, one needs to solve non-homogeneous boundary-value problems for plate systems. This, by transposition, requires a fine analysis of the boundary traces of solutions of the homogeneous system. In particular, in the class of solutions under consideration, one needs sharp estimates on the traces of the normal derivatives $\partial\Delta y/\partial\nu$, $\partial y_t/\partial\nu$ and $\partial y/\partial\nu$ (Here and henceforth, $\nu \equiv \nu(x)$ denotes the unit outward normal vector of Ω at $x \in \Gamma$), which are the complementary boundary conditions for our problem. This is done typically using multiplier techniques as in [8]. The estimates obtained this way are often referred to as "hidden regularity results". The following holds:

Lemma 2.2 *Assume that Γ_0 is an open subset of Γ, and ω is an open subset of Ω, intersection of Ω with a neighborhood of Γ_0. Given $T > 0$, $0 \le s_1 < s_0 < s_0' < s_1' \le T$ and $a \in L^\infty(0,T;L^p(\Omega;\mathbb{R}^{N\times N}))$ for some $p \in [n,\infty]$. Then*

$$\int_{s_0}^{s_0'}\int_\Omega |\nabla y_t|^2 dtdx \le \frac{CT^2(1+T^2)(1+\|a\|_p^2)}{(s_0-s_1)^2(s_1'-s_0')^2}\int_{s_1}^{s_1'}\int_\Omega |\nabla\Delta y|^2 dtdx,$$

and

$$\int_{s_0}^{s_0'}\int_{\Gamma_0}\left(\left|\frac{\partial y}{\partial\nu}\right|^2 + \left|\frac{\partial y_t}{\partial\nu}\right|^2 + \left|\frac{\partial\Delta y}{\partial\nu}\right|^2\right) dtdx$$

$$\le \frac{CT^4(1+T^2)(1+\|a\|_p^3)}{(s_0-s_1)^3(s_1'-s_0')^3}\int_\omega (|\nabla\Delta y|^2 + |\nabla y|^2) dtdx.$$

The first conclusion in the above lemma follows from the usual energy method; while the second one can be proved, as we mentioned above, by using multiplier techniques similar to those in [8, Chapter IV]. More precisely, to show the second conclusion, by multiplying the equation (or system) by $(t-s_1)(s_1'-t)\eta\cdot\nabla\Delta y$, where η is a smooth extension of the normal vector field to the interior of Ω, we obtain an estimate for $\partial\Delta y/\partial\nu$ and $\partial y_t/\partial\nu$. In order to get the estimate for $\partial y/\partial\nu$ we observe that y can be viewed as a solution of a Schrödinger equation of the form $iy_t + \Delta y = z$, satisfying Dirichlet boundary conditions. Using the multiplier $\eta\cdot\nabla y$ as above (see [9]), one gets an estimate on $\partial y/\partial\nu$ in $L^2(\Sigma)$ in terms of the $L^2(Q)$-norm of z and the $L^2(0,T;H^1(\Omega))$-norm of y. Obviously, both quantities are bounded above in terms of the energy.

2.3 Pointwise weighted estimates for the Schrödinger operator

In this section, we present some pointwise weighted estimates for the Schrödinger equation that will play a key role when deriving the sharp observability estimate for the plate system. In fact, the estimates for the plate system will be obtained applying, in an iterative manner, these pointwise estimates for the Schrödinger equation. The underlying fact is the possibility of decomposing the plate operator $\partial_t^2 + \Delta^2$ as two conjugate Schrödinger operators: $\partial_t^2 + \Delta^2 = (i\partial_t + \Delta)(-i\partial_t + \Delta)$.

First, we show a pointwise weighted estimate for the Schrödinger operator "$i\partial_t + \Delta$". For this, for any $\lambda > 0$, $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$, set

$$\ell(t,x) = \frac{\lambda}{2}\left[|x-x_0|^2 - c\left(t - \frac{T}{2}\right)^2\right]. \tag{2.5}$$

By taking $a = 0$, $b = 1$ and $(a^{jk})_{n\times n} = I$ the identity matrix, and $\theta = e^{\ell}$ with ℓ given by (2.5) in [5, Theorem 1.1], and using Hölder's inequality, one gets the following result.

Lemma 2.3 *Let* $z \in C^2(\mathbb{R}^{1+n};\mathbb{C})$, $\theta = e^{\ell}$ *and* $v = \theta z$. *Then*

$$4\lambda|\nabla v|^2 + B|v|^2 \le \theta^2|iz_t + \Delta z|^2 + M_t + \sum_{j=1}^n V_j, \tag{2.6}$$

where

$$\begin{cases} M \stackrel{\triangle}{=} \ell_t|v|^2 + i\displaystyle\sum_{j=1}^n \ell_j(\overline{v}_j v - v_j\overline{v}), \\ V_j \stackrel{\triangle}{=} \displaystyle\sum_{k=1}^n \{-i\ell_j(\overline{v}_t v - \overline{v}v_t) - i\ell_t(v_j\overline{v} - \overline{v}_j v) + \Delta\ell(v_j\overline{v} + \overline{v}_j v) \\ \qquad +2\ell_j(v_j\overline{v}_k + \overline{v}_j v_k - |\nabla v|^2) + (2\ell_j|\nabla\ell|^2 - \Delta\ell_j)|v|^2\}_j, \\ B \stackrel{\triangle}{=} 4\lambda^3|x-x_0|^2 - \lambda c. \end{cases} \tag{2.7}$$

Noting the obvious fact that $|iz_t + \Delta z| = |-iz_t + \Delta z|$, Lemma 2.3 also gives a pointwise estimate for the conjugate Schrödinger operator "$-i\partial_t + \Delta$". Note also that Lemma 2.3 simplifies a similar pointwise estimate in [11].

3 The sharp observability estimate

In this section we state the sharp observability estimate for system (1.1).

For this purpose, for any fixed $x_0 \in \mathbb{R}^n$ and $\delta > 0$, we introduce the following set:

$$\begin{cases} \omega = \mathcal{O}_\delta(\Gamma_0)\bigcap\Omega, \\ \Gamma_0 \stackrel{\triangle}{=} \{x \in \Gamma \mid (x-x_0)\cdot\nu(x) > 0\}, \end{cases} \tag{3.1}$$

where $\mathcal{O}_\delta(\Gamma_0) = \{x \in \mathbb{R}^n \mid |x - x'| < \delta \text{ for some } x' \in \Gamma_0\}$.

One of the main results in this paper is the following observability inequality with explicit dependence of the observability constant on the potential a for system (1.1):

Theorem 3.1 *Let ω be given by (3.1) for some $x_0 \in \mathbb{R}^n$ and $\delta > 0$, and $p \in [n, \infty]$. Then there is a constant $C > 0$ such that for any $T > 0$ and any $a \in L^\infty(0,T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$, the weak solution y of system (1.1) satisfies estimate (1.2) with the observability constant $P(T,a) > 0$ verifying*

$$P(T,a) \leq \exp\left[C\left(1 + \frac{1}{T} + T|||a|||_p^{\frac{1}{2-n/2p}} + \|a\|_p^{\frac{1}{3-5n/2p}}\right)\right]. \tag{3.2}$$

If the potential $a \equiv a(x) \in L^p(\Omega; \mathbb{R}^{N\times N})$ is assumed to be time-independent, then (3.2) can be improved to

$$P(T,a) \leq \exp\left[C\left(1 + \frac{1}{T} + T\|a\|_p^{\frac{1}{2-n/2p}} + \|a\|_p^{\frac{1}{3-5n/2p}}\right)\right]. \tag{3.3}$$

Inequality (3.2) provides the estimates (1.11) and (1.12) we discussed in the introduction. Note that we have used two norms on the potential a in (3.2). But, as we mentioned above, the use of the norm in $L^\infty(0,T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$ is only due to the need for performing energy estimates. In the special case of time-independent potentials, we use only one norm (see (3.3)), because, as we mentioned above, the additional regularity assumption that a belongs to $L^\infty(0,T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$ can be replaced by the fact that it belongs to $W^{1,\infty}(0,T; L^p(\Omega; \mathbb{R}^{N\times N}))$, a fact that automatically holds when $a = a(x)$ belongs to $L^p(\Omega; \mathbb{R}^{N\times N})$.

As in [11], to prove Theorem 3.1, one needs to decompose the plate equation into two Schrödinger systems and to apply the pointwise estimate for the later one in cascade. The main point in the proof of Theorem 3.1 is as follows: For simplicity, we assume that $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Hence,

$$R_1 \stackrel{\triangle}{=} \max_{x\in\Omega} |x - x_0| > R_0 \stackrel{\triangle}{=} \min_{x\in\Omega} |x - x_0| > 0.$$

First, set $z = -iy_t + \Delta y$, and note that $-ay = y_{tt} + \Delta^2 y = iz_t + \Delta z$. We see that y and z solve

$$\begin{cases} -iy_t + \Delta y = z & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \begin{cases} iz_t + \Delta z = -ay & \text{in } Q,\\ z = 0 & \text{on } \Sigma,\\ z(0) = -iy_1 + \Delta y^0 & \text{in } \Omega. \end{cases}$$

Next, choose the constant c in ℓ defined by (2.5) of section (2.3) such that

$$R_1^2 - cT^2/4 < 0.$$

This gives the desired weight function $\theta = e^\ell$. One may find T_1 and T_1' satisfying $0 < T_1 < T_1' < T$, independent of λ, so that

$$\ell(t,x) < 0, \qquad \forall\ (t,x) \in \left([0,T_1]\bigcup[T_1',T]\right) \times \Omega. \tag{3.4}$$

We now apply Lemma 2.3 to z. Integrating (2.6) in $(T_1, T_1') \times \Omega$, noting (3.4), recalling the definition of $E(t)$ in (2.1), one may deduce that

$$\begin{aligned}
&\lambda \int_{T_1}^{T_1'} \int_\Omega \theta^2 |\nabla z|^2 + \lambda^3 \int_{T_1}^{T_1'} \int_\Omega \theta^2 |z|^2 dtdx \\
&\quad \le C \left\{ \|\theta(iz_t + \Delta z)\|^2_{L^2(Q)} + \lambda[E(T_1) + E(T_1')] + \lambda e^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} \left| \frac{\partial z}{\partial \nu} \right|^2 \right\} \\
&\quad \le C \Bigg\{ \|\theta a y\|^2_{L^2(Q)} + \lambda[E(T_1) + E(T_1')] \\
&\qquad + \lambda e^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} \left(\left| \frac{\partial y_t}{\partial \nu} \right|^2 + \left| \frac{\partial \Delta y}{\partial \nu} \right|^2 \right) dtdx \Bigg\},
\end{aligned} \tag{3.5}$$

with Γ_0 being given in (3.1). Obviously, for the above to be true we need to take $\lambda > 0$ large enough so that the constant B in Lemma 2.3 is positive.

Similarly, applying Lemma 2.3 to y, we deduce that

$$\begin{aligned}
&\lambda \int_{T_1}^{T_1'} \int_\Omega \theta^2 |\nabla y|^2 + \lambda^3 \int_{T_1}^{T_1'} \int_\Omega \theta^2 |y|^2 dtdx \\
&\quad \le C \Bigg\{ \|\theta(-iy_t + \Delta y)\|^2_{L^2(Q)} + \lambda[E(T_1) + E(T_1')] \\
&\qquad + \lambda e^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 dtdx \Bigg\} \\
&\quad = C \left\{ \|\theta z\|^2_{L^2(Q)} + \lambda[E(T_1) + E(T_1')] + \lambda e^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} \left| \frac{\partial y}{\partial \nu} \right|^2 dtdx \right\}.
\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we arrive at

$$\begin{aligned}
&\lambda \int_Q \theta^2 (|\nabla y_t|^2 + |\nabla \Delta y|^2) dtdx + \lambda^4 \int_Q \theta^2 |\nabla y|^2 dtdx + \lambda^6 \int_Q \theta^2 |y|^2 dtdx \\
&\quad \le C \Bigg\{ \|\theta a y\|^2_{L^2(Q)} + \lambda^4 [E(T_1) + E(T_1')] + \lambda^6 \left[\int_0^{T_1} E(t)dt + \int_{T_1'}^T E(t)dt \right] \\
&\qquad + \lambda e^{C\lambda} \int_{T_1}^{T_1'} \int_{\Gamma_0} \left(\left| \frac{\partial y}{\partial \nu} \right|^2 + \left| \frac{\partial y_t}{\partial \nu} \right|^2 + \left| \frac{\partial \Delta y}{\partial \nu} \right|^2 \right) dtdx \Bigg\}.
\end{aligned} \tag{3.7}$$

At this level it is useful to observe that, since, without loss of generality, one may assume y to be a real function, an estimate on z as in (3.5) yields an estimate on both y_t and Δy simultaneously.

We have now to get rid of the terms on the right-hand side of (3.7):

- We consider first the term $\|\theta a y\|^2_{L^2(Q)}$.
  By [3], for any $\epsilon > 0$, we have

$$
\begin{aligned}
&\|\theta a y\|^2_{L^2(Q)}\\
&\quad\le \epsilon\lambda\|\theta y\|^2_{L^2(0,T;\,H_0^1(\Omega))} + \epsilon^{-n/(p-n)}\|a\|_p^{2p/(n-p)}\lambda^{-n/(p-n)}\|\theta y\|^2_{L^2(Q)}.
\end{aligned}
\tag{3.8}
$$

  By taking ϵ small enough the first term $\epsilon\lambda\|\theta y\|^2_{L^2(0,T;\,H_0^1(\Omega))}$ can be absorbed by the left-hand side of (3.7). Then, for this choice of ϵ and taking λ sufficiently large, the term $\epsilon^{-n/(p-n)}\|a\|_p^{2p/(n-p)}\lambda^{-n/(p-n)}\|\theta y\|^2_{L^2(Q)}$ can be absorbed similarly.
- Concerning the boundary integrals we proceed as follows. Noting the definition of ω in (3.1), using Lemma 2.2, one has

$$
\begin{aligned}
&\int_{T_1}^{T_1'}\int_{\Gamma_0}\left(\left|\frac{\partial y}{\partial\nu}\right|^2 + \left|\frac{\partial y_t}{\partial\nu}\right|^2 + \left|\frac{\partial\Delta y}{\partial\nu}\right|^2\right)dtdx\\
&\quad\le C(1+\|a\|_p^3)\int_0^T\int_\omega(|\nabla y|^2 + |\nabla\Delta y|^2)dtdx.
\end{aligned}
\tag{3.9}
$$

- The energy terms in the right-hand side can be absorbed by the terms on the left side. This can be done, once more, by taking λ large enough and exploiting the exponential growth of the weight function θ (near $t = T/2$) on λ. Note however that in this argument one has to use carefully the energy estimates, which grow exponentially with a suitable power of the norm of the potential.

4 Extension of Meshkov's construction to the bi-Laplacian equation

In this section, we construct a very special time-independent complex-valued solution u of the following bi-Laplacian equation:

$$
\Delta^2 u = qu, \qquad \text{in } \mathbb{R}^n, \tag{4.1}
$$

which decays at infinity sufficiently fast, for some bounded complex-valued potential $q \ne 0$. For x in $\mathbb{R}^n$, we shall write $r \triangleq |x|$.

The main result of this section is stated as follows:

Theorem 4.1 *Let $n \geq 2$ be even and $c_* > 0$. Then there exist two non-trivial complex-valued functions:*

$$u \in C^\infty(\mathbb{R}^n; \mathbb{C}), \quad q \in C^\infty(\mathbb{R}^n; \mathbb{C}) \bigcap L^\infty(\mathbb{R}^n; \mathbb{C})$$

such that (4.1) is satisfied, and for some constant C:

$$|u(x)| + |\nabla u(x)| + |\nabla \Delta u(x)| \leq Ce^{-c_* r^{4/3}}, \qquad \forall\, x \in \mathbb{R}^n. \tag{4.2}$$

The general case of any even dimension can easily be derived by separation of variables. The construction needed for proving Theorem 4.1 is in fact the same as in Meshkov's paper and [3]. There, a complex-valued smooth function decaying as in (4.2) is built such that $|\Delta u| \leq C|u|$ for some finite $C > 0$. The proof of the present theorem is based on the observation that, in fact, $|\Delta^2 u| \leq C'|u|$ as well for some other finite $C' > 0$.

The construction above is sharp. More precisely, one can not build non-trivial solutions u of equations (scalar or systems) of the form $\Delta^2 u = q(x)u$ with $q = q(x)$ bounded for which u decays at infinity faster than $\exp(-|x|^{4/3})$. This can be proved as in [10, Theorem 1]. The proof there uses the following Carleman inequality: For some constant $\tau_0 > 0$, and $C > 0$,

$$\tau^3 \int_{\mathbb{R}^n} |v|^2 r \exp(2\tau r^{4/3}) dx \leq C \int_{\mathbb{R}^n} |\Delta v|^2 r \exp(2\tau r^{4/3}) dx,$$
$$\forall\, \tau \geq \tau_0, \quad v \in C_0^\infty(\{r > 1\}).$$

Here and in the sequel $r = |x|$. Applying this inequality twice we get

$$\tau^6 \int_{\mathbb{R}^n} |v|^2 r \exp(2\tau r^{4/3}) dx \leq C \int_{\mathbb{R}^n} |\Delta^2 v|^2 r \exp(2\tau r^{4/3}) dx,$$
$$\forall\, \tau \geq \tau_0, \quad v \in C_0^\infty(\{r > 1\}).$$

Starting from this inequality the argument in the proof of [10, Theorem 1] shows the optimality of the decay rate (4.2) for the bi-harmonic operator, too.

It is worth noticing that, despite the different order of the bi-harmonic equation considered here, which is of order 4, the sharp superexponential decay is the same as that for the Laplacian.

5 Optimality of the observability constant for plate systems

This section is devoted to showing that when $p = \infty$, the term $\|a\|_p^{\frac{1}{3-5n/2p}}$ (i.e., $\|a\|_p^{1/3}$) in the estimate (3.3) is sharp in what concerns the dependence on the potential a in even space dimensions $n \geq 2$ for systems with at least two equations. More precisely, the following holds:

Theorem 5.1 *Assume that $n \geq 2$ is even and that $N \geq 2$. Let ω be any given open non-empty subset of Ω such that $\Omega \setminus \overline{\omega} \neq \emptyset$. Then, there exist two constants $c_1 > 0$ and $\mu > 0$, a family of time-independent potentials $\{a_R\}_{R>0} \subset L^\infty(\Omega; \mathbb{R}^{N\times N})$ satisfying*

$$\|a_R\|_\infty \to \infty, \qquad \text{as } R \to \infty$$

and a family of initial data $\{(y_R^0, y_R^1)\}_{R>0} \in Y^N \times (H_0^1(\Omega))^N$ such that the corresponding weak solutions $\{y_R\}_{R>0}$ of (1.1) satisfy

$$\lim_{R\to\infty} \left\{ \inf_{T\in I_\mu} \frac{\|\Delta y_R^0\|^2_{(H_0^1(\Omega))^N} + \|y_R^1\|^2_{(H_0^1(\Omega))^N}}{\exp(c\|a_R\|_\infty^{1/3}) \int_0^T \int_\omega (|\nabla y|^2 + |\nabla \Delta y|^2) dt dx} \right\} = \infty, \tag{5.1}$$

where $I_\mu \stackrel{\triangle}{=} (0,\ \mu\|a_R\|_\infty^{-1/6}]$.

The main idea in order to prove Theorem 5.1 is the same as that in [3]. Based on the construction of u and q in Theorem 4.1, by suitable scaling and localization arguments, one can find a family of rescaled potentials $a_R(x) = R^4 q(Rx)$ with an L^∞-norm of the order of R^4 and a family of solutions $u_R(x) = u(Rx)$ of the corresponding bi-harmonic problem, with a decay of the order of

$$|u_R(x)| \leq C\exp(-R^{4/3}|x|^{4/3}).$$

Without loss of generality we may assume that both, the observation subdomain ω and the exterior boundary Γ, are included in the region $|x| \geq 1$. This yields a sequence of solutions of the elliptic systems $\Delta^2 u_R = a_R u_R$ in which the ratio between total energy and the energy concentrated in ω and the norm of the boundary traces is of the order of $\exp(-R^{4/3})$. Taking into account that $\|a_R\|_\infty \sim R^4$, this ratio turns to be of the order of $\exp(-\|a_R\|_\infty^{1/3})$. These solutions of the above-mentioned elliptic system can be regarded also as solutions of the plate system for suitable initial data. However, they do not fully satisfy the requirements in our optimality theorem because they do not fulfill homogeneous boundary conditions. This can be compensated by subtracting the solution taking their boundary data and zero initial ones. These solutions turn out to be exponentially small in the energy space $Y^N \times (H_0^1(\Omega))^N$ during a time interval of the order of $T \leq \mu\|a_R\|_\infty^{-1/6}$. This can be shown to hold because of the estimates in Lemma 2.2 and standard energy and transposition arguments.

Note that Theorem 5.1 looks more like the situation described in [3, Theorem 1.1] for the heat system rather than that in [3, Theorem 1.2] for the wave equation. Indeed, as for the optimality for the heat observability constant $H(T, a)$, one has to take T to be small enough to compensate the time evolution of the energy and make sure that the concentration of the solution of the evolution plate system suffices for guaranteeing that (5.1) holds. This is however not necessary for the wave system. It is an unsolved problem whether one can prove Theorem 5.1 for $T > 0$ fixed and replacing (5.1) by the following

$$\lim_{R\to\infty} \frac{\|\Delta y_R^0\|^2_{(H_0^1(\Omega))^N} + \|y_R^1\|^2_{(H_0^1(\Omega))^N}}{\exp(c\|a_R\|_\infty^{1/3}) \int_0^T \int_\omega (|\nabla y|^2 + |\nabla \Delta y|^2) dt dx} = \infty. \tag{5.2}$$

6 Further remarks and open problems

In this paper we have indicated some open problems and closely related issues that remain to be clarified. We summarize here some of them:

- According to the construction in [3] one can adapt Meshkov's solutions to odd space dimensions and get the quasi-optimality of the observability estimates for heat and wave systems. One could expect the same to hold for the plate systems as well.
- The optimality result in this paper does not apply, either for scalar equations or for $1-d$ problems. The same happens for heat and wave equations. This is a completely open subject. Note however that the potentials we use depend only on x. Very likely other constructions could be made using time-dependent potentials, but this remains to be explored.
- It is very likely that our results, both with respect obtaining explicit observability estimates and their optimality, can be extended to plate systems with other boundary conditions, like, for instance, those corresponding to clamped plates: $y = \partial y/\partial \nu = 0$. A systematic analysis of this issue remains to be done.
- Similar problems arise for other plate systems, including those containing the rotational inertia term:

$$y_{tt} - \gamma \Delta y_{tt} + \Delta^2 y + ay = 0.$$

- As we mentioned above, all these questions can be analyzed in other energy spaces.
- Similar questions arise for the Schrödinger equation also. The analysis in this paper can be adapted in a straightforward way to the observability of that model in $H_0^1(\Omega)$. But the issue is more subtle and remains to be investigated in the $L^2(\Omega)$ context.

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