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A DIRECT METHOD FOR THE BOUNDARY STABILIZATION OF THE WAVE EQUATION

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ABSTRACT. — We consider the wave equation $y''-\Delta y=0$ in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary Γ, subject to mixed boundary conditions $y=0$ on Γ_1 and $\partial y/\partial\nu=F(x, y')$ on Γ_0, (Γ_0, Γ_1) being a partition of Γ. We study the boundary stabilizability of the solutions *i. e.* the existence of a partition (Γ_0, Γ_1) and of a boundary feed-back F (., .) such that every solution (corresponding to initial data with finite energy) decays exponentially in the energy space as $t\to\infty$. Several authors proved earlier that, under very strong geometrical hypothesis on Ω, the system is stabilized by the feed-back $F(x, y')=-b(x)y'$ with $b(x)\geqq b_0>0$ if the partition (Γ_0, Γ_1) is suitably chosen. We prove in this paper the stabilizability of the system without geometrical hypothesis on Ω (at least if $n\leqq 3$). The proof is based on the use of a feed-back $F(x, y')=-b(x)y'$ with $b(x)\geqq 0$ and $b(x)=0$ on the interface points $x\in cl(\Gamma_0)\cap cl(\Gamma_1)$, and on the construction of energy functionals, well adapted to the system. We derive a differential inequality for these functionals which lead to very precise estimates on the decay rates. This method is rather general and can be adapted to other evolution systems (*e.g.* models of plates, elasticity systems) as well. Also, it allows to prove the exponential decay for the solutions of the semilinear wave equation $y''-\Delta y+f(y)=0$, under natural growth and sign assumptions on the nonlinearity f.

1. Introduction and statement of the main result

Let Ω be a bounded, open, connected set in $\mathbb{R}^n$ $(n\geqq 1)$ having a boundary $\Gamma=\partial\Omega$ of class C^2. Given a partition (Γ_0, Γ_1) of Γ and a function $F:\Gamma_0\times\mathbb{R}\to\mathbb{R}$, consider the wave equation

$$y''-\Delta y=0 \quad \text{in } \Omega\times(0, \infty) \tag{1.1}$$

with the boundary and initial conditions

$$\partial y/\partial\nu=F(x, y') \quad \text{on } \Gamma_0\times(0, \infty) \tag{1.2}$$

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(1.3) $$y=0 \quad \text{on } \Gamma_1 \times (0, \infty)$$

(1.4) $$y(0)=y^0, \qquad y'(0)=y^1 \quad \text{in } \Omega$$

where ν is the unit normal vector of Γ pointing towards the exterior of Ω and $y'=\partial y/\partial t$, $y''=\partial^2 y/\partial t^2$.

If $y=y(x, t)$ is a (sufficiently smooth) solution of this system, then we define its *energy* by

(1.5) $$\mathrm{E}(y;\, t)=\frac{1}{2}\int_\Omega |y'(x, t)|^2+|\nabla y(x, t)|^2\, dx, \qquad \forall t \geqq 0.$$

We wish to *stabilize* this system, *i. e.* we seek a suitable partition (Γ_0, Γ_1) of Γ and a suitable feed-back $\mathrm{F}(x, y')$ such that for any initial data (y^0, y^1) [of finite energy $\mathrm{E}(y;\, 0) < \infty$, and satisfying the natural compatibility condition $y^0=0$ on Γ_1] the energy (1.5) of the solution y of the problem (1.1)-(1.4) *tends to zero exponentially* as $t \to \infty$.

Taking (at least formally) the derivative of the energy $\mathrm{E}(y;\, t)$, one obtains

$$\frac{d}{dt}\mathrm{E}(y;\, t)=\int_{\Gamma_0} \mathrm{F}(x, y'(x, t))\, y'(x, t)\, d\Gamma(x), \qquad \forall t \geqq 0.$$

Thus, the energy will decrease if we take a feed-back of the form

(1.6) $$\mathrm{F}(x, s)=-b(x)s, \qquad x \in \Gamma_0, \quad s \in \mathbb{R}$$

with $b \in \mathrm{L}^\infty(\Gamma_0)$, $b(x) \geqq 0$ a.e. in Γ_0.

To formulate our main result, fix a point $x^0 \in \mathbb{R}^n$ and set

(1.8) $$m(x)=x-x^0, \qquad x \in \mathbb{R}^n$$

(1.9) $$\Gamma_0=\{x \in \Gamma : m(x).\nu(x) > 0\}$$

(1.10) $$\Gamma_1=\{x \in \Gamma : m(x).\nu(x) \leqq 0\}$$

(1.11) $$\mathrm{F}(x, s)=-(m(x).\nu(x))\, s, \qquad x \in \Gamma_0, \quad s \in \mathbb{R}$$

(. denotes the scalar product in $\mathbb{R}^n$).

Let us assume that $\Gamma_1 \neq \varnothing$. (This is not a restriction on Ω, since it is always the case if we choose x^0 in the exterior of Ω.) Then

(1.12) Γ_0 and Γ_1 have non-empty interior in Γ.

Indeed, if x is a farthest (resp. nearest) point in Γ from x^0, then x is an interior point of Γ_0 (resp. Γ_1). It follows that in the subspace

(1.13) $$\mathrm{V}=\{\varphi \in \mathrm{H}^1(\Omega) : \varphi=0 \text{ on } \Gamma_1\}$$

of the Sobolev space $\mathrm{H}^1(\Omega)$, Poincaré's inequality holds, *i. e.*

(1.14) $$\exists \alpha > 0 \text{ such that } \|\varphi\|_{\mathrm{L}^2(\Omega)} \leqq \alpha \|\nabla\varphi\|_{\mathrm{L}^2(\Omega, \mathbb{R}^n)}, \qquad \forall \varphi \in \mathrm{V}.$$

In view of this inequality we may (and we shall) consider in V the equivalent scalar product $(\varphi, \psi) \mapsto \int_\Omega \nabla \varphi . \nabla \psi \, dx$.

We also note that for any fixed $s \in \mathbb{R}$, $F(x, s) \to 0$ whenever x tends to a point of $cl(\Gamma_0) \cap cl(\Gamma_1)$.

Summarizing, we shall consider the problem (1.1)-(1.4) where Γ_0, Γ_1 and F are defined by (1.8)-(1.12).

It is well-known that for any initial data

$$(y^0, y^1) \in V \times L^2(\Omega)$$

this system has a unique solution

$$y \in C([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega)), \tag{1.15}$$

and this solution decreases in the energy space, *i. e.*

$$E(y; t_1) \leqq E(y; t_2) \quad \text{if} \quad t_1 \geqq t_2. \tag{1.16}$$

Our main result is as follows.

THEOREM 1. — *Assume that* $n \leqq 3$. *Then for every constant* $C > 1$ *there exists a constant* $\omega > 0$ *such that, for any initial data*

$$(y^0, y^1) \in V \times L^2(\Omega)$$

the energy (1.5) *of the solution of the system* (1.1)-(1.4), *where* Γ_0, Γ_1 *and F are given by* (1.8)-(1.12), *satisfies the inequality*

$$E(y; t) \leqq C e^{-\omega t} E(y; 0) \quad \textit{for all } t \geqq 0. \tag{1.17}$$ ●

Remark 1.1. — The proof below will provide explicit expressions for C and ω in terms of Ω and x^0. The fact that we obtain explicit estimate on ω appears to be new. ●

Remark 1.2. — If $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$ then our proof given below remains valid without the restriction $n \leqq 3$ (Note that this assumption excludes the simply connected regions.) In this case the theorem was already proved (following the work of Quinn and Russell [26]) by Chen ([2], [3], [4]) and Lagnese [17] [without, however, explicit constants in (1.17)]. In this case the solution is sufficiently regular for proving (1.17) by a multiplier technique. These calculations are no longer valid in the general case. However, as it was proved by Grisvard ([7], [8], [9]), at least in case $n \leqq 3$, one of the crucial identities (in the application of a multiplier technique) becomes an inequality which, fortunately, is sufficient for our purposes. The case $n \geqq 4$ (without the hypothesis $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$) remains open. Provided that Grisvard's inequality extends to higher dimensions, our method should apply. However, such an extension does not seen to be proven. ●

Remark 1.3. — The choice of the special multiplier $m(x).\nu(x)$ in the feed-back (1.11) seems to be new. It will play a crucial role in the proof. •

Remark 1.4. — The case $\Gamma_1=\varnothing$ is not considered in this paper. For this case the estimate (1.17) was already proved earlier by J. Lagnese [17]. This, however, is not sufficient for the stabilization because now every constant function is a solution of energy zero. This problem is solved in E. Zuazua [35] by using a more general feed-back of the form $-(m.\nu)(y'+\alpha y)$. (For $\alpha=0$ we recover (1.11).) •

The paper is organized as follows:

- Section 2 is devoted to the proof of Theorem 1.
- In Section 3 several generalizations are given.

In particular, we will show that the proof of Theorem 1 carries over to some semilinear wave equations with minor modifications. Also, we sutdy the wave equation with a linear potential.

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2. Proof of the main result

2.1. ENERGY ESTIMATES. — It is well-known that the system (1.1)-(1.4) with the feed-back (1.11) is well-posed. The existence of strong and weak solutions may be proven either by the Galerkin method (see *e.g.* Lions-Magenes [25] or for the present case Quinn-Russell [26]) or by the theory of semigroups. Let us outline, for the sake of completeness, the semigroup approach.

Let us introduce the linear operator

$$A(y, z)=(-z, -\Delta y) \tag{2.1}$$

where

$$D(A)=\left\{(y, z)\in V\times V \mid \Delta y\in L^2(\Omega) \text{ and } \frac{\partial y}{\partial \nu}=-(m.\nu)z \text{ on } \Gamma_0\right\}; \tag{2.2}$$

here Δy is taken in the distributional sense while the equality $\partial y/\partial\nu=-(m.\nu)z$ means that

$$\int_\Omega \nabla y.\nabla v\,dx+\int_\Omega (\Delta y)\,v\,dx+\int_{\Gamma_0}(m.\nu)zv\,d\Gamma=0, \qquad \forall v\in V.$$

(Observe that $\partial y/\partial\nu$ is not necessarily defined by the usual trace theorems; however, when both definitions make sense, then they coincide.) It is clear that A maps $D(A)\subset V\times L^2(\Omega)$ linearly into $V\times L^2(\Omega)$. Using the operator A we may interprete the system (1.1)-(1.4), (1.11) in the following operational form:

$$(y, z)' + A(y, z) = 0 \quad \text{in } [0, \infty) \qquad \text{and} \qquad (y, z)(0) = (y^0, y^1). \tag{2.3}$$

It is easy to verify that A is a maximal monotone operator in the Hilbert space $V \times L^2(\Omega)$. Applying the Hille-Yosida theorem, it follows that for every initial data $(y^0, y^1) \in D(A)$ the problem (2.3) [or equivalently the system (1.1)-(1.4), (1.11)] has a unique solution

$$(y, y') \in C([0, \infty); D(A)) \cap C^1([0, \infty); V \times L^2(\Omega)). \tag{2.4}$$

Furthermore, D (A) is dense in $V \times L^2(\Omega)$ and for each fixed $t \in [0, \infty)$ the linear map

$$(y^0, y^1) \mapsto (y(t), z(t)) \tag{2.5}$$

extends to a unique contraction $S(t)$ of $V \times L^2(\Omega)$ such that $(S(t))_{t \geqq 0}$ is a strongly continuous semigroup of contractions in $V \times L^2(\Omega)$. We may therefore define for every initial data $(y^0, y^1) \in V \times L^2(\Omega)$ the weak solution of (2.3) by the formula

$$(y(t), z(t)) := S(t)(y^0, y^1), \ t \geqq 0.$$

Then

$$(y, z) \in C([0, \infty); V \times L^2(\Omega)).$$

Since the maps (2.5) are continuous for every fixed $t \geqq 0$ and since D (A) is dense in $V \times L^2(\Omega)$, it is sufficient to prove the estimate (1.17) of Theorem 1 in the special case where (y^0, y^1) belongs to D (A). We shall therefore assume in the sequel that (2.4) holds.

Remark 2.1. — It follows from (2.4) that

$$\Delta y \in C([0, \infty); L^2(\Omega)).$$

Moreover, if $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$, then by the standard regularity properties of the solutions of elliptic problems we have also

$$y \in C([0, \infty); H^2(\Omega)).$$

However, as it was shown by Lagnese [17] (*cf.* also Grisvard [6]), we have *not* $y \in C([0, \infty); H^2(\Omega))$ in general. ●

We shall need the inequality

$$\exists \beta > 0, \quad \int_{\Gamma_0} (m.\nu)\varphi^2 \, d\Gamma \leqq \beta \int_\Omega |\nabla \varphi|^2 \, dx, \qquad \forall \varphi \in V. \tag{2.6}$$

It follows easily from the Poincaré inequality (1.14) combined with the trace inequality in $H^1(\Omega)$. [Here and throughout this paper α, β, c_1, c_2, . . . denote constants depending only on Ω and eventually on x^0, but not on the initial data (y^0, y^1).]

We now fix $\varepsilon > 0$ and introduce the functional

$$\mathrm{E}_\varepsilon(y;\, t) = \mathrm{E}(y;\, t) + \varepsilon\rho(y;\, t) \tag{2.7}$$

with

$$\rho(y;\, t) = \int_\Omega y'(x,\, t)\,[2\,m(x)\,.\,\nabla y(x,\, t) + (n-1)\,y(x,\, t)]\,dx \tag{2.8}$$

for $t \geqq 0$. We shall prove the existence of positive constants ε_0, c_1, c_2 such that

$$|\mathrm{E}_\varepsilon(y;\, t) - \mathrm{E}(y;\, t)| \leqq c_1\,\varepsilon\,\mathrm{E}(y;\, t), \qquad \forall\, t \geqq 0, \quad \forall\, \varepsilon > 0, \tag{2.9}$$

and that

$$\mathrm{E}'_\varepsilon(y;\, t) := d\mathrm{E}_\varepsilon(y;\, t)/dt \leqq -c_2\,\varepsilon\,\mathrm{E}_\varepsilon(y;\, t), \qquad \forall\, t \geqq 0 \tag{2.10}$$

whenever $\varepsilon \leqq \varepsilon_0$.

From (2.9), (2.10) the theorem will follow. Indeed, for $0 < \varepsilon \leqq c_1^{-1}$, $\mathrm{E}_\varepsilon(y;\, t) \geqq 0$ by (2.9) and then integrating (2.10) we obtain

$$\mathrm{E}_\varepsilon(y;\, t) \leqq \exp(-c_2\,\varepsilon\, t)\,\mathrm{E}_\varepsilon(y;\, 0), \qquad \forall\, t \geqq 0. \tag{2.11}$$

On the other hand, for $0 < \varepsilon < c_1^{-1}(1 - \mathrm{C}^{-1/2})$ (2.9) implies that

$$\mathrm{E}(y;\, t) \leqq \sqrt{\mathrm{C}}\,\mathrm{E}_\varepsilon(y;\, t) \leqq \mathrm{C}\mathrm{E}(y;\, t), \qquad \forall\, t \geqq 0, \tag{2.12}$$

and (2.9), (2.10) yield (1.17) with $\omega = c_2\,\varepsilon$.

The proof of (2.9) is straightforward. Putting for brevity

$$\mathrm{R} = \|m\|_{\mathrm{L}^\infty(\Omega;\, \mathbb{R}^n)} \tag{2.13}$$

we have, using (1.14),

$$\begin{aligned}
\varepsilon^{-1}\,|\mathrm{E}_\varepsilon(y;\, t) - \mathrm{E}(y;\, t)| &= |\rho(y;\, t)| \\
&\leqq \|y'(t)\|_{\mathrm{L}^2(\Omega)}\,[2\,\mathrm{R}\,\|\nabla y(t)\|_{\mathrm{L}^2(\Omega;\, \mathbb{R}^n)} + (n-1)\,\|y(t)\|_{\mathrm{L}^2(\Omega)}] \\
&\leqq (2\,\mathrm{R} + (n-1)\,\alpha)\,\|y'(t)\|_{\mathrm{L}^2(\Omega)}\,\|\nabla y(t)\|_{\mathrm{L}^2(\Omega;\, \mathbb{R}^n)} \\
&\leqq (2\,\mathrm{R} + (n-1)\,\alpha)\,\mathrm{E}(y;\, t), \qquad \forall\, t \geqq 0,
\end{aligned}$$

i.e. (2.9) is satisfied with

$$c_1 = 2\,\mathrm{R} + (n-1)\,\alpha. \tag{2.14}$$

Now we turn to the proof of (2.10). In order to simplify the notations, we shall omit the variables x, t of the functions under the integral signs *i.e.* we shall write y, y', ∇y, m, ν instead of $y(t, x)$, $y'(t, x)$, $\nabla y(t, x)$, $m(x)$, $\nu(x)$, etc. We will use the following important technical result; it is a slight generalization of an inequality of Grisvard ([7], [8], [9]). The proof of the lemma will be given in the Subsection 2.2.

LEMMA 2.2. — *Assume that $n \leqq 3$ and let $(y, z) \in D(A)$. Then*

$$(2.15)\quad 2\int_\Omega (\Delta y)\, m.\nabla y\, dx \leqq (n-2)\int_\Omega |\nabla y|^2\, dx + 2\int_\Gamma (\partial y/\partial \nu)\, m.\nabla y\, d\Gamma - \int_\Gamma (m.\nu)\,|\nabla y|^2\, d\Gamma. \quad \bullet$$

Let us differentiate (1.5), (2.7) and (2.8). We obtain

$$(2.16)\quad E'_\varepsilon(y;\, t) = E'(y;\, t) + \varepsilon\rho'(y;\, t)$$

with

$$(2.17)\quad E'(y;\, t) = \int_\Omega y''\, y' + \nabla y.\nabla y'\, dx$$

and

$$(2.18)\quad \rho'(y;\, t) = \int_\Omega y''\,[2m.\nabla y + (n-1)\,y] + y'\,[2m.\nabla y' + (n-1)\,y']\, dx$$

for all $t \geqq 0$. Applying the divergence theorem and using (1.1)-(1.3), (1.8)-(1.11), we obtain

$$(2.19)\quad E'(y;\, t) = \int_\Omega (\Delta y)\, y' + \nabla y.\nabla y'\, dx = \int_\Gamma (\partial y/\partial \nu)\, y'\, d\Gamma = \int_{\Gamma_0} (\partial y/\partial \nu)\, y'\, d\Gamma = -\int_{\Gamma_0} (m.\nu)\,(y')^2\, d\Gamma,$$

$$(2.20)\quad \int_\Omega y''\, y'\, dx = \int_\Omega (\Delta y)\, y\, dx = \int_\Gamma (\partial y/\partial \nu)\, y\, d\Gamma - \int_\Omega |\nabla y|^2\, y\, dx = -\int_{\Gamma_0} (m.\nu)\, y'\, y\, d\Gamma - \int_\Omega |\nabla y|^2\, dx$$

and

$$(2.21)\quad 2\int_\Omega y'\,(m.\nabla y')\, dx = \int_\Omega m.\nabla (y')^2\, dx = \int_\Gamma (m.\nu)\,(y')^2\, d\Gamma - \int_\Omega (\operatorname{div} m)\,(y')^2\, dx = \int_{\Gamma_0} (m.\nu)\,(y')^2\, d\Gamma - n\int_\Omega (y')^2\, dx$$

for all $t \geqq 0$.

Furthermore, applying Lemma 2.2 and using (1.2)-(1.3), (1.9)-(1.11), we have

$$(2.22)\quad 2\int_\Omega y''(m.\nabla y)\,dx=2\int_\Omega(\Delta y)(m.\nabla y)\,dx$$

$$\leqq (n-2)\int_\Omega |\nabla y|^2\,dx+2\int_\Gamma (m.\nabla y)(\partial y/\partial \nu)\,d\Gamma-\int_\Gamma (m.\nu)\,|\nabla y|^2\,d\Gamma$$

$$=(n-2)\int_\Omega |\nabla y|^2\,dx+2\int_{\Gamma_1}(m.\nu)(\partial y/\partial\nu)^2\,d\Gamma-2\int_{\Gamma_0}(m.\nu)\,y'(m.\nabla y)\,d\Gamma$$

$$-\int_{\Gamma_1}(m.\nu)(\partial y/\partial\nu)^2\,d\Gamma-\int_{\Gamma_0}(m.\nu)\,|\nabla y|^2\,d\Gamma$$

$$\leqq (n-2)\int_\Omega |\nabla y|^2\,dx-\int_{\Gamma_0}(m.\nu)\,[2y'(m.\nabla y)+|\nabla y|^2]\,d\Gamma$$

for all $t \geqq 0$, because $\nabla y=(\partial y/\partial\nu)\,\nu$ on $\Gamma_1\times[0,\infty)$ and $m.\nu\leqq 0$ on Γ_1.

From (2.16)-(2.22) we conclude that

$$(2.23)\quad E'_\varepsilon(y;\,t)\leqq -2\,\varepsilon\,E_\varepsilon(y;\,t)$$

$$-\int_{\Gamma_0}(m.\nu)\,[(1-\varepsilon)(y')^2+\varepsilon\,(n-1)\,yy'+2\,\varepsilon\,y'(m.\nabla y)+\varepsilon\,|\nabla y|^2]\,d\Gamma,\qquad \forall\, t\geqq 0.$$

Next we remark that, in view of (2.6) and (2.13) we have

$$|2\,y'(m.\nabla y)|\leqq R^2(y')^2+R^{-2}\,|m.\nabla y|^2\leqq R^2(y')^2+|\nabla y|^2$$

on $\Gamma_0\times[0,\infty)$, and

$$\left|\int_{\Gamma_0}(m.\nu)(n-1)\,yy'\,d\Gamma\right|\leqq\frac{1}{2}\int_{\Gamma_0}(m.\nu)\,[(n-1)^2\,\beta\,(y')^2+\beta^{-1}\,y^2]\,d\Gamma$$

$$\leqq\frac{1}{2}(n-1)^2\,\beta\int_{\Gamma_0}(m.\nu)(y')^2\,d\Gamma+E(y;\,t),\qquad \forall\, t\geqq 0.$$

Therefore from (2.23) we conclude that

$$E'_\varepsilon(y;\,t)\leqq -\varepsilon\,E(y;\,t)-\int_{\Gamma_0}(m.\nu)\left[1-\varepsilon\left(1+\frac{1}{2}(n-1)^2\,\beta+R^2\right)\right](y')^2\,d\Gamma,\qquad \forall\, t\geqq 0.$$

If $\varepsilon\leqq(1+2^{-1}(n-1)^2\,\beta+R^2)^{-1}$, then the function under the integral sign is nonnegative because $m.\nu\geqq 0$ on Γ_0. Hence

$$E'_\varepsilon(y;\,t)\leqq -\varepsilon\,E(y;\,t),$$

and taking into account (2.9), (2.10) follows with

(2.24) $$\varepsilon_0=(1+2^{-1}(n-1)^2\beta+R^2)^{-1}$$

and

(2.25) $$c_2=(1+c_1\varepsilon_0)^{-1}.$$

Using the explicit values of c_1, c_2, ε_0 obtained above, the estimate (1.17) of Theorem 1 holds with

(2.26)
$$\omega=(2R+(n-1)\alpha+\max\{(2R+(n-1)\alpha)(1-C^{-1/2})^{-1};\ 1+2^{-1}(n-1)^2\beta+R^2\})^{-1}.$$

The theorem is thus proved. ●

Remark 2.3. — Let n be arbitrary and assume that the geometrical condition $cl(\Gamma_0)\cap cl(\Gamma_1)=\varnothing$ is satisfied. Then $(y, z)\in D(A)$ implies $y\in H^2(\Omega)$ (*cf.* Remark 2.1) and the above proof applies if we replace the inequality (1.15) of Lemma 2.2 by the following classical identity due to Rellich [28]:

(2.27) $$2\int_\Omega(\Delta y)m.\nabla y\,dx=(n-2)\int_\Omega|\nabla y|^2dx+2\int_\Gamma(\partial y/\partial\nu)m.\nabla y\,d\Gamma-\int_\Gamma(m.\nu)|\nabla y|^2d\Gamma,\qquad\forall y\in H^2(\Omega).$$

This proves the proposition formulated in Remark 1.2. ●

2.2. Proof of lemma 2.2. — In order to motivate the validity of the inequality (2.15), let us first recall briefly the proof of Rellich's identity, mentioned in Remark 2.3. Since in this case y belongs to $H^2(\Omega)$, the following calculation based on the Green's formula and on the special form (1.8) of $m(x)$ is correct:

$$\begin{aligned}2\int_\Omega(\Delta y)m.\nabla y\,dx&=2\int_\Gamma(\partial y/\partial\nu)m.\nabla y\,d\Gamma-\int_\Omega\nabla y.\nabla[2m.\nabla y]\,dx\\&=2\int_\Gamma(\partial y/\partial\nu)m.\nabla y\,d\Gamma-2\int_\Omega|\nabla y|^2dx-\int_\Omega m.\nabla|\nabla y|^2dx\\&=2\int_\Gamma(\partial y/\partial\nu)m.\nabla y\,d\Gamma-2\int_\Omega|\nabla y|^2dx\\&\quad-\int_\Gamma(m.\nu)|\nabla y|^2d\Gamma+n\int_\Omega|\nabla y|^2dx.\end{aligned}$$

Let us now turn to the proof of Lemma 2.2. If $n=1$ then obviously $cl(\Gamma_0)\cap cl(\Gamma_1)=\varnothing$ and the inequality (1.15) follows from Rellich's identity.

Let us now consider the case $n=2$. Given $(y, z)\in D(A)$ arbitrarily, let us fix a function $y_1 \in H^2(\Omega)\cap V$ such that $\partial y_1/\partial\nu = -(m.\nu)z$ on Γ and put $y_2 = y - y_1$. Then $\partial y_2/\partial\nu = 0$ on Γ_0 whence y_2 satisfies the inequality (2.15) by the results of Grisvard ([7], [8], [9]), while y_1 satisfies Rellich's identity (2.27). Therefore it is sufficient to prove the following identity:

$$\int_\Omega \Delta y_1 (m.\nabla y_2) + \Delta y_2 (m.\nabla y_1)\,dx = \int_\Gamma \frac{\partial y_1}{\partial\nu}(m.\nabla y_2) + \frac{\partial y_2}{\partial\nu}(m.\nabla y_1)\,d\Gamma - \int_\Gamma (m.\nu)(\nabla y_1 . \nabla y_2)\,dx. \tag{2.28}$$

To do this, following Grisvard ([7], [8], [9]) let us write y_2 in the form

$$y_2 = \varphi_R + \Sigma_S c_S \varphi_S$$

with $\varphi_R \in H^2(\Omega)$ and $c_S \in \mathbb{R}$, where S runs over $cl(\Gamma_0)\cap cl(\Gamma_1)$ and φ_S is a uniquely determined "singular" function associated to the point S. Furthermore, we approach the domain Ω by a family of subdomains $\Omega(\varepsilon)$, $\varepsilon > 0$, excluding the "singularity" points $S\in cl(\Gamma_0)\cap cl(\Gamma_1)$.

For each domain $\Omega(\varepsilon)$ the corresponding identity (2.28) holds. Indeed, it follows easily from Rellich's identity (2.27) because $y_2 \in H^2(\Omega(\varepsilon))$. Therefore it is sufficient to pass to the limit.

Taking into account that y_1, $y_2 \in V$ and Δy_1, $\Delta y_2 \in L^2(\Omega)$, we have

$$\int_{\Omega(\varepsilon)} \Delta y_1 (m.\nabla y_2) + \Delta y_2 (m.\nabla y_1)\,dx \to \int_\Omega \Delta y_1 (m.\nabla y_2) + \Delta y_2 (m.\nabla y_1)\,dx$$

as $\varepsilon \to 0$. On the other hand, using the estimates of the singularities, obtained in Grisvard ([7], [8], [9]), we obtain easily that

$$\int_{\Gamma\cap\Omega(\varepsilon)} \frac{\partial y_1}{\partial\nu}(m.\nabla y_2) + \frac{\partial y_2}{\partial\nu}(m.\nabla y_1) - (m.\nu)(\nabla y_1 . \nabla y_2)\,d\Gamma \to \int_\Gamma \frac{\partial y_1}{\partial\nu}(m.\nabla y_2) + \frac{\partial y_2}{\partial\nu}(m.\nabla y_1) - (m.\nu)(\nabla y_1 . \nabla y_2)\,d\Gamma$$

and

$$\int_{\Gamma^c\cap\Omega(\varepsilon)} \frac{\partial y_1}{\partial\nu}(m.\nabla y_2) + \frac{\partial y_2}{\partial\nu}(m.\nabla y_1) - (m.\nu)(\nabla y_1 . \nabla y_2)\,d\Gamma \to 0$$

as $\varepsilon \to 0$. Hence the identity (2.28) follows.

The same method applies when $n=3$. In this case we apply the inequality established in Grisvard [8]; Theorem 6.10] and the inequality (2.15) is easily deduced. ●

3. Some generalizations and other applications

3.1. A COMPACTNESS ARGUMENT AND LINKS WITH EXACT CONTRALLABILITY. — Consider first the case where (1.1) is replaced by the more general equation

$$y''-\Delta y+q(x)y=0 \quad \text{in } \Omega\times(0, \infty) \tag{3.1}$$

where $q=q(x)$ is a nonnegative function such that

$$q\in L^p(\Omega) \quad \text{with } p>n \text{ if } n\geqq 2 \quad \text{and with } p=2 \text{ if } n=1. \tag{3.2}$$

The energy of the solution of the system (3.1), (1.2)-(1.4) is now defined by

$$E(y; t)=\frac{1}{2}\int_\Omega |y'(x, t)|^2+|\nabla y(x, t)|^2+q(x)|y(x, t)|^2\,dx. \tag{3.3}$$

The aim of this section is to prove that the estimate (1.17) remains valid:

THEOREM 2. — *Assume that* $n\leqq 3$. *Then, for every constant* $C>1$ *there exists another constant* $\omega>0$ *such that, for any initial data*

$$(y^0, y^1)\in V\times L^2(\Omega)$$

the energy (3.3) *of the solution of the system* (3.1), (1.2)-(1.4), *where* Γ_0, Γ_1, F *and* $q(x)$ *are given by* (1.8)-(1.12) *and* (3.2), *satisfies the inequality*

$$E(y; t)\leqq Ce^{-\omega t}E(y; 0) \quad \textit{for all } t\geqq 0. \quad \bullet \tag{1.17}$$

Remark 3.1. — As in the case of Theorem 1, the same result holds without any restriction on n if the geometrical condition $cl(\Gamma_0)\cap cl(\Gamma_1)=\varnothing$ is satisfied. •

Let us define, as in Section 2 above,

$$E_\varepsilon(y; t)=E(y; t)+\varepsilon\rho(y; t), \qquad \forall t\geqq 0, \tag{3.4}$$

where $E(y; t)$ is given by (3.3), $\rho(y; t)$ by (2.8), and $\varepsilon>0$. The method of the proof of Theorem 1 in Section 2 allows to obtain in the present case the estimate

$$E'_\varepsilon(y; t)\leqq -c_2\varepsilon E_\varepsilon(y; t)+c_3\varepsilon\int_\Omega |q(x)y(x, t)|^2\,dx, \qquad \forall t\geqq 0, \tag{3.5}$$

for some positive constants c_2 and c_3, provided $\varepsilon>0$ is small enough. Furthermore, it is clear that the estimate (2.9) remains valid.

Remark 3.2. — The exponential decay rate of $E_\varepsilon(y; t)$ does not follow directly from (2.9) and (3.5). One can overcome this difficulty by adapting some ideas of Lagnese ([17], [18], [19]). We choose here another method motivated by the work of Rauch-Taylor [27] and using a compactness-uniqueness argument which was proved successful for several exact controllability problems (*cf.* J. L. Lions [23], E. Zuazua [32]). •

First we note that we have, as before,

$$E'(y;\,t)=-\int_{\Gamma_0}(m.\nu)(y')^2\,d\Gamma\leqq 0,\qquad \forall\, t\geqq 0. \tag{36}$$

Thus, it is sufficient to prove the existence of $T>0$ and $C>0$ such that

$$E(y;\,t)\leqq C\int_0^T\int_{\Gamma_0}(m.\nu)(y')^2\,d\Gamma\,dt \tag{3.7}$$

for any solution of finite energy $y=y(t,\,x)$ of (3.1), (1.2)-(1.4). Indeed, then integrating (3.6) for $t\in(0,\,T)$ and substituting the result in (3.7), we obtain

$$E(y;\,T)\leqq C(E(y;\,0)-E(y;\,T))$$

whence

$$E(y;\,T)\leqq C(1+C)^{-1}E(y;\,0). \tag{3.8}$$

In other words, denoting by $S(t)$ the strongly continuous semigroup in $V\times L^2(\Omega)$ associated to the problem (3.1) and (1.2)-(1.4), (3.8) implies that $\|S(T)\|<1$. This, combined with (3.6) and the semigroup property, implies

$$\|S(t)\|\leqq Ce^{-\omega t},\qquad \forall\, t\geqq 0, \tag{3.9}$$

with

$$C=\|S(T)\|^{-1}\qquad\text{and}\qquad \omega=-\log\|S(T)\|^{1/T}. \tag{3.10}$$

In order to prove the estimate (3.7) we use the multiplier technique [*cf.* e. g. L. F. Ho [11], J. L. Lions [23], V. Komornik [13], P. Grisvard ([7], [8])]. We multiply the equation (3.1) by the scalar field $m.\nabla y$ [*cf.* (1.8)]. Integrating by parts and applying Lemma 2.2 we obtain

$$\begin{aligned}\int_\Omega y'(m.\nabla y)\,dx\Big|_0^T&+\frac{1}{2}\int_0^T\int_\Omega (y')^2+|\nabla y|^2+qy^2\,dx\,dt+\frac{n-1}{2}\int_0^T\int_\Omega (y')^2-|\nabla y|^2\,dx\,dt\\&-\frac{1}{2}\int_0^T\int_\Omega qy^2\,dx\,dt+\int_0^T\int_\Omega qy(m.\nabla y)\,dx\,dt\\&\leqq\frac{1}{2}\int_0^T\int_{\Gamma_0}(m.\nu)(y')^2\,d\Gamma\,dt-\frac{1}{2}\int_0^T\int_\Gamma (m.\nu)|\nabla y|^2\,d\Gamma\,dt\\&+\int_0^T\int_\Gamma(\partial y/\partial\nu)(m.\nabla y)\,d\Gamma\,dt\leqq\frac{1}{2}\int_0^T\int_{\Gamma_0}(m.\nu)(y')^2\,d\Gamma\,dt\\&+\int_0^T\int_{\Gamma_0}(\partial y/\partial\nu)(m.\nabla y)\,d\Gamma\,dt-\frac{1}{2}\int_0^T\int_{\Gamma_0}(m.\nu)|\nabla y|^2\,d\Gamma\,dt.\end{aligned} \tag{3.11}$$

(We have used the notation $\int_\Omega f(x, t)\,dx\,|_0^T = \int_\Omega f(x, T)\,dx - \int_\Omega f(x, 0)\,dx$.) As in Section 2, we deduce easily that the right hand side is majorized by

$$C \int_0^T \int_{\Gamma_0} (m \cdot \nu)(y')^2\,dx\,dt$$

for some positive constant $C > 0$ depending only on R [*cf.* (2.13)].

On the other hand, multiplying the equation (3.1) by $y(x, t)$ we obtain

$$(3.12) \quad \int_0^T \int_\Omega (y')^2 - |\nabla y|^2\,dx\,dt = \int_0^T \int_\Omega q y^2\,dx\,dt + \int_\Omega y y'\,dx\,|_0^T - \int_0^T \int_{\Gamma_0} \frac{\partial y}{\partial \nu} y\,d\Gamma\,dt.$$

Since for every $\delta > 0$ there exists $C_1(\delta) > 0$ such that

$$\left| \int_{\Gamma_0} \frac{\partial y}{\partial \nu} y\,d\Gamma \right| \leqq \delta E(y, t) + C_1(\delta) \int_{\Gamma_0} (m \cdot \nu)\,|y'|^2\,d\Gamma,$$

from (3.11), (3.12) we deduce that

$$(3.13) \quad (1-\delta) \int_0^T E(y; t)\,dt + \int_\Omega y' \left[(m \cdot \nabla y) + \frac{n-1}{2} y \right] dx\,|_0^T + \frac{n-2}{2} \int_0^T \int_\Omega q y^2\,dx\,dt$$
$$+ \int_0^T \int_\Omega q y (m \cdot \nabla y)\,dx\,dt \leqq (C + C_1(\delta)) \int_0^T \int_{\Gamma_0} (m \cdot \nu)(y')^2\,d\Gamma\,dt.$$

Using (3.6) we have clearly

$$(3.14) \quad \left| \int_0^T y'(m \cdot \nabla y)\,dx\,|_0^T \right| \leqq R(E(y; T) + E(y; 0))$$
$$= 2RE(y; T) + R \int_0^T \int_{\Gamma_0} (m \cdot \nu)(y')^2\,d\Gamma\,dt.$$

On the other hand, taking (3.2) into account we have for some $s < 1$ the estimate

$$(3.15) \quad \left| \frac{n-1}{2} \int_0^T y' y\,dx\,|_0^T + \frac{n-2}{2} \int_0^T \int_\Omega q y^2\,dx\,dt + \int_0^T \int_\Omega q y (m \cdot \nabla y)\,dx\,dt \right|$$
$$\leqq \delta \int_0^T E(y; t)\,dt + C_2(\delta)\,\|y\|^2_{L^\infty(0, T; H^s(\Omega))}$$

with $C_2(\delta) > 0$ large enough.

Combining (3.13)-(3.15) we conclude that

$$((1-2\delta)T-2R)E(y;T) \leqq (1-\delta)\int_0^T E(y;t)\,dt - 2RE(y;T)$$
$$\leqq C_3(\delta)\left(\int_0^T \int_{\Gamma_0} (m.\nu)(y')^2\,dx\,dt + \|y\|^2_{L^\infty(0,T;H^s(\Omega))}\right)$$

and then, for every $T > 2R$ there exists a positive constant $C > 0$ such that

$$E(y;T) \leqq C\left(\int_0^T \int_{\Gamma_0} (m.\nu)(y')^2\,d\Gamma\,dt + \|y\|^2_{L_\infty(0,T;H^s(\Omega))}\right). \tag{3.16}$$

Now we remark that

$$\|y\|^2_{L^\infty(0,T;H^s(\Omega))} \leqq C\left(\int_0^T \int_{\Gamma_0} (m.\nu)(y')^2\,d\Gamma\,dt + \|y\|^2_{H^{-1}(\Omega\times(0,T))}\right) \tag{3.17}$$

for some constant C large enough.

Indeed, if (3.17) is not satisfied, then there exists a sequence (y_k) of solutions of finite energy of (3.1), (1.2), (1.3) such that

$$\|y_k\|_{L^\infty(0,T;H^s(\Omega))} = 1, \qquad \forall k \geqq 1,$$
$$\int_0^T \int_{\Gamma_0} (m.\nu)(y_k')^2\,d\Gamma\,dt + \|y_k\|^2_{H^{-1}(\Omega\times(0,T))} \to 0 \text{ as } k \to \infty.$$

From (3.16) we deduce that

$$(y_k) \text{ is bounded in } L^\infty(0,T;V) \cap W^{1,\infty}(0,T;L^2(\Omega)) \tag{3.18}$$

and then (*cf.* Simon [30])

$$(y_k) \text{ is relatively compact in } L^\infty(0,T;H^S(\Omega)). \tag{3.19}$$

Thus, for a subsequence [that we still denote by (y_k)] we have

$$y_k \to y \text{ strongly in } L^\infty(0,T;H^S(\Omega))$$

and

$$\|y\|_{L^\infty(0,T;H^S(\Omega))} = 1. \tag{3.20}$$

But on the other hand we have

$$y_k \to 0 \quad \text{in } H^{-1}(\Omega\times(0,T)).$$

Hnece $y \equiv 0$ which contradicts (3.20).

From (3.16) and (3.17) we deduce that

$$\text{(3.21)} \qquad E(y; T) \leqq C\left(\int_0^T \int_{\Gamma_0} (m.\nu)(y')^2 \, d\Gamma \, dt + \|y\|^2_{H^{-1}(\Omega \times (0, T))}\right).$$

Therefore it is sufficient to establish the existence of a positive constant C such that

$$\text{(3.22)} \qquad \|y\|^2_{H^{-1}(\Omega \times (0, T))} \leqq C \int_0^T \int_{\Gamma_0} (m.\nu)(y')^2 \, d\Gamma \, dt.$$

We argue by contradiction. If (3.22) is not satisfied, then there exists a sequence of solutions y_k of (3.1), (1.2), (1.3) of finite energy such that

$$\text{(3.23)} \qquad \|y_k\|_{H^{-1}(\Omega \times (0, T))} = 1, \qquad \forall k$$
and
$$\int_0^T \int_{\Gamma_0} (m.\nu)(y')^2 \, d\Gamma \, dt \to 0 \quad \text{as } k \to \infty.$$

The estimate (3.21) implies (3.18) and then

$$\text{(3.24)} \qquad (y_k) \text{ is relatively compact in } H^{-1}(\Omega \times (0, T)).$$

Choosing a subsequence if needed, we may assume that

$$\text{(3.25)} \qquad y_k \to y \text{ weakly* in } L^\infty(0, T; V)$$
and
$$y'_k \to y' \text{ weakly* in } L^\infty(0, T; L^2(\Omega)).$$

From (3.23), (3.24) we conclude that

$$\text{(3.26)} \qquad \|y\|_{H^{-1}(\Omega \times (0, T))} = 1$$

and

$$\text{(3.27)} \qquad \partial y/\partial \nu = -(m.\nu)\, y' = 0 \quad \text{on } \Gamma_0 \times (0, T).$$

On the other hand $y = y(x, t)$ is a weak solution of (3.1), (1.2) and (1.3).

Applying a unique continuation theorem of Kenig-Ruiz-Sogge [12], we are going to show that (3.26) and (3.27) are in contradiction. We shall prove in fact that $y \equiv 0$. In view of (1.3) it is sufficient to prove that $y' \equiv 0$.

Let us consider the function $z := y'$. Then z belongs to $L^\infty(0, T; L^2(\Omega))$, and z is a weak solution of (3.1) satisfying [by (3.27)] the boundary conditions

$$\text{(3.28)} \qquad z = 0 \quad \textit{on } \Gamma \times (0, T)$$
and
$$\partial z/\partial \nu = 0 \quad \text{on } \Gamma_0 \times (0, T).$$

Therefore the Corollary 3.3 in Kenig-Ruiz-Sogge [12] applies provided

$$z \in W^{2,\, 2n/(n+2)}(\Omega \times (0, T)), \tag{3.29}$$

and gives $z \equiv 0$. In order to prove (3.29) we remark that $w := z'$ is a weak solution of (3.1), (3.28) in the class $H^{-1}(\Omega \times (0, T))$. From (3.21) and taking the property $w' = 0$ on $\Gamma_0 \times (0, T)$ into account, we then deduce that w has finite energy *i.e.*

$$z' \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; L^2(\Omega)),$$

whence, taking into account that z satisfies (3.1), the property (3.29) follows. This completes the proof of Theorem 2. ●

Remark 3.3. — The use of compactness arguments and unique continuation results make the proof of Theorem 2 rather technical. In V. Komornik [15] a general constructive method was introduced which, based on an estimation method of A. Haraux [10], permits to avoid the indirect arguments of this type. ●

Remark 3.4. — Theorem 2 and its proof remains valid if we assume, instead of the nonnegativity of q that

$$\exists \alpha > 0 \quad \text{such that} \quad (1-\alpha)\int_\Omega |\nabla u|^2\,dx + \int_\Omega q\,|u|^2\,dx \geqq 0, \qquad \forall u \in V. \tag{3.30}$$ ●

Remark 3.5. — It is well-known that stabilizability implies exact controlability, *cf.* D. L. Russell [29]. From the above stabilizatbility results the following exact controllability results may be obtained.

Let $q(x)$ be a function satisfying the hypothesis of Theorem 2, and let T_0 be such that $\|S(T_0)\| < 1$. Then for every $T > T_0$ and for every $(y^0, y^1) \in V \times L^2(\Omega)$ there exists a boundary control

$$v \in H^1(0, T; L^2(\Gamma_0))$$

such that the solution of the problem

$$\begin{gathered} y'' - \Delta y + q(x)\,y = 0 \quad \text{in } \Omega \times (0, T) \\ \partial y/\partial \nu = v \quad \text{on } \Gamma_0 \times (0, T) \\ y = 0 \quad \text{on } \Gamma_1 \times (0, T) \\ y(0) = y^0, \qquad y'(0) = y^1 \quad \text{on } \Omega \end{gathered}$$

satisfies $y(T) = y'(T) = 0$. The above estimates show that $\|S(T)\| < 1$ if $T > 2R$; this implies the exact controllability provided $T > 2R$.

However, using directly the Hilbert Uniqueness Method due to J. L. Lions ([22], [23], [24]), one obtains the same exact controllability result without the hypothesis (3.30). ●

3.2. Other results and remarks. — In order to improve the estimate (1.17), we may try to use more general feed-backs of the form

$$F(x, y') = -a(x)\, y'(x, t)$$

with $a(x) \in L^\infty(\Gamma_0)$, $a(x) \geqq 0$ *a.e.* on Γ_0. It turns out however, that the method given in Sections 2 and 3 no longer works unless

$$c_4\, m(x).\nu(x) \leqq a(x) \leqq c_5\, m(x).\nu(x) \quad \text{on } \Gamma_0$$

for some positive constants $c_4, c_5 > 0$.

We may slightly improve the value of ω obtained in Section 2 by taking $a(x) = b(x)(m(x).\nu(x))$ with a suitably chosen function $b \in L^\infty(\Gamma_0)$, $b(x) = b_0 > 0$ *a.e.* on Γ_0, but $\omega(b)$ may not be taken arbitrarily large. The reason of this, as it was pointed out to us by J. Lagnese and J. L. Lions, is that the system (1.1)-(1.4) becomes (at least formally) conservative as $b_0 \to \infty$.

The method of Section 2 allows us to obtain exponential decay rates for nonlinear feed-backs $F(x, y') = -(m(x).\nu(x)) f(y')$, too, where $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function such that $f(0) = 0$ and

$$c_6 |s| \leqq |f(s)| \leqq c_7 |s|, \qquad \forall s \in \mathbb{R}$$

for some positive constants $c_6, c_7 > 0$.

It has been proved by Lagnese [19] that the special radial vector field $m(x) = x - x_0$ may be replaced by a more general vector field of the form

$$h(x) = (h_1(x), \ldots, h_n(x)) \in C^2(cl(\Omega))$$

where

$$h(x).\nu(x) \leqq 0 \quad \text{on } \Gamma_0, \qquad h(x).\nu(x) \geqq 0 \quad \text{on } \Gamma_1$$

and

$$(\partial_i h_j + \partial_j h_i)(x) \text{ is positive definite on } cl(\Omega),$$

provided that the corresponding generalization of Grisvard's inequality (*cf.* Lemma 2.4) holds. Since, for the time being it does not seem to be proved, these results are formal unless $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$.

In Lagnese [20] it was pointed out that the utilisation of feed-backs containing the multiplicator $m(x).\nu(x)$ permits to remove (at least formally) the classical hypothesis $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$ also in the stabilization problem of some models of plates.

We note that this type of feed-backs allows us to remove (at least formally) the hypothesis $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$ in the stabilization of elastodynamic systems (*cf.* J. Lagnese [18]), too. However, the corresponding generalization of Lemma 2.4 does not seem to be proved.

In a recent work of C. Bardos-G. Lebeau-J. Rauch [1] other boundary partitions (Γ_0, Γ_1) have been also used for the stabilizability of the wave equation. Their proofs are based on microlocal analysis. Assuming that $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$ and that Γ_0 "controls geometrically" the domain Ω, they prove the exponential decay rate of the solutions of (1.1)-(1.4) by taking $F(x, y') = -b(x)y'$ with $b \in C^\infty(\Gamma_0)$ and $b(x) \geqq b_0 > 0$ on Γ_0. The class of boundary regions Γ_0 which "control geometrically" Ω is much larger than the class considered in this paper [*cf.* (1.9)]. Unfortunately, the additional hypothesis $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$ is needed again.

I. Lasiecka and R. Triggiani [21] proved the stabilizability of the wave equation with Dirichlet boundary feed-back under a rather strict geometrical hypothesis on Ω.

3.3. SEMILINEAR WAVE EQUATION. — The aim of this section is to extend Theorem 1 to the semilinear wave equation

$$y'' - \Delta y + g(y) = 0 \quad \text{in } \Omega \times (0, \infty). \tag{3.31}$$

Assume that $g \in W^{1,\infty}_{loc}(\mathbb{R})$ is a locally Lipschitz continuous function satisfying the following conditions:

$$g(s)s \geqq 0, \qquad \forall s \in \mathbb{R}, \tag{3.32}$$

$$\exists c > 0, \quad |g(x) - g(y)| \leqq C(1 + |x|^{p-1} + |y|^{p-1})|x - y|, \qquad \forall x, y \in \mathbb{R} \tag{3.33}$$

for some $1 < p \leqq n/(n-2)$,

$$\exists \delta > 0, \quad g(s)s \geqq (2+\delta)G(s), \qquad \forall s \in \mathbb{R}, \qquad \text{where } G(s) := \int_0^s g(t)\,dt. \tag{3.34}$$

We define the energy of the solution of the system (3.31), (1.2)-(1.4) by

$$E(y; t) = \frac{1}{2} \int_\Omega |y'(x, t)|^2 + |\nabla y(x, t)|^2 + G(y(x, t))\,dx. \tag{3.35}$$

Then the estimate (1.17) remains valid:

THEOREM 3. — *Assume that $n \leqq 3$. Then, to every constant $C > 1$ there exists another constant $\omega > 0$ such that, for any initial data (y^0, y^1) of finite energy, the solution of the system* (3.31), (1.2)-(1.4), *where Γ_0, Γ_1, F and $g \in W^{1,\infty}_{loc}(\mathbb{R})$ are given by* (1.8)-(1.12) *and* (3.32)-(3.34), *satisfies the inequality*

$$E(y; t) \leqq C e^{-\omega t} E(y; 0) \quad \textit{for all } t \geqq 0. \quad \bullet$$

Remark 3.6. — As in the case of Theorems 1 and 2, the same result holds without any restriction on n if the geometrical condition $cl(\Gamma_0) \cap cl(\Gamma_1) = \varnothing$ is satisfied. $\bullet$

Proof of Theorem 3. — It is easy to prove that under the hypothesis (3.32)-(3.33) there exists a unique solution $y = y(x, t)$ of the system (3.31), (1.2)-(1.4) in the class

$$C([0, \infty); V) \cap C^1([0, \infty); L^2(\Omega)) \tag{3.36}$$

for every pair of initial data $(y^0, y^1) \in V \times L^2(\Omega)$. Furthermore we have

$$E(y; t) \leqq E(y; 0), \qquad \forall t \geqq 0. \tag{3.37}$$

Given any initial data $(y^0, y^1) \in D(A)$, the solution $y = y(x, t)$ has the additional regularity property

$$(y, y') \in C([0, \infty); D(A))$$

and then

$$\frac{dE}{dt}(y; t) = E'(y; t) = -\int_{\Gamma_0} (m.\nu)(y')^2 \, d\Gamma, \qquad \forall t \geqq 0. \tag{3.38}$$

We have in particular, as in the proof of Theorem 1, the property

$$\Delta y \in C([0, \infty); L^2(\Omega))$$

and then Grisvard's inequality applies.

As we have observed in Section 2, $D(A)$ is dense in $V \times L^2(\Omega)$. Furthermore, one can easily show that

$$(y_k^0, y_k^1) \to (y^0, y^1) \quad \text{in } V \times L^2(\Omega)$$

implies

$$E(y_k; t) \to E(y; t) \quad \text{in } L^\infty_{\text{loc}}(0, \infty).$$

Thus, it is sufficient to prove the estimate of the theorem for the solutions corresponding to initial data in $D(A)$.

We define for each $\varepsilon > 0$

$$E_\varepsilon(y; t) = E(y; t) + \varepsilon \rho(y; t) \tag{3.39}$$

with

$$\rho(y; t) = \int_\Omega y'(x, t)\,[2m(x).\nabla y(x, t) + \theta y(x, t)]\, dx \tag{3.40}$$

where $\theta \in (n-2, n)$ is such that

$$\exists \gamma > 0, \quad (2n+\gamma)\, G(s) \leqq \theta g(s) s, \qquad \forall s \in \mathbb{R}.$$

This choice of θ is possible because

$$(2n+\gamma)\, G(s) \leqq \frac{2n+\gamma}{2+\delta} g(s) s$$

and $(2n+\gamma)/(2+\delta) < n$ for $0 < \gamma < \delta n$.

As in Section 2, it is easy to show that (2.9) is satisfied. On the other hand, we have

$$(3.41) \qquad \rho'(y;\,t)=\int_\Omega 2y''(m.\nabla y)+2y'(m.\nabla y')+\theta y'' y+\theta (y')^2\,dx, \qquad \forall t \geqq 0.$$

We also have

$$(3.42) \qquad \int_\Omega y'' y\,dx=\int_\Omega (\Delta y-g(y))\,y\,dx$$

$$=-\int_{\Gamma_0}(m.\nu)\,y' y\,d\Gamma-\int_\Omega |\nabla y|^2\,dx-\int_\Omega g(y)\,y\,dx, \qquad \forall t \geqq 0$$

and

$$(3.43) \qquad 2\int_\Omega y'(m.\nabla y')\,dx=\int_\Omega m.\nabla (y')^2\,dx$$

$$=-n\int_\Omega |y'|^2\,dx+\int_{\Gamma_0}(m.\nu)\,|y'|^2\,d\Gamma, \qquad \forall t \geqq 0.$$

Furthermore, using Grisvard's inequality (*cf.* Lemma 2.4) we obtain

$$(3.44) \qquad 2\int_\Omega y''(m.\nabla y)\,dx=2\int_\Omega (\Delta y-g(y))(m.\nabla y)\,dx$$

$$\leqq (n-2)\int_\Omega |\nabla y|^2\,dx-\int_{\Gamma_0}(m.\nu)\,[2y'(m.\nabla y)+|\nabla y|^2]\,d\Gamma-2\int_\Omega m.\nabla G(y)\,dx$$

$$=(n-2)\int_\Omega |\nabla y|^2\,dx-\int_{\Gamma_0}(m.\nu)\,[2y'(m.\nabla y)+|\nabla y|^2]\,d\Gamma$$

$$+2n\int_\Omega G(y)\,dx-2\int_\Gamma (m.\nu)\,G(y)\,d\Gamma$$

$$\leqq (n-2)\int_\Omega |\nabla y|^2\,dx-\int_{\Gamma_0}(m.\nu)\,[2y'(m.\nabla y)+|\nabla y|^2]\,d\Gamma+2n\int_\Omega G(y)\,dx, \qquad \forall t \geqq 0.$$

Combining (3.38), (3.41)-(3.44) we deduce, as in Section 2, for sufficiently small positive ε (which is independent of y) the inequality

$$E'_\varepsilon(y;\,t) \leqq \varepsilon\left[(n-2-\theta)\int_\Omega |\nabla y|^2\,dx+(\theta-n)\int_\Omega (y')^2\,dx-\int_\Omega g(y)\,y\,dx+2n\int_\Omega G(y)\,dx\right].$$

Taking into account the choice of θ this yields the inequality

$$E'_\varepsilon(y;\,t) \leqq -\varepsilon \min\{2(\theta-n+2),\,2(n-\theta),\,\gamma\}\,E(y;\,t), \qquad \forall t \geqq 0;$$

combining with (2.9) this implies (2.10) and the proof may be completed as in Section 2. ●

Remark 3.7. — Theorem 3 applies in particular to the nonlinearities

$$g(s)=|s|^{p-1}s \quad \text{with} \quad p>1 \quad \text{if } n \leqq 2 \quad \text{and} \quad 1<p<n/(n-2) \quad \text{if } n \geqq 3. \quad ●$$

Remark 3.8. — The argument (*cf.* D. L. Russell [29]) which proves exact controllability from stabilizability is no longer valid for semilinear equations. Hence other methods are required to obtain exact controllability results in the semilinear framework. For these problems we refer to D. L. Russell [29] and E. Zuazua [34]. ●

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