On the observability of time-discrete conservative linear systems

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Abstract. We consider various time discretization schemes of abstract conservative evolution equations of the form $\dot{z} = Az$, where A is a skew-adjoint operator. We analyze the problem of observability through an operator B. More precisely, we assume that the pair (A, B) is exactly observable for the continuous model, and we derive uniform observability inequalities for suitable time-discretization schemes within the class of conveniently filtered initial data. The method we use is mainly based on the resolvent estimate given in [2]. We present some applications of our results to time-discrete schemes for wave, Schrödinger and KdV equations and fully discrete approximation schemes for wave equations.

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1. Introduction

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator with compact resolvent. Let us consider the following abstract system:

$$\dot{z}(t) = Az(t), \quad z(0) = z_0. \tag{1.1}$$

Here and henceforth, a dot ($\dot{}$) denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, and $z = z(t)$ is the *state* of the system. Such systems are often used as models of vibrating systems (e.g., the wave equation), electromagnetic phenomena (Maxwell's equations) or in quantum mechanics (Schrödinger's equation).

Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed with the classical operator norm. Let $B \in \mathfrak{L}(\mathcal{D}(A), Y)$ be an observation operator and define the output function

$$y(t) = Bz(t). \tag{1.2}$$

In order to give a sense to (1.2), we make the assumption that B is an admissible observation operator in the following sense (see [26]):

Definition 1.1. The operator B is an admissible observation operator for system (1.1)-(1.2) if for every $T > 0$ there exists a constant $K_T > 0$ such that

$$\int_0^T \|y(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2, \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{1.3}$$

Note that if B is *bounded* in X, i.e. if it can be extended such that $B \in \mathfrak{L}(X, Y)$, then B is obviously an admissible observation operator. However, in applications, this is often not the case, and the admissibility condition is then a consequence of a suitable "hidden regularity" property of the solutions of the evolution equation (1.1).

The exact observability property of system (1.1)-(1.2) can be formulated as follows:

Definition 1.2. System (1.1)-(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$k_T \|z_0\|_X^2 \leq \int_0^T \|y(t)\|_Y^2 \, dt, \qquad \forall \, z_0 \in \mathcal{D}(A). \tag{1.4}$$

Moreover, (1.1)-(1.2) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Note that observability issues arise naturally when dealing with controllability and stabilization properties of linear systems (see for instance the textbook [15]). Indeed, controllability and observability are dual notions, and therefore each statement concerning observability has its counterpart in controllability. In the sequel, we mainly focus on the observability properties of (1.1)-(1.2).

It was proved in [2] and [17] that system (1.1)-(1.2) is exactly observable if and only if the following assertion holds:

$$\begin{cases} \text{There exist constants } \; M, m > 0 \; \text{ such that} \\ M^2 \|(i\omega I - A)z\|^2 + m^2 \|Bz\|_Y^2 \geq \|z\|^2, \qquad \forall \; \omega \in \mathbb{R}, \;\; z \in \mathcal{D}(A). \end{cases} \tag{1.5}$$

This spectral condition can be viewed as a Hautus-type test, and generalizes the classical Kalman rank condition, see for instance [25]. To be more precise, if (1.5) holds, then system (1.1)-(1.2) is exactly observable in any time $T > T_0 = \pi M$ (see [17]).

There is an extensive literature providing observability results for wave, plate, Schrödinger and elasticity equations, among other models and by various methods including microlocal analysis, multipliers and Fourier series, etc. Our goal in this paper is to develop a theory allowing to get results for time-discrete systems as a direct consequence of those corresponding to the time-continuous ones.

Let us first present a natural discretization of the continuous system. For any $\triangle t > 0$, we denote by z^k and y^k respectively the approximations of the solution z and the output function y of system (1.1)–(1.2) at time $t_k = k\triangle t$ for $k \in \mathbb{Z}$. Consider the following *implicit midpoint* time discretization of system (1.1):

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\triangle t} = A\Big(\dfrac{z^{k+1} + z^k}{2}\Big), \qquad \text{in } \; X, \qquad k \in \mathbb{Z}, \\ z^0 \text{ given.} \end{cases} \tag{1.6}$$

The output function of (1.6) is given by

$$y^k = Bz^k, \qquad k \in \mathbb{Z}. \tag{1.7}$$

Note that (1.6)–(1.7) is a discrete version of (1.1)–(1.2).

Taking into account that the spectrum of A is purely imaginary, it is easy to show that $\|z^k\|_X$ is conserved in the discrete time variable $k \in \mathbb{Z}$, i.e. $\|z^k\|_X = \|z^0\|_X$. Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting.

The uniform exact observability problem for system (1.6) is formulated as follows: *To find a positive constant* $\tilde{k}_T$*, independent of* $\triangle t$*, such that the solutions* z^k *of system* (1.6) *satisfy:*

$$\tilde{k}_T \left\|z^0\right\|_X^2 \leq \triangle t \sum_{k\in(0,T/\triangle t)} \left\|y^k\right\|_Y^2, \tag{1.8}$$

for all initial data z^0 *in an appropriate class.*

Clearly, (1.8) is a discrete version of (1.4).

Note that this type of observability inequalities appears naturally when dealing with stabilization and controllability problems (see, for instance, [15], [25] and [29]). For numerical approximation processes, it is important that these inequalities hold uniformly with respect to the discretization parameter(s) (here $\triangle t$ only) to recover uniform stabilization properties or the convergence of discrete controls to the continuous ones. We refer to the review article [29] and the references therein for more precise statements. To our knowledge, there are very few results addressing the observability issues for time semi-discrete schemes. We refer to [18], where the uniform controllability of a fully discrete approximation scheme of the 1-d wave equation is analyzed, and to [27], where a time discretization of the wave equation is analyzed using multiplier techniques. Especially, the results in [27] may be viewed as a particular instance of the abstract models we address here.

In the sequel, we are interested in understanding under which assumptions inequality (1.8) holds uniformly on $\triangle t$. One expects to do it so that, when letting $\triangle t \to 0$, one recovers the observability property of the continuous model.

It can be done by means of a spectral filtering mechanism. More precisely, since A is skew-adjoint with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j : \ j \in \mathbb{N}\}$, where $(\mu_j)_{j\in\mathbb{N}}$ is a sequence of real numbers. Set $(\Phi_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\mathbb{N}}$, that is:

$$A\Phi_j = i\mu_j\Phi_j. \tag{1.9}$$

Moreover, we define

$$\mathcal{C}_s = \operatorname{span}\{\Phi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s\}. \tag{1.10}$$

We will prove that inequality (1.8) holds uniformly (with respect to $\triangle t > 0$) in the class $\mathcal{C}_{\delta/\triangle t}$ for any $\delta > 0$ and for T_δ large enough, depending on the filtering parameter δ.

This result will be obtained as a consequence of the following theorem:

Theorem 1.3. *Let* $\delta > 0$.

Assume that we have a family of vector spaces $X_{\delta,\triangle t} \subset X$ *and a family of unbounded operators* $(A_{\triangle t}, B_{\triangle t})$ *depending on the parameter* $\triangle t > 0$ *such that*

(H1) *For each $\triangle t > 0$, the operator $A_{\triangle t}$ is skew-adjoint on $X_{\delta,\triangle t}$, and the vector space $X_{\delta,\triangle t}$ is globally invariant by $A_{\triangle t}$. Moreover,*

$$\|A_{\triangle t} z\|_X \leq \frac{\delta}{\triangle t} \|z\|_X, \qquad \forall z \in X_{\delta,\triangle t}, \ \forall \triangle t > 0. \tag{1.11}$$

(H2) *There exists a positive constant C_B such that*

$$\|B_{\triangle t} z\|_Y \leq C_B \|A_{\triangle t} z\|_X, \qquad \forall z \in X_{\delta,\triangle t}, \ \forall \triangle t > 0. \tag{1.12}$$

(H3) *There exist two positive constants M and m such that*

$$\begin{aligned} &M^2 \|(A_{\triangle t} - i\omega I)z\|_X^2 + m^2 \|B_{\triangle t} z\|_Y^2 \geq \|z\|_X^2, \\ &\qquad\qquad \forall z \in X_{\delta,\triangle t} \cup \mathcal{D}(A_{\triangle t}), \forall \omega \in \mathbb{R}, \ \forall \triangle t > 0. \end{aligned} \tag{1.13}$$

Then there exists a time T_δ such that for all time $T > T_\delta$, there exists a constant $k_{T,\delta}$ such that for $\triangle t$ small enough, the solution of

$$\frac{z^{k+1} - z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^{k+1} + z^k}{2}\Big), \qquad in \ \ X_{\delta,\triangle t}, \qquad k \in \mathbb{Z},. \tag{1.14}$$

with initial data $z^0 \in X_{\delta,\triangle t}$ satisfies

$$k_{T,\delta} \left\|z^0\right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\|B_{\triangle t} z^k\right\|_Y^2, \qquad \forall\, z^0 \in X_{\delta,\triangle t}. \tag{1.15}$$

Moreover, T_δ can be taken to be such that

$$T_\delta = \pi \Big[\Big(1 + \frac{\delta^2}{4}\Big)^2 M^2 + m^2 C_B^2 \frac{\delta^4}{16}\Big]^{1/2}, \tag{1.16}$$

where C_B is as in (2.1).

As we shall see in Theorem 2.1, taking $A_{\triangle t} = A$, $B_{\triangle t} = B$ and $X_{\delta/\triangle t} = \mathcal{C}_{\delta/\triangle t}$, Theorem 1.3 provides an observability result within the class $\mathcal{C}_{\delta/\triangle t}$ for system (1.6)-(1.7), as a consequence of assumption (1.5) and since $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.

Theorem 1.3 is also useful to address observability issues for more general time-discretization schemes of (1.1)-(1.2) than (1.6). For instance, one can consider time semi-discrete schemes of the form

$$z^{k+1} = \mathbb{T}_{\triangle t} z^k, \qquad y^k = B z^k, \tag{1.17}$$

where $\mathbb{T}_{\triangle t}$ is a linear operator with the same eigenvectors as the operator A. We will prove that, under some general assumptions on $\mathbb{T}_{\triangle t}$, inequality (1.8) holds uniformly on $\triangle t$ for solutions of (1.17) when the initial data are taken in the class $\mathcal{C}_{\delta/\triangle t}$, as we shall see in Theorem 3.1.

We can also consider second order in time systems such as

$$\begin{aligned} &\ddot{u}(t) + A_0 u(t) = 0, \\ &u(0) = u_0, \quad \dot{u}(0) = v_0, \end{aligned} \tag{1.18}$$

where A_0 is a positive self-adjoint operator. Of course, such systems can be written in the same first-order form as (1.1). However, there are time-discretization schemes such as the Newmark method which cannot be put in the form (1.17).

Hence we present a specific analysis of the Newmark method for (1.18), still based on Theorem 1.3.

One of the interesting applications of our results is that it allows us to develop a two-step strategy to study the observability of fully discrete approximation schemes of (1.1)-(1.2). First, one uses the observability properties for *space* semi-discrete approximation schemes, uniformly with respect to the space mesh-size parameter, as it has already been done in many cases (see [3], [6], [7], [10], [19], [20], [28] and [29] for more references). Second, from the results of this paper on time discretizations, the uniform observability (with respect to both the time and space mesh-sizes) for the *fully* discrete approximation schemes is derived. To our knowledge, the observability issues for fully discrete approximation schemes have been studied only in [18], in the very particular case of the 1-d wave equation. The results we present here can be applied to a much wider class of systems and time-discretization schemes.

To complete our analysis of the discretizations of system (1.1)-(1.2), we also analyze admissibility properties for the time semi-discrete systems introduced throughout this paper. They are useful when deriving controllability results out of the observability ones. More precisely, it allows proving controllability results by means of duality arguments combined with observability and admissibility results (see for instance the textbook [15] and the survey article [29]). In particular, we prove that the admissibility inequality (1.3) can be interpreted in terms of the behavior of wave packets. From this wave packet estimate, we will deduce admissibility inequalities for the time semi-discrete schemes. This part can be read independently from the rest of the article.

The outline of this paper is as follows.

In Section 2 we prove Theorem 1.3, from which we deduce the uniform observability property (1.8) for system (1.6)-(1.7), assuming that the initial data are taken in some subspace of filtered data $\mathcal{C}_{\delta/\triangle t}$ for arbitrary $\delta > 0$. Our proof of Theorem 1.3 is mainly based on the resolvent estimate (1.13), combined with standard Fourier arguments adapted to the time-discrete setting. In Section 3, we show how to apply Theorem 1.3 to obtain similar results for time semi-discrete approximation schemes such as (1.17) and the Newmark approximation schemes, for which we prove that a uniform observability inequality holds as well, provided the initial data belong to $\mathcal{C}_{\delta/\triangle t}$. In Section 4, we give some applications to the observability of some classical conservative equations, such as the Schrödinger equation or the linearized KdV equation, etc. In Section 5, we give some applications of our main results to fully discrete schemes for skew-adjoint systems as (1.1). In Section 6, we present admissibility results similar to (1.3) for the time semi-discrete schemes used along the article. We end the paper by stating some further comments and open problems.

2. The implicit mid-point scheme

In this section we show the uniform observability of system (1.6)-(1.7), which can be seen as a direct consequence of Theorem 1.3. In other words, its proof is a simplified version of the one of Theorem 1.3. To avoid the duplication of the process, we only give the proof of the latter one, which is more general.

Let us first introduce some notations and definitions.

The Hilbert space $\mathcal{D}(A)$ is endowed with the norm of the graph of A, which is equivalent to $\|A\cdot\|$ since A has a compact resolvent. It follows that $B \in \mathfrak{L}(\mathcal{D}(A), Y)$ implies

$$\|Bz\|_Y \leq C_B \|Az\|_X, \qquad \forall z \in \mathcal{D}(A). \tag{2.1}$$

We are now in position to claim the following theorem based on the resolvent estimate (1.5):

Theorem 2.1. *Assume that (A, B) satisfy (1.5) and that $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.*

Then, for any $\delta > 0$, there exist T_δ and $\triangle t_0 > 0$ such that for any $T > T_\delta$ and $\triangle t \in (0, \triangle t_0)$, there exists a positive constant $k_{T,\delta}$, independent of $\triangle t$, such that the solution z^k of (1.6) satisfies

$$k_{T,\delta} \left\|z^0\right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\|Bz^k\right\|_Y^2, \qquad \forall\, z^0 \in \mathcal{C}_{\delta/\triangle t}. \tag{2.2}$$

Moreover, T_δ can be taken to be such that

$$T_\delta = \pi\Big[M^2\Big(1 + \frac{\delta^2}{4}\Big)^2 + m^2 C_B^2 \frac{\delta^4}{16}\Big]^{1/2}, \tag{2.3}$$

where C_B is as in (2.1).

Remark 2.2. If we filter at a scale smaller than $\triangle t$, for instance in the class $\mathcal{C}_{\delta/(\triangle t)^\alpha}$, with $\alpha < 1$, then δ in (2.3) vanishes as $\triangle t$ tends to zero. In that case the uniform observability time T_0 we obtain is $T_0 = \pi M$, which coincides with the time obtained by the resolvent estimate (1.5) in the continuous setting (see [17]). Note that, however, even in the continuous setting, in general πM is not the optimal observability time.

Proof of Theorem 2.1. Theorem 2.1 can be seen as a direct consequence of Theorem 1.3, which will be proved below. Indeed, one can easily verify that (H1)–(H3) hold by taking $A_{\triangle t} = A$, $B_{\triangle t} = B$ and $X_{\delta,\triangle t} = \mathcal{C}_{\delta/\triangle t}$. $\Diamond$

Before getting into the proof of Theorem 1.3, let us first introduce the discrete Fourier transform at scale $\triangle t$, which is one of the main ingredients of the proof of Theorem 1.3.

Definition 2.3. Given any sequence $(u^k) \in l^2(\triangle t\mathbb{Z})$, we define its Fourier transform as:

$$\hat{u}(\tau) = \triangle t \sum_{k \in \mathbb{Z}} u^k \exp(-i\tau k \triangle t), \quad \tau \triangle t \in (-\pi, \pi]. \tag{2.4}$$

For any function $v \in L^2(-\pi/\triangle t, \pi/\triangle t)$, we define the inverse Fourier transform at scale $\triangle t > 0$:

$$\tilde{v}^k = \frac{1}{2\pi} \int_{-\pi/\triangle t}^{\pi/\triangle t} v(\tau) \exp(i\tau k \triangle t) \, d\tau, \quad k \in \mathbb{Z}. \tag{2.5}$$

According to Definition 2.3,

$$\tilde{\hat{u}} = u, \quad \hat{\tilde{v}} = v, \tag{2.6}$$

and the Parseval identity holds

$$\frac{1}{2\pi} \int_{-\pi/\triangle t}^{\pi/\triangle t} |\hat{u}(\tau)|^2 \, d\tau = \triangle t \sum_{k \in \mathbb{Z}} |u^k|^2. \tag{2.7}$$

These properties will be used in the sequel.

Proof of Theorem 1.3. The proof is split into three parts.

Step 1: Estimates in the class $X_{\delta,\triangle t}$. Let us take $z^0 \in X_{\delta,\triangle t}$. Then the solution of (1.14) has constant norm since $A_{\triangle t}$ is skew-adjoint (see (H1)). Indeed,

$$z^{k+1} = \Big(\frac{I + \frac{\triangle t}{2} A_{\triangle t}}{I - \frac{\triangle t}{2} A_{\triangle t}} \Big) z^k := \mathbb{T}_{\triangle t} z^k,$$

where the operator $\mathbb{T}_{\triangle t}$ is obviously unitary.

Further, since

$$\frac{z^k + z^{k+1}}{2} = \frac{1}{2} \Big(I + \mathbb{T}_{\triangle t} \Big) z^k = \Big(\frac{I}{I - \frac{\triangle t}{2} A_{\triangle t}} \Big) z^k,$$

we get that for any k,

$$\left\| \frac{z^0 + z^1}{2} \right\|_X^2 = \left\| \frac{z^k + z^{k+1}}{2} \right\|_X^2 \geq \frac{1}{1 + \left(\frac{\delta}{2}\right)^2} \left\| z^0 \right\|_X^2, \tag{2.8}$$

as a consequence of (1.11) and the skew-adjointness assumption (H1) of $A_{\triangle t}$.

Step 2: The resolvent estimate. Set $\chi \in H^1(\mathbb{R})$ and $\chi^k = \chi(k\triangle t)$. Let $g^k = \chi^k z^k$, and

$$f^k = \frac{g^{k+1} - g^k}{\triangle t} - A_{\triangle t} \Big(\frac{g^{k+1} + g^k}{2} \Big). \tag{2.9}$$

One can easily check that

$$\begin{aligned} f^k &= \frac{\chi^{k+1}-\chi^k}{\triangle t}\frac{z^{k+1}+z^k}{2} + \frac{\chi^{k+1}+\chi^k}{2}\frac{z^{k+1}-z^k}{\triangle t} \\ &\quad -A_{\triangle t}\Big(\frac{\chi^{k+1}+\chi^k}{2}\frac{z^{k+1}+z^k}{2} + \frac{\chi^{k+1}-\chi^k}{2}\frac{z^{k+1}-z^k}{2}\Big) \\ &= \frac{\chi^{k+1}-\chi^k}{\triangle t}\Big(\frac{z^k+z^{k+1}}{2} - \frac{(\triangle t)^2}{4}A_{\triangle t}\Big(\frac{z^{k+1}-z^k}{\triangle t}\Big)\Big) \\ &= \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)\Big(I - \frac{(\triangle t)^2}{4}A_{\triangle t}^2\Big)\Big(\frac{z^k+z^{k+1}}{2}\Big). \end{aligned} \tag{2.10}$$

Especially, recalling (2.8) and (1.11), (2.10) implies

$$\left\|f^k\right\|_X^2 \le \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 \left\|\frac{z^0+z^1}{2}\right\|_X^2 \Big(1+\frac{\delta^2}{4}\Big). \tag{2.11}$$

In particular, $f^k \in l^2(\triangle t\mathbb{Z}; X)$.

Taking the Fourier transform of (2.9), for all $\tau \in (-\pi/\triangle t, \pi/\triangle t)$, we get

$$\begin{aligned} \hat{f}(\tau) &= \triangle t \sum_{k\in\mathbb{Z}} f^k \exp(-ik\triangle t\tau) \\ &= \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{g^{k+1}-g^k}{\triangle t} - A_{\triangle t}\Big(\frac{g^{k+1}+g^k}{2}\Big)\Big)\exp(-ik\triangle t\tau) \\ &= \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\exp(i\triangle t\tau)-1}{\triangle t} - A_{\triangle t}\Big(\frac{\exp(i\triangle t\tau)+1}{2}\Big)\Big) g^k \exp(-ik\triangle t\tau) \\ &= \Big(i\frac{2}{\triangle t}\tan\Big(\frac{\tau\triangle t}{2}\Big)I - A_{\triangle t}\Big)\hat{g}(\tau)\exp\Big(i\frac{\tau\triangle t}{2}\Big)\cos\Big(\frac{\tau\triangle t}{2}\Big). \end{aligned} \tag{2.12}$$

We claim the following Lemma:

Lemma 2.4. *The solution* (z^k) *in* (1.14) *satisfies*

$$\begin{aligned} &(1+\alpha)m^2\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 \left\|B_{\triangle t}\Big(\frac{z^k+z^{k+1}}{2}\Big)\right\|_Y^2 \\ &\ge \left\|\frac{z^0+z^1}{2}\right\|_X^2 \left[a_1\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 - a_2\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\right], \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} a_1 &= \Big(1-\frac{1}{\beta}\Big), \\ a_2 &= M^2\Big(1+\frac{\delta^2}{4}\Big)^2 + m^2C_B^2\Big(1+\frac{1}{\alpha}\Big)\frac{\delta^4}{16} + \frac{(\triangle t)^2}{16}\delta^2(\beta-1), \end{aligned} \tag{2.14}$$

for any $\alpha > 0$ *and* $\beta > 1$*, where* C_B, M, m *are as in* (1.12)-(1.13).

Proof of Lemma 2.4. Let

$$G(\tau)=\hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos(\frac{\tau\triangle t}{2}). \tag{2.15}$$

By its definition and the fact that $z^k\in X_{\delta,\triangle t}$, it is obvious that $G(\tau)\in X_{\delta,\triangle t}$.

In view of (2.12), applying the resolvent estimate (1.13) to $G(\tau)$, integrating on τ from $-\pi/\triangle t$ to $\pi/\triangle t$, it holds

$$M^2\int_{-\pi/\triangle t}^{\pi/\triangle t}\left\|\hat{f}(\tau)\right\|_X^2 d\tau+m^2\int_{-\pi/\triangle t}^{\pi/\triangle t}\left\|B_{\triangle t}G(\tau)\right\|_Y^2 d\tau \geq \int_{-\pi/\triangle t}^{\pi/\triangle t}\left\|G(\tau)\right\|_X^2 d\tau. \tag{2.16}$$

Applying Parseval's identity (2.7) to (2.16), and noticing that

$$\tilde{G}^k=\frac{g^k+g^{k+1}}{2},\quad i.e.\;\; G(\tau)=\widehat{\Big(\frac{g^k+g^{k+1}}{2}\Big)}(\tau),$$

we get

$$M^2\triangle t\sum_{k\in\mathbb{Z}}\left\|f^k\right\|_X^2+m^2\triangle t\sum_{k\in\mathbb{Z}}\left\|B_{\triangle t}\Big(\frac{g^k+g^{k+1}}{2}\Big)\right\|_Y^2 \geq \triangle t\sum_{k\in\mathbb{Z}}\left\|\frac{g^k+g^{k+1}}{2}\right\|_X^2. \tag{2.17}$$

Now we estimate the three terms in (2.17). The first term can be bounded above in view of (2.11).

Second, since

$$\frac{g^{k+1}+g^k}{2}=\frac{\chi^{k+1}+\chi^k}{2}\frac{z^{k+1}+z^k}{2}+\frac{\triangle t}{2}\frac{\chi^{k+1}-\chi^k}{\triangle t}\frac{z^{k+1}-z^k}{2}, \tag{2.18}$$

using

$$\|a+b\|^2\leq(1+\alpha)\|a\|^2+\Big(1+\frac{1}{\alpha}\Big)\|b\|^2,$$

we deduce that

$$\begin{aligned}\left\|B_{\triangle t}\Big(\frac{g^{k+1}+g^k}{2}\Big)\right\|_Y^2 &\leq(1+\alpha)\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\left\|B_{\triangle t}\Big(\frac{z^{k+1}+z^k}{2}\Big)\right\|_Y^2\\ &\quad+\Big(1+\frac{1}{\alpha}\Big)\frac{(\triangle t)^4}{16}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\left\|B_{\triangle t}\Big(\frac{z^{k+1}-z^k}{\triangle t}\Big)\right\|_Y^2\\ &\leq(1+\alpha)\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\left\|B_{\triangle t}\Big(\frac{z^{k+1}+z^k}{2}\Big)\right\|_Y^2\\ &\quad+\Big(1+\frac{1}{\alpha}\Big)\frac{\delta^4}{16}C_B^2\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\left\|\frac{z^0+z^1}{2}\right\|_X^2.\end{aligned} \tag{2.19}$$

In (2.19) we use the fact that (recalling (1.11) and (1.12))

$$\Big\|B_{\triangle t}A_{\triangle t}\Big(\frac{z^k+z^{k+1}}{2}\Big)\Big\|_Y \le C_B\,\Big\|A^2_{\triangle t}\Big(\frac{z^k+z^{k+1}}{2}\Big)\Big\|_X \le \frac{\delta^2 C_B}{(\triangle t)^2}\,\Big\|\frac{z^0+z^1}{2}\Big\|_X .$$

Finally, for any $\beta > 1$, recalling (2.8), (1.11) and (2.18), we get

$$\begin{aligned}\Big\|\frac{g^{k+1}+g^k}{2}\Big\|_X^2 \quad &\ge \Big(1-\frac{1}{\beta}\Big)\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\Big\|\frac{z^{k+1}+z^k}{2}\Big\|_X^2 \\ &\quad -(\beta-1)\Big(\frac{\triangle t}{2}\Big)^2\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\Big\|\frac{z^{k+1}-z^k}{2}\Big\|_X^2 \\ &\ge \Big(1-\frac{1}{\beta}\Big)\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\Big\|\frac{z^0+z^1}{2}\Big\|_X^2 \\ &\quad -(\beta-1)\Big(\frac{\triangle t}{2}\Big)^4\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\Big\|A_{\triangle t}\Big(\frac{z^0+z^1}{2}\Big)\Big\|_X^2, \\ &\ge \Big(1-\frac{1}{\beta}\Big)\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\Big\|\frac{z^0+z^1}{2}\Big\|_X^2 \\ &\quad -(\beta-1)\Big(\frac{\delta\triangle t}{4}\Big)^2\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\Big\|\Big(\frac{z^0+z^1}{2}\Big)\Big\|_X^2\end{aligned} \tag{2.20}$$

where we used

$$\|a+b\|^2 \ge \Big(1-\frac{1}{\beta}\Big)\|a\|^2 - \Big(\beta-1\Big)\|b\|^2 .$$

Applying (2.11), (2.19) and (2.20) to (2.17), we complete the proof of Lemma 2.4. $\Diamond$

Step 3: The observability estimate. This step is aimed to derive the observability estimate (1.15) stated in Theorem 1.3 from Lemma 2.4 with explicit estimates on the optimal time T_δ.

First of all, let us recall the following classical Lemma on Riemann sums:

Lemma 2.5. *Let $\chi(t) = \phi(t/T)$ with $\phi \in H^2 \cap H^1_0(0,1)$, extended by zero outside $(0,T)$. Recalling that $\chi^k = \chi(k\triangle t)$, the following estimates hold:*

$$\begin{aligned}&\Big|\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 - T\,\|\phi\|^2_{L^2(0,1)}\Big| \le 2T\triangle t\,\|\phi\|_{L^2(0,1)}\,\big\|\dot\phi\big\|_{L^2(0,1)}, \\ &\Big|\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 - \frac{1}{T}\,\big\|\dot\phi\big\|^2_{L^2(0,1)}\Big| \le \frac{2}{T}\triangle t\,\big\|\dot\phi\big\|_{L^2(0,1)}\,\big\|\ddot\phi\big\|_{L^2(0,1)} .\end{aligned} \tag{2.21}$$

Sketch of the proof of Lemma 2.5. It is easy to show that for all $f = f(t) \in C^1(0,T)$ and sequence $\tau_k \in [k\triangle t,(k+1)\triangle t]$, it holds

$$\begin{aligned}\Big|\int_0^T f(t)dt - \triangle t\sum_{k\in(0,T/\triangle t)} f(\tau_k)\Big| \quad &\le \quad \sum_{k\in(0,T/\triangle t)}\iint_{[k\triangle t,(k+1)\triangle t]^2}|\dot f(s)|\;ds\;dt \\ &\le \quad \triangle t\int_0^T|\dot f|\;dt.\end{aligned} \tag{2.22}$$

Replacing f by ϕ^2 we get the first inequality (2.21). Similarly, replacing f by $\dot{\phi}^2$, the second one can be proved too. $\Diamond$

Taking Lemma 2.4 and 2.5 into account, the coefficient of $\left\|(z^0+z^1)/2\right\|_X^2$ in (2.13) tends to

$$k_{T,\delta,\alpha,\beta,\phi} = \frac{1}{m^2(1+\alpha)}\Big[\Big(1-\frac{1}{\beta}\Big)T\,\|\phi\|^2_{L^2(0,1)} - \Big(M^2\Big(1+\frac{\delta^2}{4}\Big)^2 + m^2C_B^2\Big(1+\frac{1}{\alpha}\Big)\frac{\delta^4}{16}\Big)\frac{1}{T}\left\|\dot{\phi}\right\|^2_{L^2(0,1)}\Big],$$

when $\triangle t \to 0$.

Note that $k_{T,\delta,\alpha,\beta,\phi}$ is an increasing function of T tending to $-\infty$ when $T\to 0^+$ and to $+\infty$ when $T\to\infty$. Let $T_{\delta,\alpha,\beta,\phi}$ be the unique positive solution of $k_{T,\delta,\alpha,\beta,\phi}=0$. Then, for any time $T > T_{\delta,\alpha,\beta,\phi}$, choosing a positive $k_{T,\delta}$ such that

$$0 < k_{T,\delta} < k_{T,\delta,\alpha,\beta,\phi},$$

there exists $\triangle t_0 > 0$ such that for any $\triangle t < \triangle t_0$, the following holds:

$$k_{T,\delta}\left\|\frac{z^0+z^1}{2}\right\|_X^2 \le \triangle t \sum_{k\in(0,T/\triangle t)} \left\|B_{\triangle t} z^k\right\|_Y^2. \tag{2.23}$$

This combined with (2.8) yields (1.15).

This construction yields the following estimate on the time T_δ in Theorem 1.3. Namely, for any $\alpha>0$, $\beta>1$ and smooth function ϕ, compactly supported in $[0,1]$:

$$T_\delta \le \frac{\left\|\dot{\phi}\right\|_{L^2}}{\|\phi\|_{L^2}}\Big[\frac{\beta}{\beta-1}\Big]^{1/2}\Big[M^2\Big(1+\frac{\delta^2}{4}\Big)^2 + m^2C_B^2\Big(1+\frac{1}{\alpha}\Big)\frac{\delta^4}{16}\Big]^{1/2}.$$

We optimize in α,β and ϕ by choosing $\alpha=\infty$, $\beta=\infty$ and

$$\phi(t) = \begin{cases} \sin(\pi t), & t\in(0,1) \\ 0, & \text{elsewhere}, \end{cases} \tag{2.24}$$

which is well-known to minimize the ratio

$$\frac{\left\|\dot{\phi}\right\|_{L^2}}{\|\phi\|_{L^2}}.$$

For this choice of ϕ, this quotient equals π, and thus we recover the estimate (1.16). This completes the proof of Theorem 1.3.$\Diamond$

Theorem 2.1 has many applications. Indeed, it roughly says that, for any continuous conservative system, which is observable in finite time, there exists a time semi-discretization which uniformly preserves the observability property in finite time, provided the initial data are filtered at a scale $1/\triangle t$. Later, using formally some microlocal tools, we will explain why this filtering scale is the optimal one. Note that in Theorem 7.1 of [27] this scale was proved to be optimal for a particular time-discretization scheme on the wave equation.

Besides, as we will see in Section 3, Theorem 1.3 is a key ingredient to address observability issues.

3. General time-discrete schemes

3.1. General time-discrete schemes for first order systems

In this section, we deal with more general time-discretization schemes of the form (1.17). We will show that, under some appropriate assumptions on the operator $\mathbb{T}_{\triangle t}$, inequality (1.8) holds uniformly on $\triangle t$ for solutions of (1.17) when the initial data are taken in the class $\mathcal{C}_{\delta/\triangle t}$.

More precisely, we assume that (1.17) is conservative in the sense that there exist real numbers $\lambda_{j,\triangle t}$ such that

$$\mathbb{T}_{\triangle t}\Phi_j = \exp(i\lambda_{j,\triangle t}\triangle t)\Phi_j. \tag{3.1}$$

Moreover, we assume that there is an explicit relation between $\lambda_{j,\triangle t}$ and μ_j (as in (1.9)) of the following form:

$$\lambda_{j,\triangle t} = \frac{1}{\triangle t}\, h(\mu_j \triangle t), \tag{3.2}$$

where $h : [-\delta, \delta] \mapsto [-\pi, \pi]$ is a smooth strictly increasing function, i.e.

$$|h(\eta)| \leq \pi, \qquad \inf\{h'(\eta),\ |\eta| \leq \delta\} > 0. \tag{3.3}$$

Roughly speaking, the first part of (3.3) reflects the fact that one cannot measure frequencies higher than $\pi/\triangle t$ in a mesh of size $\triangle t$. The second part is a non-degeneracy condition on the group velocity (see [21]) of solutions of (1.17) which is necessary to guarantee the propagation of solutions that is required for observability to hold.

We also assume

$$\frac{h(\eta)}{\eta} \longrightarrow 1 \quad \text{as} \quad \eta \to 0. \tag{3.4}$$

This guarantees the consistency of the time-discrete scheme with the continuous model (1.1).

We have the following Theorem:

Theorem 3.1. *Assume that* (A, B) *satisfy* (1.5) *and that* $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.

Under assumptions (3.1), (3.2), (3.3) *and* (3.4), *for any* $\delta > 0$, *there exists a time* T_δ *such that for all* $T > T_\delta$, *there exists a constant* $k_{T,\delta} > 0$ *such that for all* $\triangle t$ *small enough, any solution of* (1.17) *with initial value* $z^0 \in \mathcal{C}_{\delta/\triangle t}$ *satisfies*

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B\Big(\frac{z^k + z^{k+1}}{2}\Big)\right\|_Y^2 . \tag{3.5}$$

Besides, we have the following estimate on T_δ:

$$T_\delta \le \pi \Bigg[M^2 \Big(1+\tan^2\Big(\frac{h(\delta)}{2}\Big)\Big)^2 \sup_{|\eta|\le\delta} \Big\{ \frac{\cos^4(h(\eta)/2)}{h'(\eta)^2} \Big\} + m^2 C_B^2 \sup_{|\eta|\le\delta} \Big\{ \frac{2}{\eta} \tan\Big(\frac{h(\eta)}{2}\Big) \Big\}^2 \tan^4\Big(\frac{h(\delta)}{2}\Big) \Bigg]^{1/2}, \quad (3.6)$$

where C_B is as in (2.1).

Proof. The main idea is to use Theorem 1.3. Hence we introduce an operator $A_{\triangle t}$ such that the solution of (1.17) coincides with the solution of the linear system

$$\frac{z^{k+1}-z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^k+z^{k+1}}{2}\Big), \quad z^0 = z_0. \quad (3.7)$$

This can be done defining the action of the operator $A_{\triangle t}$ on each eigenfunction:

$$A_{\triangle t}\Phi_j = i k_{\triangle t}(\mu_j)\Phi_j, \quad (3.8)$$

where

$$k_{\triangle t}(\omega) = \frac{2}{\triangle t}\tan\Big(\frac{h(\omega\triangle t)}{2}\Big). \quad (3.9)$$

Indeed, if

$$z_0 = \sum a_j \Phi_j,$$

then the solution of (1.17) can be written as

$$z^k = \sum a_j \phi_j \exp(i\lambda_j k \triangle t) = \sum a_j \phi_j \exp(i h(\mu_j \triangle t) k)$$

and the definition of $A_{\triangle t}$ follows naturally.

Obviously, when the scheme (1.17) under consideration is the one of Section 2, that is (1.6), the operator $A_{\triangle t}$ is precisely the operator A.

Then (3.5) would be a straightforward consequence of Theorem 1.3, if we could prove the resolvent estimate for $A_{\triangle t}$. We will see in the sequel that a weak form of the resolvent estimate holds, and that this is actually sufficient to get the desired observability inequality. In the sequel, δ is a given positive number, determining the class of filtered data under consideration.

Step 1: A weak form of the resolvent estimate. By hypothesis (1.5),

$$M^2 \|(A-i\omega)z\|_X^2 + m^2 \|Bz\|_Y^2 \ge \|z\|_X^2, \quad z \in \mathcal{D}(A),\ \omega \in \mathbb{R}. \quad (3.10)$$

For $z \in \mathcal{C}_{\delta/\triangle t}$, that is

$$z = \sum_{|\mu_j|\le\delta/\triangle t} a_j \phi_j, \quad (3.11)$$

one can easily check that

$$\|(A-i\omega)z\|_X^2 = \sum |a_j|^2 \Big(\mu_j - \omega\Big)^2$$

and

$$\|(A_{\triangle t} - i\omega)z\|_X^2 = \sum |a_j|^2 \Big(k_{\triangle t}(\mu_j) - \omega\Big)^2.$$

Especially, for any $\omega \in \mathbb{R}$, this last estimate takes the form

$$\|(A_{\triangle t} - ik_{\triangle t}(\omega))z\|_X^2 = \sum |a_j|^2 \Big(k_{\triangle t}(\mu_j) - k_{\triangle t}(\omega)\Big)^2$$

with $k_{\triangle t}$ as in (3.9). Thus, taking $\varepsilon > 0$, it follows that for any $\omega < (\delta+\varepsilon)/\triangle t$,

$$\|(A_{\triangle t} - ik_{\triangle t}(\omega))z\|_X^2 \geq \Big(\inf_{|\omega|\triangle t \leq \delta+\varepsilon} \Big\{|k'_{\triangle t}(\omega)|\Big\}\Big)^2 \|(A - i\omega)z\|_X^2 .$$

Hence, setting

$$\alpha_{\triangle t,\varepsilon} = k_{\triangle t}\Big(\frac{\delta+\varepsilon}{\triangle t}\Big), \quad C_{\delta,\varepsilon} = \Big(\inf\{k'_{\triangle t}(\omega) \ : \ |\omega|\triangle t \leq \delta+\varepsilon\}\Big)^{-1}, \tag{3.12}$$

which is finite in view of (3.3), we get the following weak resolvent estimate:

$$C_{\delta,\varepsilon}^2 M^2 \left\|\Big(A_{\triangle t} - i\omega\Big)z\right\|_X^2 + m^2 \|Bz\|_Y^2 \geq \|z\|_X^2, \quad z \in \mathcal{C}_{\delta/\triangle t},\ |\omega| \leq \alpha_{\triangle t,\varepsilon}. \tag{3.13}$$

Our purpose is now to show that this is enough to get the time-discrete observability estimate. We emphasize that the main difference between (3.13) and (1.13) is that (1.13) is assumed to hold for all $\omega \in \mathbb{R}$ while (3.13) only holds for $|\omega| \leq \alpha_{\triangle t,\varepsilon}$.

Step 2: Improving the resolvent estimate (3.13)**.** Here we prove that (3.13) can be extended to all $\omega \in \mathbb{R}$. Indeed, consider ω such that $|\omega| \geq \alpha_{\triangle t,\varepsilon}$ and $z \in \mathcal{C}_{\delta/\triangle t}$ as in (3.11). Then

$$\begin{aligned}
\|(A_{\triangle t} - i\omega)z\|_X^2 \quad &\geq \sum_{|\mu_j|\leq \delta/\triangle t} \Big(k_{\triangle t}(\mu_j) - k_{\triangle t}\Big(\frac{\delta+\varepsilon}{\triangle t}\Big)\Big)^2 a_j^2 \\
&\geq \sum_{|\mu_j|\leq \delta/\triangle t} \Big(k_{\triangle t}\Big(\frac{\delta}{\triangle t}\Big) - k_{\triangle t}\Big(\frac{\delta+\varepsilon}{\triangle t}\Big)\Big)^2 a_j^2 \\
&\geq \Big(\frac{\varepsilon}{\triangle t}\Big)^2 \Big(\inf_{\omega\triangle t \in [\delta,\delta+\varepsilon]} k'_{\triangle t}(\omega)\Big)^2 \|z\|^2 .
\end{aligned}$$

Using the explicit expression (3.9) of $k_{\triangle t}$, we get

$$\|(A_{\triangle t} - i\omega)z\|_X^2 \geq \Big(\frac{\varepsilon}{\triangle t}\Big)^2 \inf_{\eta\in[\delta,\delta+\varepsilon]} \{h'(\eta)\}^2 \|z\|^2 . \tag{3.14}$$

Therefore, for each $\varepsilon > 0$, in view of (3.3) and (3.12), there exists $(\triangle t)_\varepsilon > 0$ such that, for $\triangle t \leq (\triangle t)_\varepsilon$

$$C_{\delta,\varepsilon}^2 M^2 \left\|\Big(A_{\triangle t} - i\omega\Big)z\right\|_X^2 + m^2 \|Bz\|_Y^2 \geq \|z\|_X^2, \quad z \in \mathcal{C}_{\delta/\triangle t},\ \omega \in \mathbb{R}. \tag{3.15}$$

Step 3: Application of Theorem 1.3. First, one easily checks from (3.8)-(3.9) that

$$\triangle t \left\| A_{\triangle t} z \right\|_X \leq \tilde{\delta} \left\| z \right\|_X, \qquad z \in \mathcal{C}_{\delta/\triangle t}, \tag{3.16}$$

with $\tilde{\delta} = 2\tan(h(\delta)/2)$.

Second, we check that there exists a constant $C_{B,\delta}$ such that

$$\left\| Bz \right\|_Y \leq C_{B,\delta} \left\| A_{\triangle t} z \right\|_X, \qquad z \in \mathcal{C}_{\delta/\triangle t}, \tag{3.17}$$

where C_B is as in (2.1). Indeed, for $z \in \mathcal{C}_{\delta/\triangle t}$,

$$\left\| Az \right\|_X \leq \sup_{|\omega|\triangle t \leq \delta} \left\{ \left| \frac{k_{\triangle t}(\omega)}{\omega} \right| \right\} \left\| A_{\triangle t} z \right\|_X,$$

and therefore one can take

$$C_{B,\delta} = \beta_\delta C_B, \tag{3.18}$$

where

$$\beta_\delta = \sup_{|\eta| \leq \delta} \left\{ \frac{2}{\eta} \tan\left(\frac{h(\eta)}{2} \right) \right\},$$

which is finite from hypothesis (3.3) and (3.4).

Third, the resolvent estimate (3.15) holds.

Then Theorem 1.3 can be applied and proves the observability inequality (3.5) for the solutions of (1.17) with initial data in $\mathcal{C}_{\delta/\triangle t}$. Besides, we have the following estimate on the observability time $T_{\delta,\varepsilon}$:

$$T_{\delta,\varepsilon} = \pi \left[\left(1 + \frac{\tilde{\delta}^2}{4} \right)^2 M^2 C_{\delta,\varepsilon}^2 + m^2 C_B^2 \beta_\delta^2 \frac{\tilde{\delta}^4}{16} \right]^{1/2}.$$

In the limit $\varepsilon \to 0$, $T_{\delta,\varepsilon}$ converges to an observability time T_δ. Besides, using the explicit form of the constants $C_{\delta,\varepsilon}, \tilde{\delta}$ and β_δ one gets (3.6). $\diamondsuit$

3.2. The Newmark method for second order in time systems

In this subsection we investigate observability properties for time-discrete schemes for the second order in time evolution equation (1.18).

Let H be a Hilbert space endowed with the norm $\|\cdot\|_H$ and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint positive operator with compact resolvent. We consider the initial value problem (1.18), which can be seen as a generic model for the free vibrations of elastic structures such as strings, beams, membranes, plates or three-dimensional elastic bodies.

The energy of (1.18) is given by

$$E(t) = \left\| \dot{u}(t) \right\|_H^2 + \left\| A_0^{1/2} u(t) \right\|_H^2, \tag{3.19}$$

which is constant in time.

We consider the output function

$$y(t) = B_1 u(t) + B_2 \dot{u}(t), \tag{3.20}$$

where B_1 and B_2 are two observation operators satisfying $B_1 \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ and $B_2 \in \mathfrak{L}(\mathcal{D}(A_0^{1/2}), Y)$. In other words, we assume that there exist two constants $C_{B,1}$ and $C_{B,2}$, such that

$$\|B_1 u\|_Y \le C_{B,1} \|A_0 u\|_H, \qquad \|B_2 v\|_Y \le C_{B,2} \left\|A_0^{1/2} v\right\|. \tag{3.21}$$

In the sequel, we assume either $B_1 = 0$ or $B_2 = 0$. This assumption is needed for technical reasons, as we shall see in Remark 3.3 and in the proof of Theorem 3.2.

System (1.18)–(3.20) can be put in the form (1.1)–(1.2). Indeed, setting

$$z_1(t) = \dot{u} + iA_0^{1/2} u, \qquad z_2(t) = \dot{u} - iA_0^{1/2} u, \tag{3.22}$$

equation (1.18) is equivalent to

$$\dot{z} = Az, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad A = \begin{pmatrix} iA_0^{1/2} & 0 \\ 0 & -iA_0^{1/2} \end{pmatrix}, \tag{3.23}$$

for which the energy space is $X = H \times H$ with the domain $\mathcal{D}(A) = \mathcal{D}(A_0^{1/2}) \times \mathcal{D}(A_0^{1/2})$. Moreover, the energy $E(t)$ given in (3.19) coincides with half of the norm of z in X.

Note that the spectrum of A is explicitly given by the spectrum of A_0. Indeed, if $(\mu_j^2)_{j\in\mathbb{N}^*}$ ($\mu_j > 0$) is the sequence of eigenvalues of A_0, i.e.

$$A_0 \phi_j = \mu_j^2 \phi_j, \qquad j \in \mathbb{N}^*,$$

with corresponding eigenvectors ϕ_j, then the eigenvalues of A are $\pm i\mu_j$, with corresponding eigenvectors

$$\Phi_j = \begin{pmatrix} \phi_j \\ 0 \end{pmatrix}, \qquad \Phi_{-j} = \begin{pmatrix} 0 \\ \phi_j \end{pmatrix}, \qquad j \in \mathbb{N}^*. \tag{3.24}$$

Besides, in the new variables (3.22), the output function is given by

$$y(t) = Bz(t) = B_1 A_0^{-1/2} \Big(\frac{iz_2(t) - iz_1(t)}{2}\Big) + B_2 \Big(\frac{z_1(t) + z_2(t)}{2}\Big). \tag{3.25}$$

Recalling the assumptions on B_1 and B_2 in (3.21), the admissible observation B belongs to $\mathfrak{L}(\mathcal{D}(A), Y)$.

In the sequel, we assume that the system (1.18)–(3.20) is exactly observable. As a consequence of this we obtain that system (3.23)–(3.25) is exactly observable and therefore the resolvent estimate (1.5) holds.

We now introduce the time-discrete schemes we are interested in. For any $\triangle t > 0$ and $\beta > 0$, we consider the following Newmark time-discrete scheme for system (1.18):

$$\begin{cases} \dfrac{u^{k+1} + u^{k-1} - 2u^k}{(\triangle t)^2} + A_0\Big(\beta u^{k+1} + (1-2\beta)u^k + \beta u^{k-1}\Big) = 0, \\ \Big(\dfrac{u^0 + u^1}{2}, \dfrac{u^1 - u^0}{\triangle t}\Big) = (u_0, v_0) \in \mathcal{D}(A_0^{\frac{1}{2}}) \times H. \end{cases} \tag{3.26}$$

The energy of (3.26) is given by

$$E^{k+1/2} = \left\| A_0^{1/2}\Big(\frac{u^k+u^{k+1}}{2}\Big)\right\|^2 + \left\|\frac{u^{k+1}-u^k}{\triangle t}\right\|^2 + (4\beta-1)\frac{(\triangle t)^2}{4}\left\| A_0^{1/2}\Big(\frac{u^{k+1}-u^k}{\triangle t}\Big)\right\|^2, \ k\in\mathbb{Z}, \quad (3.27)$$

which is a discrete counterpart of the continuous energy (3.19). Multiplying the first equation of (3.26) by $(u^{k+1}-u^{k-1})/2$ and using integration by parts, it is easy to show that (3.27) remains constant with respect to k. Furthermore, we assume in the sequel that $\beta \geq 1/4$ to guarantee that system (3.26) is unconditionally stable.

The output function is given by the following discretization of (3.20):

$$y^{k+1/2} = B_1\Big(\frac{u^k+u^{k+1}}{2}\Big) + B_2\Big(\frac{u^{k+1}-u^k}{\triangle t}\Big), \quad (3.28)$$

where, as in (3.20), we assume that either B_1 or B_2 vanishes.

For any $s>0$, we define $\mathcal{C}_s$ as in (1.10). Note that this space is invariant under the actions of the discrete semi-groups associated to the Newmark time-discrete schemes (3.26).

We have the following theorem:

Theorem 3.2. *Let $\beta \geq 1/4$ and $\delta > 0$. We assume that either $B_1 \equiv 0$ or $B_2 \equiv 0$.*

Then there exists a time T_δ such that for all $T > T_\delta$, there exists a positive constant $k_{T,\delta}$, such that for $\triangle t$ small enough, the solution of (3.26) with initial data $(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}$ satisfies

$$k_{T,\delta} E^{1/2} \leq \triangle t \sum_{k\triangle t \in (0,T)} \left\| y^{k+1/2}\right\|_Y^2, \quad (3.29)$$

where $y^{k+1/2}$ is defined in (3.28) and B_1, B_2 satisfy (3.21).

Besides, T_δ can be chosen as

$$T_{\delta,1} = \pi\Big[(1+\beta\delta^2)^2\Big(1+\Big(\beta-\frac{1}{4}\Big)\delta^2\Big)^2 M^2 + m^2 C_{B,1}^2 \frac{\delta}{16}^4\Big]^{1/2}, \quad (3.30)$$

if $B_2 = 0$ and as

$$T_{\delta,2} = \pi\Big[(1+\beta\delta^2)^2\Big(1+\Big(\beta-\frac{1}{4}\Big)\delta^2\Big) M^2 + m^2 C_{B,2}^2 \frac{\delta^4}{16}\Big]^{1/2}, \quad (3.31)$$

if $B_1 = 0$.

Remark 3.3. This result, and especially the time estimates (3.30) and (3.31) on the observability time need further comments.

As in Theorem 2.1, we see that, if we filter at a scale smaller than $\triangle t$, for instance in the class $\mathcal{C}_{\delta/(\triangle t)^\alpha}$, with $\alpha < 1$, then the uniform observability time T_0 is given by $T_0 = \pi M$, which coincides with the value obtained by the resolvent estimate (1.5) in the continuous setting.

Note that the estimates (3.30) and (3.31) do not have the same growth in δ when δ goes to ∞. This fact does not seem to be natural because the observability time is expected to depend on the group velocity (see [21]) and not on the form of the observation operator.

By now we could not avoid the assumption that either B_1 or B_2 vanishes, the special case $\beta = 1/4$ being excepted. However, we can deal with an observable of the form

$$y^{k+1/2} = B_1\Big(I + (\beta - 1/4)(\triangle t)^2 A_0\Big)^{1/2}\Big(\frac{u^k + u^{k+1}}{2}\Big) + B_2\Big(\frac{u^{k+1} - u^k}{\triangle t}\Big), \quad (3.32)$$

with both non-trivial B_1 and B_2. Indeed, in this case, the operator $B_{\triangle t}$ arising in the proof of Theorem 3.2 does not depend on $\triangle t$ and therefore the proof works as in the case $B_1 = 0$, and yields the time estimate (3.31). However, this observation operator, which compares to the continuous one (3.20) when $\delta \to 0$, does not seem to be the most natural discretization of (3.25).

When $\beta = 1/4$, both (3.30) and (3.31) have the same form. Besides, one can easily adapt the proof to show that when $\beta = 1/4$, we can deal with a general observation operator B as in (3.20). Actually, the Newmark scheme (3.26) with $\beta = 1/4$ is equivalent to a midpoint scheme, and therefore Theorem 2.1 applies.

Proof. **Step** 1. We first transform system (3.26) into a first order time-discrete scheme similar to (3.23). For this, we define

$$A_{0,\triangle t} = A_0[I + (\beta - 1/4)(\triangle t)^2 A_0]^{-1}. \quad (3.33)$$

Then (3.26) can be rewritten as

$$\frac{u^{k+1} + u^{k-1} - 2u^k}{(\triangle t)^2} + A_{0,\triangle t}\Big(\frac{u^{k-1} + 2u^k + u^{k+1}}{4}\Big) = 0. \quad (3.34)$$

As in (3.22), using the following change of variables

$$\begin{cases} z_1^{k+1/2} = \dfrac{u^{k+1} - u^k}{\triangle t} + iA_{0,\triangle t}^{1/2}\Big(\dfrac{u^k + u^{k+1}}{2}\Big), \\ z_2^{k+1/2} = \dfrac{u^{k+1} - u^k}{\triangle t} - iA_{0,\triangle t}^{1/2}\Big(\dfrac{u^k + u^{k+1}}{2}\Big), \end{cases} \quad (3.35)$$

system (3.26) (and also system (3.34)) is equivalent to

$$\frac{z^{k+1/2} - z^{k-1/2}}{\triangle t} = A_{\triangle t}\Big(\frac{z^{k-1/2} + z^{k+1/2}}{2}\Big), \quad (3.36)$$

with

$$A_{\triangle t} = \begin{pmatrix} iA_{0,\triangle t}^{1/2} & 0 \\ 0 & -iA_{0,\triangle t}^{1/2} \end{pmatrix}, \quad z^{k+1/2} = \begin{pmatrix} z_1^{k+1/2} \\ z_2^{k+1/2} \end{pmatrix}. \quad (3.37)$$

Consequently, the observation operator $y^{k+1/2}$ in (3.28) is given by

$$\begin{aligned} y^{k+1/2} &= B_1 A_{0,\triangle t}^{-1/2}\Big(\frac{iz_2^{k+1/2} - iz_1^{k+1/2}}{2}\Big) + B_2\Big(\frac{z_1^{k+1/2} + z_2^{k+1/2}}{2}\Big) \\ &\triangleq B_{\triangle t} z^{k+1/2}. \end{aligned} \tag{3.38}$$

Step 2. We now verify that system (3.36)–(3.38) satisfies the hypothesis of Theorem 1.3.

We first check (H1). It is obvious that the eigenvectors of $A_{\triangle t}$ are the same as those of A (see (3.24)). Moreover, for any Φ_j we compute

$$A_{\triangle t}\Phi_j = \ell_j \Phi_j, \quad \text{with} \quad \ell_j = \frac{i\mu_j}{\sqrt{1 + (\beta - 1/4)(\triangle t)^2 \mu_j^2}}. \tag{3.39}$$

In other words, we are close to the situation considered in Subsection 3.1, and the time semi-discrete approximation scheme (3.36) satisfies the hypothesis (3.1), (3.2), (3.3), (3.3) and (3.4) with the function h defined by

$$h(\eta) = 2\arctan\Big(\frac{\eta}{2}\frac{1}{\sqrt{1 + (\beta - 1/4)\eta^2}}\Big). \tag{3.40}$$

In particular, this implies that (3.16) holds in the class $\mathcal{C}_{\delta/\triangle t}$, and takes the form

$$\triangle t \left\|A_{\triangle t} z\right\|_X \leq \frac{\delta}{\sqrt{1 + (\beta - 1/4)\delta^2}} \left\|z\right\|_X, \qquad z \in \mathcal{C}_{\delta/\triangle t}. \tag{3.41}$$

Second, we check hypothesis (H2):

$$\begin{aligned} \left\|B_{\triangle t} z\right\|_Y &\leq \left\|A_{\triangle t} z\right\|_H \Big(C_{B,1}\left\|A_0 A_{0,\triangle t}^{-1}\right\|_{\mathfrak{L}(\mathcal{C}_{\delta/\triangle t}, H)} \\ &\qquad\qquad + C_{B,2}\left\|A_0^{1/2} A_{0,\triangle t}^{-1/2}\right\|_{\mathfrak{L}(\mathcal{C}_{\delta/\triangle t}, H)}\Big) \\ &\leq \left\|A_{\triangle t} z\right\|_H \Big((1 + (\beta - 1/4)\delta^2) C_{B,1} + \sqrt{1 + (\beta - 1/4)\delta^2} C_{B,2}\Big) \\ &\leq C_{B,\delta} \left\|A_{\triangle t} z\right\|_H. \end{aligned} \tag{3.42}$$

The third point is more technical. Following the proof of Theorem 3.1, for any $\varepsilon > 0$, we obtain the following resolvent estimate:

$$C_{\delta,\varepsilon}^2 M^2 \left\|\Big(A_{\triangle t} - i\omega\Big) z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \quad z \in \mathcal{C}_{\delta/\triangle t},\ \omega \in \mathbb{R}, \tag{3.43}$$

where $C_{\delta,\varepsilon}$ is given by (3.12), with

$$k_{\triangle t}(\omega) = \frac{\omega}{\sqrt{1 + (\beta - 1/4)(\omega \triangle t)^2}}.$$

Straightforward computations show that, actually,

$$C_{\delta,\varepsilon} = \Big(1 + (\beta - 1/4)(\delta + \varepsilon)^2\Big)^{3/2}. \tag{3.44}$$

Our goal now is to derive from (3.43) the resolvent estimate (H3) given in (1.13). Here, we will handle separately the two cases $B_1 = 0$ and $B_2 = 0$.

The case $B_1 = 0$. Under this assumption, $B_{\triangle t} = B$, and therefore, (3.43) is the resolvent estimate (H3) we need.

The case $B_2 = 0$. In this case, we observe that

$$B_{\triangle t} z = B R_{\triangle t} z, \quad \text{where } R_{\triangle t} = \begin{pmatrix} A_0^{1/2} A_{0,\triangle t}^{-1/2} & 0 \\ 0 & A_0^{1/2} A_{0,\triangle t}^{-1/2} \end{pmatrix} = A A_{\triangle t}^{-1}.$$

Note that the operator $R_{\triangle t}$ commutes with $A_{\triangle t}$, maps $\mathcal{C}_{\delta/\triangle t}$ into itself, and is invertible. Then, applying (3.43) to $R_{\triangle t} z$, we obtain that

$$C_{\delta,\varepsilon}^2 M^2 \left\| R_{\triangle t} \Big(A_{\triangle t} - i\omega \Big) z \right\|_X^2 + m^2 \left\| B_{\triangle t} z \right\|_Y^2 \geq \left\| R_{\triangle t} z \right\|_X^2, \qquad \forall\, z \in \mathcal{C}_{\delta/\triangle t}, \quad \omega \in \mathbb{R}. \tag{3.45}$$

We now compute explicitly the norm of $R_{\triangle t}$ and $R_{\triangle t}^{-1}$ in the class $\mathcal{C}_{\delta/\triangle t}$. Since

$$A_0 A_{0,\triangle t}^{-1} = 1 + (\beta - 1/4)(\triangle t)^2 A_0,$$

one easily checks that

$$\left\| R_{\triangle t} \right\|_\delta^2 = 1 + (\beta - 1/4)\delta^2, \qquad \left\| R_{\triangle t}^{-1} \right\|_\delta^2 = 1, \tag{3.46}$$

where $\|\cdot\|_\delta$ denotes the operator norm from $\mathcal{C}_{\delta/\triangle t}$ into itself. Applying (3.46) into (3.45), we obtain

$$C_{\delta,\varepsilon}^2 M^2 \Big(1 + (\beta - 1/4)\delta^2 \Big) \left\| \Big(A_{\triangle t} - i\omega \Big) z \right\|_X^2 + m^2 \left\| B_{\triangle t} z \right\|_Y^2 \geq \|z\|_X^2, \qquad \forall\, z \in \mathcal{C}_{\delta/\triangle t}, \quad \omega \in \mathbb{R}. \tag{3.47}$$

Thus, in both cases, we can apply Theorem 1.3, which gives the existence of a time $T_{\delta,\varepsilon}$ such that for $T > T_{\delta,\varepsilon}$, there exists a positive $k_{T,\delta}$ such that any solution of (3.36) with initial data $z^{1/2} \in \mathcal{C}_{\delta/\triangle t}$ satisfies

$$k_{T,\delta} \left\| z^{1/2} \right\|_X^2 \leq \sum_{k=0}^{T/\triangle t} \left\| B_{\triangle t} z^{k+1/2} \right\|_Y^2.$$

Besides, the estimates of Theorem 1.3 allow to estimate the observability time $T_{\delta,\varepsilon}$:

$$T_{\delta,\varepsilon} = \begin{cases} \pi \Big[(1 + \beta\delta^2)^2 \dfrac{(1 + (\beta - 1/4)(\delta + \varepsilon)^2)^3}{1 + (\beta - 1/4)\delta^2} M^2 + m^2 C_{B,1}^2 \dfrac{\delta}{16}^4 \Big]^{1/2}, \\ \qquad\qquad if\ B_2 = 0, \\ \pi \Big[(1 + \beta\delta^2)^2 \dfrac{(1 + (\beta - 1/4)(\delta + \varepsilon)^2)^3}{(1 + (\beta - 1/4)\delta^2)^2} M^2 + m^2 C_{B,2}^2 \dfrac{\delta^4}{16} \Big]^{1/2}, \\ \qquad\qquad if\ B_1 = 0. \end{cases}$$

Letting $\varepsilon \to 0$, we obtain the estimates (3.30)-(3.31).

To complete the proof we check that if the initial data $z^{1/2}$ is taken within the class $\mathcal{C}_{\delta/\triangle t}$, the solution of (3.26) satisfies

$$\left\|z^{1/2}\right\|_X^2 = \left\|z^{k+1/2}\right\|_X^2 \geq \frac{2}{1+(\beta-1/4)\delta^2} E^{k+1/2},$$

which can be deduced from the explicit expression of the energy (3.27) and the formula (3.35). ◊

4. Applications

4.1. Application of Theorem 2.1

4.1.1. Boundary observation of the Schrödinger equation. The goal of this subsection is to present a straightforward application of Theorem 2.1 to the observability properties of the Schrödinger equation based on the results in [13].

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Consider the equation

$$\left\{ \begin{array}{ll} iu_t = \Delta_x u, & (t,x) \in (0,T)\times\Omega, \\ u(0) = u_0,\ x\in\Omega, & u(t,x) = 0,\ (t,x)\in(0,T)\times\partial\Omega. \end{array} \right. \tag{4.1}$$

where $u_0 \in L^2(\Omega)$ is the initial data. Equation (4.1) obviously has the form (1.1) with $A = -i\Delta_x$ of domain $\mathcal{D}(A) = H_0^1(\Omega)\cap H^2(\Omega)$.

Let $\Gamma_0 \subset \partial\Omega$ be an open subset of $\partial\Omega$ and define the output

$$y(t) = \frac{\partial u(t)}{\partial \nu}\Big|_{\Gamma_0}.$$

Using Sobolev's embedding theorems, one can easily check that this defines a continuous observation operator B from $\mathcal{D}(A)$ to $L^2(\Gamma_0)$.

Let us assume that Γ_0 satisfies in some time T the *Geometric Control Condition* (GCC) introduced in [1], which asserts that all the rays of Geometric Optics in Ω touch the sub-boundary Γ_0 in a time smaller than T. In this case, the following observability result is known ([13]) :

Theorem 4.1. *For any $T>0$, there exist positive constants $k_T>0$ and $K_T>0$ such that for any $u_0\in L^2(\Omega)$, the solution of* (4.1) *satisfies*

$$k_T \left\|u_0\right\|^2_{L^2(\Omega)} \leq \int_0^T \int_{\Gamma_0} \left|\frac{\partial u(t)}{\partial\nu}\right|^2 d\Gamma_0 dt \leq K_T \left\|u_0\right\|^2_{L^2(\Omega)}. \tag{4.2}$$

We introduce the following time semi-discretization of system (4.1):

$$\left\{ \begin{array}{lll} i\dfrac{u^{k+1}-u^k}{\triangle t} &= \Delta_x\Big(\dfrac{u^{k+1}+u^k}{2}\Big), & x\in\Omega,\ k\in\mathbb{N} \\ u^k(x) &= 0, & x\in\partial\Omega,\ k\in\mathbb{N} \\ u^0(x) &= u_0(x), & x\in\Omega, \end{array} \right. \tag{4.3}$$

that we observe through

$$y^k = \frac{\partial u^k}{\partial\nu}\Big|_{\Gamma_0}.$$

Then Theorem 2.1 implies the following result:

Theorem 4.2. *For any* $\delta > 0$*, there exists a time* T_δ *such that for any time* $T > T_\delta$*, there exists a positive constant* $k_{T,\delta} > 0$ *such that for* $\triangle t$ *small enough, the solution of* (4.3) *satisfies*

$$k_T \left\| u_0 \right\|^2_{L^2(\Omega)} \leq \triangle t \sum_{k \in (0,T/\triangle t)} \int_{\Gamma_0} \left| \frac{\partial u^k}{\partial \nu} \right|^2 d\Gamma_0 \tag{4.4}$$

for any $u_0 \in \mathcal{C}_{\delta/\triangle t}$*.*

Note that we do not know if inequality (4.4) holds in any time $T > 0$ as in the continuous case (see (4.2)). This quesion is still open.

Remark 4.3. Note that in the present section, we do not state any admissibility result for the time-discrete systems under consideration. However, uniform (with respect to $\triangle t > 0$) admissibility results hold for all the examples presented in this article. These results will be derived in Section 6 using the admissibility property of the continuous system (1.1)-(1.2).

4.1.2. Boundary observation of the linearized KdV equation. We now present an application of Theorem 2.1 to the boundary observability of the linear KdV equation.

We consider the following initial-value boundary problem for the KdV equation:

$$\left\{ \begin{array}{rcll} u_t + u_{xxx} & = & 0, & (t,x) \in (0,T) \times (0,2\pi), \\ u(t,0) & = & u(t,2\pi), & t \in (0,T), \\ u_x(t,0) & = & u_x(t,2\pi), & t \in (0,T), \\ u_{xx}(t,0) & = & u_{xx}(t,2\pi), & t \in (0,T), \\ u(0,x) & = & u_0(x), & x \in (0,2\pi). \end{array} \right. \tag{4.5}$$

For any integer k we set

$$H^k_p \triangleq \left\{ u \in H^k(0,2\pi); \;\; \partial^j_x u(0) = \partial^j_x u(2\pi) \;\text{ for }\; 0 \leq j \leq k-1 \right\}, \tag{4.6}$$

where $H^k(0,2\pi)$ denotes the classical Sobolev spaces on the interval $(0,2\pi)$. The initial data of (4.5) are taken in the space $X \triangleq H^2_p(0,2\pi)$, endowed with the classical $H^2(0,2\pi)$-norm.

Let A denote the operator $Au = -\partial^3_x u$ with domain $\mathcal{D}(A) = H^5_p$. As shown in [23], A is a skew-adjoint operator with compact resolvent. Moreover, its spectrum is given by $\sigma(A) = \{i\mu_j \quad \text{with } \mu_j = j^3, \; j \in \mathbb{Z}\}$. The output function $y(t)$ and the corresponding operator $B : \mathcal{D}(A) \longrightarrow Y$ is given by

$$y(t) \triangleq Bu(t) = \begin{pmatrix} u(t,0) \\ u_x(t,0) \\ u_{xx}(t,0) \end{pmatrix},$$

with the norm $\|Bu\|^2_Y = |u(0)|^2 + |u_x(0)|^2 + |u_{xx}(0)|^2$. Note that $B \in \mathfrak{L}(H^5_p, \mathbb{R}^3)$.

The following observability inequality for system (4.5) is well-known (Prop. 2.2 of [22]):

Lemma 4.4. *Let $T > 0$. Then there exist positive numbers k_T and K_T such that for every $u_0 \in H_p^2(0, 2\pi)$,*

$$k_T \|u_0\|_{H_p^2}^2 \leq \int_0^T \Big(|u(t,0)|^2 + |u_x(t,0)|^2 + |u_{xx}(t,0)|^2\Big) dt \leq K_T \|u_0\|_{H_p^2}^2 . \quad (4.7)$$

We now introduce the following time semi-discretization of system (4.5):

$$\begin{cases} \dfrac{u^{k+1} - u^k}{\triangle t} + \dfrac{u_{xxx}^{k+1} + u_{xxx}^k}{2} = 0, & x \in (0, 2\pi),\ k \in \mathbb{N} \\ u^k(0) = u^k(2\pi), & k \in \mathbb{N} \\ u_x^k(0) = u_x^k(2\pi), & k \in \mathbb{N} \\ u_{xx}^k(0) = u_{xx}^k(2\pi), & k \in \mathbb{N} \\ u^0(x) = u_0(x), & x \in (0, 2\pi). \end{cases} \quad (4.8)$$

It is easy to show that the eigenfunctions of A are given by $\{\Phi_j = e^{ijx}\}_{j\in\mathbb{Z}}$ with the corresponding eigenvalues $\{ij^3\}_{j\in\mathbb{Z}}$. Hence, for any $\delta > 0$, we have

$$\mathcal{C}_{\delta/\triangle t} = \text{span}\,\{\Phi_j, j^3 \leq \delta/\triangle t\}. \quad (4.9)$$

As a direct consequence of Theorem 2.1 we have the following uniform observability result for system (4.8):

Theorem 4.5. *For any $\delta > 0$, there exists a time T_δ such that for any $T > T_\delta$, there exists a positive constant $k_{T,\delta} > 0$ such that for $\triangle t$ small enough, the solution u^k of* (4.8) *satisfies*

$$k_{T,\delta} \|u_0\|_{H_p^2}^2 \leq \triangle t \sum_{k\triangle t \in (0,T)} \Big(|u^k(0)|^2 + |u_x^k(0)|^2 + |u_{xx}^k(0)|^2\Big), \quad (4.10)$$

for any initial data $u^0 \in \mathcal{C}_{\delta/\triangle t}$.

As in Theorem 4.2, we do not know if the observability estimate (4.10) holds in any time $T > 0$ as in the continuous case (see Lemma 4.4).

4.2. Application of Theorem 3.1

Let us present an application of Theorem 3.1 to the so-called fourth order Gauss method discretization of equation (1.1) (see for instance [8]-[9]). This fourth order Gauss method is a special case of the Runge-Kutta time approximation schemes, which corresponds to the only conservative scheme within this class.

Consider the following discrete system:

$$\begin{cases} \kappa_i = A\Big(z^k + \triangle t \displaystyle\sum_{j=1}^2 \alpha_{ij}\kappa_i\Big), & i = 1, 2, \\ z^{k+1} = z^k + \dfrac{\triangle t}{2}(\kappa_1 + \kappa_2), & (\alpha_{ij}) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} \\ \frac{1}{4} + \frac{\sqrt{3}}{6} & \frac{1}{4} \end{pmatrix}. \\ z^0 \in \mathcal{C}_{\delta/\triangle t} \text{ given,} & \end{cases} \quad (4.11)$$

The scheme is unstable for the eigenfunctions corresponding to the eigenvalues μ_j such that $\mu_j \triangle t \geq 2\sqrt{3}$ ([8]-[9]). Thus we immediately impose the following restriction on the filtering parameter :

$$\delta < 2\sqrt{3}.$$

To use Theorem 3.1, we only need to check the behavior of the semi-discrete scheme (4.11) on the eigenvectors. If $z^0 = \Phi_j$, an easy computation shows that

$$z^1 = \exp(i\ell_j \triangle t) z^0,$$

where

$$\ell_j = \frac{2}{\triangle t} \arctan\Big(\frac{\mu_j \triangle t}{2 - (\mu_j \triangle t)^2/6}\Big). \tag{4.12}$$

In other words, $\ell_j \triangle t = h(\mu_j \triangle t)$, where

$$h(\eta) = 2 \arctan\Big(\frac{\eta}{2 - \eta^2/6}\Big).$$

Then, a simple application of Theorem 3.1 gives :

Theorem 4.6. *Assume that B is an observation operator such that (A, B) satisfy* (1.5) *and $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.*

For any $\delta \in (0, 2\sqrt{3})$, there exists a time $T_\delta > 0$ such that for any $T > T_\delta$, there exists $\triangle t_0 > 0$ such that for all $0 < \triangle t < \triangle t_0$, there exists a constant $k_{T,\delta} > 0$, independent of $\triangle t$, such that the solutions of system (4.11) *satisfy*

$$k_{T,\delta} \left\| z^0 \right\|_X^2 \leq \triangle t \sum_{k \in (0, T/\triangle t)} \left\| B z^k \right\|_Y^2, \quad \forall\, z^0 \in \mathcal{C}_{\delta/\triangle t}. \tag{4.13}$$

Note that Theorem 3.1 also provides an estimate on T_δ by using (3.6).

In particular, this provides another possible time-discretization of (4.5), for which the observability inequality holds uniformly in $\triangle t$ provided the initial data are taken in $\mathcal{C}_{\delta/\triangle t}$, with $\delta < 2\sqrt{3}$, where $\mathcal{C}_{\delta/\triangle t}$ is as in (4.9).

4.3. Application of Theorem 3.2

There are plenty of applications of Theorem 3.2. We present here an application to the boundary observability of the wave equation.

Consider a smooth nonempty open bounded domain $\Omega \subset \mathbb{R}^d$ and let Γ_0 be an open subset of $\partial\Omega$. We consider the following initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta_x u = 0, & x \in \Omega, \quad t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, \ \ t \geq 0, \\ u(x,0) = u_0, \quad u_t(x,0) = v_0, & x \in \Omega \end{cases} \tag{4.14}$$

with the output

$$y(t) = \frac{\partial u}{\partial \nu}\Big|_{\Gamma_0}. \tag{4.15}$$

This system is conservative and the energy of (4.14)

$$E(t) = \frac{1}{2}\int_\Omega \Big[|u_t(t,x)|^2 + |\nabla u(t,x)|^2\Big] dx, \tag{4.16}$$

remains constant, i.e.

$$E(t) = E(0), \qquad \forall\, t \in [0,T]. \tag{4.17}$$

The boundary observability property for system (4.14) is as follows: *For some constant $C = C(T,\Omega,\Gamma_0) > 0$, solutions of* (4.14) *satisfy*

$$E(0) \le C \int_0^T \int_{\Gamma_0} \Big|\frac{\partial u}{\partial \nu}\Big|^2 d\Gamma_0 dt, \qquad \forall\ (u_0,v_0) \in H_0^1(\Omega)\times L^2(\Omega). \tag{4.18}$$

Note that this inequality holds true for all triplets (T,Ω,Γ_0) satisfying the *Geometric Control Condition* (GCC) introduced in [1], see Subsection 4.1.1. In this case, (4.18) is established by means of micro-local analysis tools (see [1]). From now, we assume this condition to hold.

We then introduce the following time semi-discretization of (4.14):

$$\begin{cases} \dfrac{u^{k+1}+u^{k-1}-2u^k}{(\triangle t)^2} = \Delta_x\Big(\beta u^{k+1} + (1-2\beta)u^k + \beta u^{k-1}\Big), & \text{in } \Omega\times\mathbb{Z}, \\ u^k = 0, & \text{in } \partial\Omega\times\mathbb{Z}, \\ \Big(\dfrac{u^0+u^1}{2}, \dfrac{u^1-u^0}{\triangle t}\Big) = (u_0,\ v_0)\ \in H_0^1(\Omega)\times L^2(\Omega), \end{cases} \tag{4.19}$$

where β is a given parameter satisfying $\beta \ge \frac{1}{4}$.

The output functions y^k are given by

$$y^k = \frac{\partial u^k}{\partial \nu}\Big|_{\Gamma_0}. \tag{4.20}$$

System (4.14)–(4.15) (or system (4.19)–(4.20)) can be written in the form (1.18) (or (3.26)) with observation operator (3.20) by setting:

$$H = L^2(\Omega), \quad \mathcal{D}(A_0) = H^2(\Omega)\cap H_0^1(\Omega), \quad Y = L^2(\Gamma_0),$$
$$A_0\varphi = -\Delta_x\varphi \quad \forall \varphi \in \mathcal{D}(A_0),$$
$$B_1\varphi = \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_0}, \varphi \in \mathcal{D}(A_0).$$

One can easily check that A_0 is self-adjoint in H, positive and boundedly invertible and

$$\mathcal{D}(A_0^{1/2}) = H_0^1(\Omega), \qquad \mathcal{D}(A_0^{1/2})^* = H^{-1}(\Omega).$$

Proposition 4.7. *With the above notation, $B_1 \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ is an admissible observation operator, i.e. for all $T > 0$ there exists a constant $K_T > 0$ such that: If u satisfies* (4.14) *then*

$$\int_0^T \int_{\Gamma_0} \Big|\frac{\partial u}{\partial\nu}\Big|^2 d\Gamma_0 dt \le K_T\Big(\ \|u_0\|^2_{H_0^1(\Omega)} + \|v_0\|^2_{L^2(\Omega)}\ \Big)$$

for all $(u_0,v_0) \in H_0^1(\Omega)\times L^2(\Omega)$.

The above proposition is classical (see, for instance, p. 44 of [15]), so we skip the proof.

Hence we are in the position to give the following theorem:

Theorem 4.8. *Set* $\beta \geq 1/4$.

For any $\delta > 0$, *system* (4.19) *is uniformly observable with* $(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}$. *More precisely, there exists* T_δ, *such that for any* $T > T_\delta$, *there exists a positive constant* $k_{T,\delta}$ *independent of* $\triangle t$, *such that for* $\triangle t > 0$ *small enough, the solutions of system* (4.19) *satisfy*

$$k_{T,\delta}\Big(\left\|\nabla u_0\right\|^2 + \left\|v_0\right\|^2 \Big) \leq \triangle t \sum_{k \in (0, T/\triangle t)} \int_{\Gamma_0} \Big|\frac{\partial u^k}{\partial \nu}\Big|^2 d\Gamma_0, \tag{4.21}$$

for any $(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}$.

Proof. The scheme proposed here is a Newmark discretization. Hence this result is a direct consequence of Theorem 3.1. ◊

Remark 4.9. One can use Fourier analysis and microlocal tools to discuss the optimality of the filtering condition as in [27]. The symbol of the operator in (4.19), that can be obtained by taking the Fourier transform of the differential operator in space-time is of the form (see for instance [16])

$$\frac{4}{\triangle t^2}\sin^2\Big(\frac{\tau\triangle t}{2}\Big) - \Big|\xi\Big|^2\Big(1 - 4\beta\sin^2\Big(\frac{\tau\triangle t}{2}\Big)\Big).$$

Note that this symbol is not hyperbolic in the whole range $(\tau, \xi) \in (-\pi/\triangle t, \pi/\triangle t) \times \mathbb{R}^n$. However, the Fourier transform of any solution of (4.19) is supported in the set of (τ, ξ) satisfying $1 - 4\beta\sin^2(\tau\triangle t/2) > 0$, where the symbol is hyperbolic.

As in the continuous case, one expects the optimal observability time to be the time needed by all the rays to meet Γ_0. Along the bicharacteristic rays associated to this hamiltonian the following identity holds

$$|\tau| = \frac{2}{\triangle t}\arctan\left(\frac{|\xi|\triangle t}{2}\,\frac{1}{\sqrt{1 + (\beta - 1/4)|\xi|^2(\triangle t)^2}}\right).$$

These rays are straight lines as in the continuous case, but their velocity is not 1 anymore. Indeed, one can prove that along the rays corresponding to $|\xi| < \delta/\triangle t$, the velocity of propagation is given by

$$\Big|\frac{dx}{dt}\Big| = \frac{1}{1 + \beta(|\xi|\triangle t)^2}\,\frac{1}{\sqrt{1 + (\beta - 1/4)(\xi\triangle t)^2}} \geq \frac{1}{(1 + \beta\delta^2)\sqrt{1 + (\beta - 1/4)\delta^2}}.$$

In other words, in the class $\mathcal{C}_{\delta/\triangle t}$, the velocity of propagation of the rays concentrated in frequency around $\delta/\triangle t$ is $(1 + \delta^2/4)^{-1}$ times that of the continuous wave equation. Therefore we expect the optimal observability time T^*_δ in the class $\mathcal{C}_{\delta/\triangle t}$ to be

$$T^*_\delta = T^*_0(1 + \beta\delta^2)\sqrt{1 + \Big(\beta - \frac{1}{4}\Big)\delta^2}, \tag{4.22}$$

where T_0^* is the optimal observability time for the continuous system. According to this, the estimate $T_{\delta,2}$ in (3.31) on the time of observability has the good growth rate when $\delta \to \infty$. Besides, when δ goes to ∞, we have that

$$T_{\delta,2} \simeq \pi M(1+\beta\delta^2)\sqrt{1+\Big(\beta-\frac{1}{4}\Big)\delta^2}. \tag{4.23}$$

Recall that $\pi M = T_0$ is the time of observability that the resolvent estimate (1.5) in the continuous setting yields (see [17]). The similarity between (4.22) and (4.23) indicates that the resolvent method accurately measures the group velocity.

Note however that πM is not the expected sharp observability time T_0^* in (4.22) in the continuous setting. This is one of the drawbacks of the method based on the resolvent estimates we use in this paper. Even at the continuous level the observability time one gets this way is far from being the optimal one that Geometric Optics yields.

5. Fully discrete schemes

5.1. Main statement

In this section, we deal with the observability properties for time-discretization systems such as (1.1)-(1.2) depending on an extra parameter, for instance the *space* mesh-size, or the size of the microstructure in homogenization.

To this end, it is convenient to introduce the following class of operators:

Definition 5.1. For any $(m, M, C_B) \in (\mathbb{R}_+^*)^3$, we define $\mathfrak{C}(m, M, C_B)$ as the class of operators (A, B) satisfying:

(A1) The operator A is skew-adjoint on some Hilbert space X, and has a compact resolvent.

(A2) The operator B is defined from $\mathcal{D}(A)$ with values in a Hilbert space Y, and satisfies (2.1) with C_B.

(A3) The pair of operators (A, B) satisfies the resolvent estimate (1.5) with constants m and M.

In this class, Theorems 2.1-3.1-3.2 apply and provide uniform observability results for any of the time semi-discrete approximation schemes (1.6)-(1.7), (1.17), and (1.18). Indeed, this can be deduced by the explicit form of the constants T_δ and $k_{T,\delta}$ which only depend on m, M and C_B. Note that this definition does not depend on the spaces X and Y.

When considering families of pairs of operators (A, B), it is not easy, in general, to show that they belong to the same class $\mathfrak{C}(m, M, C_B)$ for some choice of the constants (m, M, C_B). Indeed, item (A3) is not obvious in general. Therefore, in the sequel, we define another class included in some $\mathfrak{C}(m, M, C_B)$ and that is easier to handle in practice.

Definition 5.2. For any $(C_B, T, k_T, K_T) \in (\mathbb{R}_+^*)^4$, we define $\mathfrak{D}(C_B, T, k_T, K_T)$ as the class of operators (A, B) satisfying (A1), (A2) and:

(B1) The admissibility inequality

$$\int_0^T \left\|B\exp(tA)z^0\right\|_Y^2 \, dt \le K_T \left\|z^0\right\|_X^2, \tag{5.1}$$

where $\exp(tA)$ stands for the semigroup associated to the equation

$$\dot{z} = Az, \quad z(0) = z^0 \in X. \tag{5.2}$$

(B2) The observability inequality

$$k_T \left\|z^0\right\|_X^2 \le \int_0^T \left\|B\exp(tA)z^0\right\|_Y^2 \, dt. \tag{5.3}$$

As we will see below, assumptions (B1)-(B2) imply (A3):

Lemma 5.3. *If the pair (A, B) belongs to $\mathfrak{D}(C_B, T, k_T, K_t)$, then there exist m and M such that $(A, B) \in \mathfrak{C}(m, M, C_B)$.*

Besides m and M can be chosen as

$$m = \sqrt{\frac{2T}{k_T}}, \qquad M = T\sqrt{\frac{K_T}{2k_T}}. \tag{5.4}$$

Proof. We only need to prove (A3). This is actually already done in [17] or in [25]. Indeed, it was proved that once the admissibility inequality (1.3) and the observability inequality (1.4) hold for some time T, then the resolvent estimate (1.5) hold with m and M as in (5.4).$\Diamond$

Note that assumptions (B1)-(B2) are related to the *continuous* systems (5.2).

Now we consider a sequence of operators (A_p, B_p) depending on a parameter $p \in P$, which are in some $\mathfrak{L}(X_p) \times \mathfrak{L}(\mathcal{D}(A_p), Y_p)$ for each p, where X_p and Y_p are Hilbert spaces. We want to address the observability problem for a time-discretization scheme of

$$\dot{z} = A_p z, \quad z(0) = z^0 \in X_p, \qquad y(t) = B_p z(t) \in Y_p. \tag{5.5}$$

In applications, we need the observability to be uniform in both $p \in P$ and $\triangle t > 0$ small enough. The previous analysis and the properties of the class $\mathfrak{D}(C_B, T, k_T, K_T)$ suggest the following two-steps strategy:

1. Study the continuous system (5.5) for every parameter p and prove the uniform admissibility (5.1) and observability (5.3).
2. Apply one of the Theorems 2.1, 3.1 and 3.2 to obtain uniform observability estimates (1.8) for the corresponding time-discrete approximation schemes.

This allows dealing with fully discrete approximation schemes. In that setting the parameter p is actually the standard parameter $h > 0$ associated with the space mesh-size. In this way one can use automatically the existing results for space semi-discretizations as, for instance, [3], [6], [7], [10], [19], [20], [28] and [29], etc.

5.2. Applications

5.2.1. The fully discrete wave equation. Let us consider the wave equation (4.14) in a 2-d square. More precisely, let $\Omega = (0,\pi)\times(0,\pi) \subset \mathbb{R}^2$ and Γ_0 be a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1,\pi) : x_1 \in (0,\pi)\} \cup \{(\pi, x_2) : x_2 \in (0,\pi)\} \triangleq \Gamma_1 \cup \Gamma_2.$$

As in (4.15), the output function $y(t) = Bu(t)$ is given by

$$Bu = \frac{\partial u}{\partial \nu}\Big|_{\Gamma_0} = \frac{\partial}{\partial x_2} u(x_1,\pi)\Big|_{\Gamma_1} + \frac{\partial}{\partial x_1} u(\pi, x_2)\Big|_{\Gamma_2}.$$

Let us first consider the finite-difference semi-discretization of (4.14). The following can be found in [28]. Given $J, K \in \mathbb{N}$ we set

$$h_1 = \frac{\pi}{J+1}, \qquad h_2 = \frac{\pi}{K+1}. \tag{5.6}$$

We denote by $u_{jk}(t)$ the approximation of the solution u of (4.14) at the point $x_{jk} = (jh_1, kh_2)$. The space semi-discrete approximation scheme of (4.14) is as follows:

$$\begin{cases} \ddot{u}_{jk} - \dfrac{u_{j+1k} + u_{j-1k} - 2u_{jk}}{h_1^2} - \dfrac{u_{jk+1} + u_{jk-1} - 2u_{jk}}{h_2^2} = 0 \\ \qquad\qquad 0 < t < T,\ j = 1, \cdots, J;\ k = 1, \cdots, K \\ u_{jk} = 0, \quad 0 < t < T,\ j = 0, J+1;\ k = 0, K+1 \\ u_{jk}(0) = u_{jk,0}, \quad \dot{u}_{jk}(0) = u_{jk,1},\ j = 1, \cdots, J;\ k = 1, \cdots, K. \end{cases} \tag{5.7}$$

System (5.7) is a system of JK linear differential equations. Moreover, if we denote the unknown

$$U(t) = (u_{11}(t), u_{21}(t), \cdots, u_{J1}(t), \cdots, u_{1K}(t), u_{2K}(t), \cdots, u_{JK}(t))^T,$$

then system (5.7) can be rewritten in vector form as follows

$$\begin{cases} \ddot{U}(t) + A_{0,h} U(t) = 0, \qquad 0 < t < T. \\ U(0) = U_{h,0}, \quad \dot{U}(0) = U_{h,1}, \end{cases} \tag{5.8}$$

where $(U_{h,0}, U_{h,1}) = (u_{jk,0}, u_{jk,1})_{1\le j\le J, 1\le k\le K} \in \mathbb{R}^{2JK}$ are the initial data. The corresponding solution of (5.7) is given by $(U_h, \dot{U}_h) = (u_{jk}, \dot{u}_{jk})_{1\le j\le J, 1\le k\le K}$. Note that the entries of $A_{0,h}$ belonging to $\mathcal{M}_{JK}(\mathbb{R})$ may be easily deduced from (5.7).

As a discretization of the output, we choose

$$B_h U = \Big(\Big(\frac{u_{jK}}{h_2}\Big)_{j\in\{1,\cdots,J\}}, \Big(\frac{u_{Jk}}{h_1}\Big)_{k\in\{1,\cdots K\}} \Big). \tag{5.9}$$

The corresponding norm for the observation operator B_h is given by

$$\|B_h U(t)\|_{Y_h}^2 = h_1 \sum_{j=1}^{J} \Big|\frac{u_{jK}(t)}{h_2}\Big|^2 + h_2 \sum_{k=1}^{K} \Big|\frac{u_{Jk}(t)}{h_1}\Big|^2.$$

Besides, the energy of the system (5.8) is given by

$$E_h(t) = \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \Big(|\dot{u}_{jk}(t)|^2 + \Big| \frac{u_{j+1k}(t) - u_{jk}(t)}{h_1} \Big|^2 + \Big| \frac{u_{jk+1}(t) - u_{jk}(t)}{h_2} \Big|^2 \Big). \quad (5.10)$$

As in the continuous case, this quantity is constant.

$$E_h(t) = E_h(0), \qquad \forall \ 0 < t < T.$$

In order to prove the uniform observability of (5.8), we have to filter the high frequencies. To do that we consider the eigenvalue problem associated with (5.8):

$$A_{0,h}\varphi = \lambda^2 \varphi. \quad (5.11)$$

As in the continuous case, it is easy to show that the eigenvalues λ^{j,k,h_1,h_2} are purely imaginary. Let us denote by φ^{j,k,h_1,h_2} the corresponding eigenvectors.

Let us now introduce the following classes of solutions of (5.8) for any $0 < \gamma < 1$:

$$\widehat{\mathcal{C}_\gamma}(h) = \text{span}\,\{\varphi^{j,k,h_1,h_2} \text{ such that } |\lambda^{j,k,h_1,h_2}| \max(h_1,h_2) \le 2\sqrt{\gamma}\}.$$

The following Lemma holds (*see* [28]):

Lemma 5.4. *Let $0 < \gamma < 1$. Then there exist T_γ such that for all $T > T_\gamma$ there exist $k_{T,\gamma} > 0$ and $K_{T,\gamma} > 0$ such that*

$$k_{T,\gamma} E_h(0) \le \int_0^T \|B_h U(t)\|^2_{Y_h}\, dt \le K_{T,\gamma} E_h(0) \quad (5.12)$$

holds for every solution of (5.8) *in the class $\widehat{\mathcal{C}_\gamma}(h)$ and every h_1, h_2 small enough satisfying*

$$\sup \Big| \frac{h_1}{h_2} \Big| < \sqrt{\frac{\gamma}{4-\gamma}}.$$

Now we present the time discrete schemes we are interested in. For any $\triangle t > 0$, we consider the following time Newmark approximation scheme of system (5.8):

$$\begin{cases} \dfrac{U^{k+1} + U^{k-1} - 2U^k}{(\triangle t)^2} + A_{0,h}\Big(\beta U^{k+1} + (1-2\beta)U^k + \beta U^{k-1}\Big) = 0, \\ \Big(\dfrac{U^0 + U^1}{2}, \dfrac{U^1 - U^0}{\triangle t}\Big) = (U_{h,0}, U_{h,1}), \end{cases} \quad (5.13)$$

with $\beta \ge 1/4$.

The energy of (5.13) given by

$$E^k = \frac{1}{2}\left\| A_{0,h}^{\frac{1}{2}}\Big(\frac{U^k+U^{k+1}}{2}\Big)\right\|^2 + \frac{1}{2}\left\|\frac{U^{k+1}-U^k}{\triangle t}\right\|^2 + (4\beta-1)\frac{(\triangle t)^2}{8}\left\| A_{0,h}^{\frac{1}{2}}\Big(\frac{U^{k+1}-U^k}{\triangle t}\Big)\right\|^2, \quad k\in\mathbb{Z} \tag{5.14}$$

which is a discrete counterpart of the time continuous energy (3.19) and remains constant (see (3.27) as well).

In view of (5.12), conditions (B1) and (B2) are satisfied. Besides, conditions (A1) and (A2) are straightforward. Therefore the following theorem can be obtained as a direct consequence of Theorem 3.2:

Theorem 5.5. *Set $\beta \geq 1/4$. Set $0<\gamma<1$. Assume that the mesh sizes h_1, h_2 and $\triangle t$ tend to zero and*

$$\sup\Big|\frac{h_1}{h_2}\Big| < \sqrt{\frac{\gamma}{4-\gamma}}, \qquad \frac{\max\{h_1,h_2\}}{\triangle t} \leq \tau, \tag{5.15}$$

where τ is a positive constant.

Then, for any $0<\delta\leq 2\sqrt{\gamma}/\tau$, there exist $T_\delta>0$ such that for any $T>T_\delta$, there exists $k_{T,\delta,\gamma}>0$ such that the observability inequality

$$k_{T,\delta,\gamma}E^k \leq \triangle t \sum_{k\triangle t\in(0,T)} \left\| B_h U^k\right\|_{Y_h}^2$$

holds for every solution of (5.13) *with initial data in the class $\mathcal{C}_{\delta/\triangle t}$ for $h_1, h_2, \triangle t$ small enough satisfying* (5.15).

Proof. We are in the setting given before and thus Lemma 5.3 applies. Hence, to apply Theorem 3.1, we only need to verify that $\mathcal{C}_{\delta/\triangle t}\subset\widehat{\mathcal{C}_\gamma}(h)$. But

$$|\lambda| < \frac{\delta}{\triangle t} \Rightarrow |\lambda| \leq 2\frac{\sqrt{\gamma}}{\tau\triangle t} \leq 2\frac{\sqrt{\gamma}}{\max\{h_1,h_2\}}.$$

and this completes the proof. $\Diamond$

5.2.2. The 1-d string with rapidly oscillating density. In this paragraph, we consider a one-dimensional wave equation with rapidly oscillating density, which provides another example where the model under consideration depends on an extra parameter.

Let us state the problem. Let $\rho\in L^\infty(\mathbb{R})$ be a periodic function such that $0<\rho_m\leq\rho(x)\leq\rho_M<\infty$, *a.e.* $x\in\mathbb{R}$. Given $\varepsilon>0$, set $\rho^\varepsilon(x)=\rho(x/\varepsilon)$ and consider the one-dimensional wave equation

$$\begin{cases} \rho^\varepsilon(x)\ddot{u}^\varepsilon - \partial_{xx}^2 u^\varepsilon = 0, & (x,t)\in(0,1)\times(0,T),\\ u^\varepsilon(0,t)=u^\varepsilon(1,t)=0, & t\in(0,T),\\ u^\varepsilon(x,0)=u_0(x), \quad \dot{u}^\varepsilon(x,0)=v_0(x), & x\in(0,1). \end{cases} \tag{5.16}$$

We consider the observation operator

$$Bu^\varepsilon(t) = \partial_x u^\varepsilon(1,t). \tag{5.17}$$

The mathematical setting is the same as in Subsection 4.3 and therefore we do not recall it.

The eigenvalue problem for (5.16) reads

$$\rho^\varepsilon(x)\lambda_n^\varepsilon \Phi_n^\varepsilon + \partial_{xx}^2 \Phi_n^\varepsilon = 0, \quad x \in (0,1); \qquad \Phi_n^\varepsilon(0) = \Phi_n^\varepsilon(1) = 0. \tag{5.18}$$

For each $\varepsilon > 0$, there exists a sequence of eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon < \cdots < \lambda_n^\varepsilon < \cdots \to \infty$$

and a sequence of associated eigenfunctions $(\Phi_n^\varepsilon)_n$ which can be chosen to constitute an orthonormal basis in $L^2(0,1)$ with respect to the norm

$$\|\phi\|_{L^2}^2 = \int_0^1 \rho^\varepsilon(x)|\phi(x)|^2\, dx.$$

In [4], the following is proved:

Theorem 5.6 ([4]). *There exists a positive number $D > 0$, such that the following holds:*

Let $T > 2\sqrt{\bar\rho}$, where $\bar\rho$ denotes the mean value of ρ. Then there exist two positive constants k_T and K_T such that for any initial data (u_0, v_0) in

$$\widetilde{\mathcal{C}}_{D/\varepsilon} = \operatorname{span}\{\Phi_n^\varepsilon \ : \ n < D/\varepsilon\},$$

the solution u^ε of (5.16) verifies

$$k_T \|(u_0,v_0)\|_{H_0^1(0,1)\times L^2(0,1)}^2 \le \int_0^T |u_x^\varepsilon(1,t)|^2 dt \le K_T \|(u_0,v_0)\|_{H_0^1(0,1)\times L^2(0,1)}^2 .$$

Given $\beta \ge 1/4$, let us consider the following time semi-discretization of (5.16)

$$\rho^\varepsilon(x)\Big(\frac{u^{\varepsilon,k+1} - 2u^{\varepsilon,k} + u^{\varepsilon,k-1}}{(\triangle t)^2}\Big) - \partial_{xx}^2\Big((1-2\beta)u^{\varepsilon,k} + \beta(u^{\varepsilon,k-1} + u^{\varepsilon,k+1})\Big) = 0, \qquad (x,k) \in (0,1)\times\mathbb{N}, \tag{5.19}$$

completed with the following boundary conditions and initial data

$$\begin{cases} u^{\varepsilon,k}(0) = u^{\varepsilon,k}(1) = 0, & k \in \mathbb{N}, \\ \Big(\dfrac{u^{\varepsilon,0} + u^{\varepsilon,1}}{2}\Big)(x) = u_0(x), \quad \Big(\dfrac{u^{\varepsilon,1} - u^{\varepsilon,0}}{\triangle t}\Big)(x) = v_0(x), & x \in (0,1). \end{cases} \tag{5.20}$$

Since conditions (A1)-(A2)-(B1)-(B2) hold, we get the following result as a consequence of Theorem 3.2:

Theorem 5.7. *Let $\delta > 0$ and $\beta \ge 1/4$. Assume that the parameters $\triangle t$ and ε tend to zero.*

Then there exists a time T_δ such that for any $T > T_\delta$, there exists a positive constant $k_{T,\delta}$ such that the observability inequality

$$k_{T,\delta} \left\| (u_0, v_0) \right\|^2_{H^1_0(0,1)\times L^2(0,1)} \leq \triangle t \sum_{k\triangle t \in (0,T)} |u_x^{\varepsilon,k}(1)|^2 \tag{5.21}$$

holds for every solution of (5.19)-(5.20) *with initial data* (u_0, v_0) *in the class* $\mathcal{C}_{\delta/\triangle t} \cap \widetilde{\mathcal{C}}_{D/\varepsilon}$, *independently of* $\triangle t$ *and* ε.

6. On the admissibility condition

The goal of this section is to provide admissibility results for the time-discrete schemes used throughout the paper. These results are complementary to the observability results proved in Theorems 2.1, 3.1 and 3.2 when dealing with controllability problems (see [15]).

6.1. The time-continuous setting

Let us assume that system (1.1)-(1.2) is admissible. By definition, there exists a positive constant K_T such that :

$$\int_0^T \left\| y(t) \right\|_Y^2 dt \leq K_T \left\| z_0 \right\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{6.1}$$

The goal of this section is to prove that this property can be read on the wave packets setting as well.

Proposition 6.1. *System* (1.1)-(1.2) *is admissible if and only if*

$$\begin{cases} \text{There exists } r > 0 \text{ and } D > 0 \text{ such that} \\ \text{for all } n \in \Lambda \text{ and for all } z = \displaystyle\sum_{l \in J_r(\mu_n)} c_l \Phi_l : \quad \left\| Bz \right\|_Y \leq D \left\| z \right\|_X , \end{cases} \tag{6.2}$$

where

$$J_r(\mu) = \{ l \in \mathbb{N}, \text{ such that } |\mu_l - \mu| \leq r \}. \tag{6.3}$$

Proof. We will prove separately the two implications.

First let us assume that system (1.1)-(1.2) is admissible.
Denote by

$$V(\omega, \varepsilon) = \text{span}\{\Phi_j \text{ such that } |\mu_j - \omega| \leq \varepsilon\}.$$

Then the following lemma holds:

Lemma 6.2. *Let us define $K(\omega, \varepsilon)$ as*

$$K(\omega, \varepsilon) = \left\| B(A - i\omega I)^{-1} \right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*, Y)} .$$

Then for any $\varepsilon > 0$, $K(\omega, \varepsilon)$ is uniformly bounded in ω, that is

$$K(\varepsilon) = \sup_{\omega \in \mathbb{R}} K(\omega, \varepsilon) < \infty. \tag{6.4}$$

Besides, the following estimate holds

$$K(\varepsilon) \leq \sqrt{\frac{K_1}{1-\exp(-1)}}\Big(1+\frac{1}{\varepsilon}\Big), \tag{6.5}$$

where K_1 is the admissibility constant in (1.3).

Proof of Lemma 6.2.

Let us first notice these resolvent identities:

$$\begin{aligned} (A - i\omega I) - I &= A - (1+i\omega)I, \\ (A-(1+i\omega)I)^{-1}(I-(A-i\omega I)^{-1}) &= (A-i\omega I)^{-1}. \end{aligned}$$

Hence

$$K(\omega,\varepsilon) \leq \left\| B(A-(1+i\omega)I)^{-1}\right\|_{\mathfrak{L}(X,Y)} \ \left\|(I-(A-i\omega I)^{-1})\right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*,X)}.$$

Obviously

$$\left\|(I-(A-i\omega I)^{-1})\right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*,X)} \leq 1+\frac{1}{\varepsilon}$$

Hence we restrict ourselves to the study of

$$\left\| B(A-(1+i\omega)I)^{-1}\right\|_{\mathfrak{L}(X,Y)}.$$

Let us remark that for all $z = \sum a_j \Phi_j \in X$,

$$\begin{aligned} (A-(1+i\omega)I)^{-1}z &= \sum \frac{1}{i(\mu_j-\omega)-1} a_j\Phi_j \\ &= \int_0^\infty \exp(-(1+i\omega)t)z(t)\ dt, \end{aligned}$$

where $z(t)$ is the solution of (1.1) with initial value z. This implies that

$$\begin{aligned} \left\| B(A-(1+i\omega)I)^{-1}z\right\|_Y^2 &= \left\| \int_0^\infty \exp(-1-i\omega t)Bz(t)\ dt\right\|_Y^2 \\ &\leq \int_0^\infty \Big|\exp(-1-2i\omega t)\Big|\ dt\ \Big(\int_0^\infty \exp(-t)\left\|Bz(t)\right\|_Y^2\ dt\Big) \\ &\leq \int_0^\infty \exp(-t)\left\|Bz(t)\right\|_Y^2\ dt. \end{aligned}$$

But using the admissibility property of the operator B, we obtain

$$\begin{aligned} \int_0^\infty \exp(-t)\left\|Bz(t)\right\|_Y^2\ dt &\leq \sum_{k\in\mathbb{N}} \exp(-k)\int_k^{k+1}\left\|Bz(t)\right\|_Y^2\ dt \\ &\leq \Big(\sum_{k\in\mathbb{N}}\exp(-k)\Big)K_1\left\|z\right\|_X^2 \\ &\leq \frac{K_1}{1-\exp(-1)}\left\|z\right\|_X^2. \end{aligned}$$

The estimate (6.5) follows.$\Diamond$

Let us now consider a wave packet $z_0 = \sum_{l \in J_1(\mu_n)} c_l \Phi_l$. Then taking $\varepsilon = 1$ in Lemma 6.2, one gets that

$$\begin{aligned} \|Bz\|_Y &\leq \left\| B(A - i(\mu_n - 2)I)^{-1} \right\|_{\mathfrak{L}(V(\mu_n - 2, 1)^*, Y)} \|(A - i(\mu_n - 2)I)z\| \\ &\leq K(1)\Big(\max_{l \in J_1(\mu_n)} |\mu_l - \mu_n| + 1 \Big) \|z\| \\ &\leq 3K(1) \|z\| . \end{aligned}$$

Now we assume that estimate (6.2) holds for some $r > 0$ and $D > 0$. Set $z_0 \in \mathcal{D}(A)$, and expand z_0 as

$$z_0 = \sum_{k \in \mathbb{Z}} z_k, \qquad z_k = \sum_{l \in J_r(2kr)} c_l \Phi_l.$$

We need a special test function whose existence is established in the following Lemma:

Lemma 6.3. *There exists a time T and a function M satisfying*

$$\begin{cases} M(t) \geq 0, & |t| \geq T/2, \\ M(t) \geq 1, & |t| \leq T/2, \\ \text{Supp } \hat{M} \subseteq (-2r, 2r). \end{cases} \tag{6.6}$$

The proof is postponed to the end of this section. Note that functions satisfying similar properties appear naturally in the proofs of various Ingham's type inequalities, see [11] and [25].

Taking Lemma 6.3 into account, we estimate

$$\begin{aligned} \int_0^T \|Bz(t)\|_Y^2 &\leq \int_{\mathbb{R}} M(t - T/2) \|Bz(t)\|_Y^2 \, dt \\ &\leq \sum_{k_1, k_2} \int_{\mathbb{R}} M(t - T/2) < Bz_{k_1}(t), Bz_{k_2}(t) >_{Y \times Y} \, dt. \end{aligned}$$

But these scalar products vanish most of the time. Indeed, if $|k_1 - k_2| \geq 2$, from (6.6), we get

$$\begin{aligned} &\int_{\mathbb{R}} M(t - T/2) < Bz_{k_1}(t), Bz_{k_2}(t) >_Y \, dt \\ &\qquad = \sum_{(l_1, l_2) \in J_r(2k_1 r) \times J_r(2k_2 r)} \hat{M}(\mu_{l_1} - \mu_{l_2}) < a_{l_1} B\Phi_{l_1}, a_{l_2} B\Phi_{l_2} >_Y = 0. \end{aligned}$$

This implies that

$$\begin{aligned}\int_0^T \|Bz(t)\|_Y^2 &\leq \int_{\mathbb{R}} M(t-T/2)\sum_k \Big(\|Bz_k(t)\|_Y^2 + 2\mathrm{Re} < Bz_k(t), Bz_{k+1}(t) >_{Y\times Y}\Big) dt \\ &\leq 3\int_{\mathbb{R}} M(t-T/2)\sum_k \|Bz_k(t)\|_Y^2 \ dt \\ &\leq 3D\int_{\mathbb{R}} M(t-T/2)\sum_k \|z_k(t)\|_X^2 \ dt \\ &\leq 3D\hat{M}(0)\,\|z_0\|_X^2 .\end{aligned}$$

This completes the proof, since admissibility at time T is obviously equivalent to admissibility in any time.$\Diamond$

Proof of Lemma 6.3.

In this proof, we do not care about the value of the parameters r and T that can be handled through a scaling argument.
Let us consider the function

$$f(t) = \frac{1}{\pi}\mathrm{sinc}(t) = \frac{\sin(t)}{\pi t}.$$

It is well-known that its Fourier transform is $\hat{f}(\tau) = \chi_{(-1,1)}(\tau)$, where $\chi_{(-1,1)}$ denotes the characteristic function of $(-1,1)$.

Hence, the function

$$M(t) = f(t)^2 = \frac{\mathrm{sinc}^2(t)}{\pi^2}$$

satisfies the following properties

$$M(t) \geq \frac{2}{\pi^3}, \quad |t| < \frac{\pi}{4}; \quad M(t)\geq 0, \quad t\in\mathbb{R}; \quad \hat{M}(\tau) = (2-|\tau|)_+, \quad \tau\in\mathbb{R}$$

and the proof is complete. For instance, for $r>0$, one can take the function $M_r(t)$ as

$$M_r(t) = \frac{\pi^2}{8}\mathrm{sinc}^2(rt) \tag{6.7}$$

which satisfies (6.6) with $T = \pi/2r$. $\Diamond$

Remark 6.4. In the context of families of pairs (A,B), according to Proposition 6.1, the uniform admissibility condition (5.1) is equivalent to a uniform wave packet estimate similar to (6.2). To be more precise, if $(\Phi_j^p)_{j\in\mathbb{N}}$ denotes the eigenvectors of A_p associated to the eigenvalues $(\lambda_j^p)_{j\in\mathbb{N}}$, that is $A_p\Phi_j^p = \lambda_j^p\Phi_j^p$, the uniform admissibility condition is equivalent to:

$$\left\{\begin{array}{l}\text{There exists } r>0 \text{ and } D>0 \text{ such that for all } p,\ n\in\mathbb{N} \\ \text{and for all } z = \displaystyle\sum_{l\in J_r(\lambda_n^p)} c_l\Phi_l^p : \qquad \|B_p z\|_{Y_p} \leq D\,\|z\|_{X_p}.\end{array}\right.$$

6.2. The time-discrete setting

This subsection is aimed to prove that if the continuous system (1.1)-(1.2) is admissible, in the sense of Definition 1.1, then its time semi-discrete approximation will be admissible as well under suitable assumptions. In this part, we will focus on the particular discretization given in Subsection 3.1, but everything works as well in all the time semi-discretization schemes considered in the article.

More precisely, we assume that the continuous system (1.1)-(1.2) is admissible, that is, from Proposition 6.1, the wave packet estimate (6.2) holds.

Then we claim that, under the assumptions (1.17), (3.1), (3.2), (3.3) and (3.4), the following discrete admissibility inequality holds:

Theorem 6.5. *Assume that system* (1.1)-(1.2) *is admissible. Set* $\delta > 0$*. For any* $T > 0$*, there exists a constant* $K_{T,\delta} > 0$ *such that for all* $\triangle t$ *small enough, the solution of equation* (1.17) *with initial data in* $\mathcal{C}_{\delta/\triangle t}$ *satisfies*

$$\triangle t \sum_{k=0}^{T/\triangle t} \left\| B z^k \right\|_Y^2 \leq K_{T,\delta} \left\| z^0 \right\|_X^2 . \tag{6.8}$$

Proof. The proof follows the one given in the continuous case. First of all, let us remark the following straightforward fact: There exists $r_\delta > 0$ such that for all $n \in \mathbb{Z}$ satisfying $\triangle t |\lambda_{n,\triangle t}| \leq \delta$, for all $\triangle t > 0$, the set

$$\tilde{J}_{r_\delta}(\lambda_{n,\triangle t}) = \{ l \in \mathbb{Z}, \text{ such that } |\lambda_{l,\triangle t} - \lambda_{n,\triangle t}| \leq r_d \},$$

where $\lambda_{l,\triangle t}$ is as in (3.2), is a subset of $J_r(\mu_n)$ (recall (6.3)). Besides, one can take:

$$r_\delta = r \inf\{ |h'(\eta)|, \ |\eta| \leq \delta \}.$$

Note that condition (3.3) implies the positivity of the right hand side.

Given $\triangle t > 0$, assume that there is a time T and a function $M^{\triangle t} \in l^2(\triangle t \mathbb{Z})$ such that

$$\begin{cases} M^{\triangle t, k} \geq 0, & |k \triangle t| \geq T/2, \\ M^{\triangle t, k} \geq 1, & |t| \leq T/2, \\ \text{Supp } \hat{M}^{\triangle t} \subseteq (-2r_\delta, 2r_\delta), \end{cases} \tag{6.9}$$

where this time $\hat{M}^{\triangle t}$ denotes the discrete Fourier transform at scale $\triangle t$ defined in Definition 2.3. One can easily check that we can take $M^{\triangle t} = M_{r_\delta}$ for all $\triangle t > 0$ where M_{r_δ} is as in (6.7).

With this definition, the proof of inequality (6.8) consists in rewriting the one of Proposition 6.1 by replacing the continuous integrals and the Fourier transform by their discrete versions. Since all the steps are independent of $\triangle t$, the admissibility inequality holds uniformly. $\Diamond$

Note that this proof can be applied to derive uniform admissibility results for families of operators (A, B) within the class $\mathfrak{D}(C_B, T, k_T, K_T)$ for the fully discrete schemes. Indeed, in the setting of Section 5, according to Remark 6.4, the proof presented above directly implies uniform admissibility properties for operators in the class $\mathfrak{D}(C_B, T, k_T, K_T)$ when the initial data are taken in the filtered class $\mathcal{C}_{\delta/\triangle t}$.

7. Further comments and open problems

1. The resolvent estimate is a useful tool to analyze time-discrete approximation schemes, as we have seen in this paper. However, although this method is quite robust, it does not allow to deal with observability inequalities with loss, arising, for instance, when dealing with networks (see [5], Chapter 4) or for the wave equation in the absence of the Geometric Control Conditions (see [12], [14]). In those cases one only needs a weaker version of the observability inequality (1.4), in which the observed norm is weaker than $\|\cdot\|_X$. Actually, this question is also open at the continuous level.

2. As said in Remark 4.9, we are not able to recover the optimal value of the time of observability for systems (1.1)–(1.2) and their time-discrete approximation schemes. This is a drawback of the method based on the resolvent estimate. Indeed, even in the continuous setting, to our knowledge, this method does not allow to recover the optimal time of observability.

3. There are several different methods to derive uniform observability inequalities for systems (4.19). In [27], a discrete multiplier technique is developed to derive the uniform observability of the time semi-discrete wave equation in a bounded domain. There, the same order of filtering parameter $\delta/(\triangle t)$ is attained but a smallness condition on δ is imposed. Theorem 3.2 generalizes this result to any $\delta > 0$, as the dispersion diagram analysis in [27] suggests.

4. Along the paper, we derived uniform observability inequalities and admissibility results for time-discretization schemes of abstract first order and second order (in time) systems. As it is well-known in controllability theory, they imply uniform controllability results as well. For instance, in the context of the time-discrete wave equation analyzed in [27], combining the duality arguments in it and the results of this paper, one can immediately deduce the uniform (with respect to $\triangle t > 0$) controllability of projections on the classes of filtered space $\mathcal{C}_{\delta/\triangle t}$, for $T > T_\delta$ large enough and $\delta > 0$ arbitrary. This improves the results in [27] that required the filtering parameter $\delta > 0$ to be small enough.

The same duality arguments combined with the uniform observability and admissibility results we have presented in this paper allow proving uniform controllability results in a number of other cases including the time-discrete KdV and Schrödinger equations, the fully discrete wave equation, the time-discretization of wave equations with rapidly oscillating coefficients, etc.

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