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# Dispersive numerical schemes for linear and nonlinear Schrödinger equations

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## MOTIVATION:

TO BUILD CONVERGENT NUMERICAL SCHEMES FOR **NON-LINEAR** PARTIAL DIFFERENTIAL EQUATIONS (PDE).

**Example:** SCHRÖDINGER EQUATION.

Similar problems for other dispersive equations: Korteweg-de-Vries, wave equation, ...

**Goal:** To cover the classes of **NONLINEAR** Schrödinger equations that can be solved nowadays with **fine tools** from **PDE theory** and **Harmonic analysis**.

**Key point:** To handle nonlinearities one needs to decode the intrinsic hidden properties of the underlying linear differential operators (Kato, Strichartz, Ginibre, Velo, Cazenave, Weissler, Saut, Bourgain, Kenig, Ponce, Saut, Vega, Burq, Gérard, Tzvetkov, ...)

This has been done successfully for the PDE models.

What about Numerical schemes?

**FROM FINITE TO INFINITE DIMENSIONS IN PURELY  
CONSERVATIVE SYSTEMS.....**

## UNDERLYING MAJOR PROBLEM:

Reproduce in the computer the dynamics in Continuum and Quantum Mechanics, avoiding spurious numerical solutions.

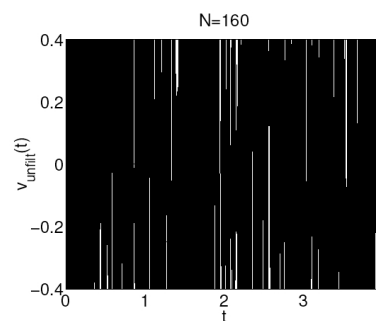
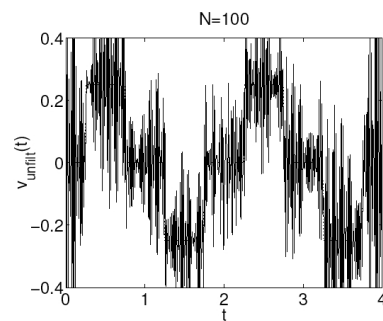
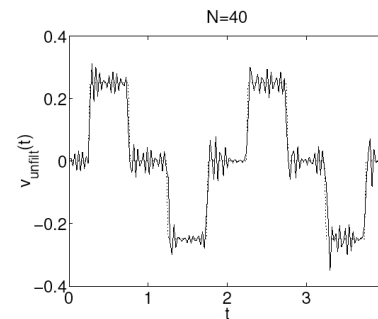
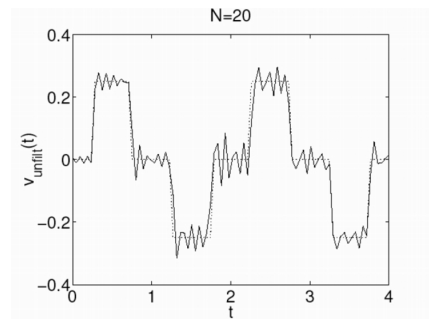
The issue can only be understood by adapting at the discrete numerical level the techniques developed in the continuous context.

## WARNING!

NUMERICS = CONTINUUM + (POSSIBLY) SPURIOUS

Strongly inspired in our previous work on the **CONTROL OF WAVE PHENOMENA**

E. Z. **SIAM Review**, 47 (2) (2005), 197-243.



The conclusions of that analysis were:

- Most stable numerical schemes for solving the initial boundary value problem for the wave equation are **unstable for boundary control problems**;
- This is due to the fact that, adding boundary controls, **excites all numerical frequencies** simultaneously. Consequently we can not apply the classical "consistency+stability" analysis that reduces the problem to dealing with data with a finite number of Fourier components;
- A number of **remedies** have been developed: numerical viscosity, two-grid filtering, mixed finite elements, etc.

## PRELIMINARIES ON CLASSICAL NUMERICAL ANALYSIS

$$\frac{du}{dt}(t) = Au(t), \quad t \geq 0; \quad u(0) = u_0.$$

$A$  an unbounded operator in a Hilbert (or Banach) space  $H$ , with domain  $D(A) \subset H$ . The solution is given by

$$u(t) = e^{At}u_0.$$

Semigroup theory provides conditions under which  $e^{At}$  is well defined. Roughly  $A$  needs to be *maximal* ( $A + I$  is invertible) and *dissipative* ( $A \leq 0$ ).

Most of the *linear* PDE from Mechanics enter in this general frame: wave, heat, Schrödinger equations,...

**Nonlinear problems** are solved by using *fixed point arguments* on the *variation of constants formulation* of the PDE:

$$u_t(t) = Au(t) + f(u(t)), \quad t \geq 0; \quad u(0) = u_0.$$

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}f(u(s))ds.$$

Assuming  $f : H \rightarrow H$  is *locally Lipschitz*, allows proving local (in time) existence and uniqueness in

$$u \in C([0, T]; H).$$

But, often in applications, the property that  $f : H \rightarrow H$  is *locally Lipschitz FAILS*.

For instance  $H = L^2(\Omega)$  and  $f(u) = |u|^{p-1}u$ , with  $p > 1$ .



Then, one needs to discover other properties of the underlying linear equation (smoothing, dispersion): IF  $e^{At}u_0 \in X$ , then look for solutions of the nonlinear problem in

$$C([0, T]; H) \cap X.$$

One then needs to investigate whether

$$f : C([0, T]; H) \cap X \rightarrow C([0, T]; H) \cap X$$

is locally Lipschitz. This requires extra work: We need to check the behavior of  $f$  in the space  $X$ . But the the class of functions to be tested is restricted to those belonging to  $X$ .

Typically in applications  $X = L^r(0, T; L^q(\Omega))$ . This allows enlarging the class of solvable nonlinear PDE in a significant way.

IF WORKING IN  $C([0, T]; : H) \cap X$  IS NEEDED FOR SOLVING THE PDE, FOR PROVING CONVERGENCE OF A NUMERICAL SCHEME WE WILL NEED TO MAKE SURE THAT IT FULFILLS SIMILAR STABILITY PROPERTIES IN  $X$  (OR  $X_h$ ).

THIS OFTEN FAILS!

## THE 1 – D LINEAR SCHRÖDINGER EQUATION

Consider the **Linear Schrödinger Equation (LSE)**:

$$\begin{cases} iu_t + u_{xx} = 0 & x \in \mathbf{R}, t > 0, \\ u(0, x) = \varphi & x \in \mathbf{R}. \end{cases} \quad (1)$$

It may be written in the abstract form:

$$u_t = Au, \quad A = i\Delta = i\partial^2 \cdot / \partial x^2.$$

Accordingly, the LSE generates a group of isometries  $e^{i\Delta t}$  in  $L^2(\mathbf{R})$ , i. e.

$$\|u(t)\|_{L^2(\mathbf{R})} = \|\varphi\|_{L^2(\mathbf{R})}, \quad \forall t \geq 0.$$

The fundamental solution is explicit  $G(x, t) = (4i\pi t)^{-1/2} \exp(-|x|^2/4i\pi t)$ .

**Dispersive properties:** Fourier components with different wave numbers propagate with different velocities.

- $L^1 \rightarrow L^\infty$  decay.

$$\|u(t)\|_{L^\infty(\mathbf{R})} \leq (4\pi t)^{-\frac{1}{2}} \|\varphi\|_{L^1(\mathbf{R})}.$$

$$\|u(t)\|_{L^p(\mathbf{R})} \leq (4\pi t)^{-\left(\frac{1}{2} - \frac{1}{p}\right)} \|\varphi\|_{L^{p'}(\mathbf{R})}, \quad 2 \leq p \leq \infty.$$

- **Local gain of 1/2-derivative:** If the initial datum  $\varphi$  is in  $L^2(\mathbf{R})$ , then  $u(t)$  belongs to  $H_{loc}^{1/2}(\mathbf{R})$  for a.e.  $t \in \mathbf{R}$ .

These properties are not only relevant for a better understanding of the dynamics of the linear system but also to derive [well-posedness](#) and [compactness results for nonlinear Schrödinger equations \(NLS\)](#).

## The three-point finite-difference scheme

Consider the finite difference approximation

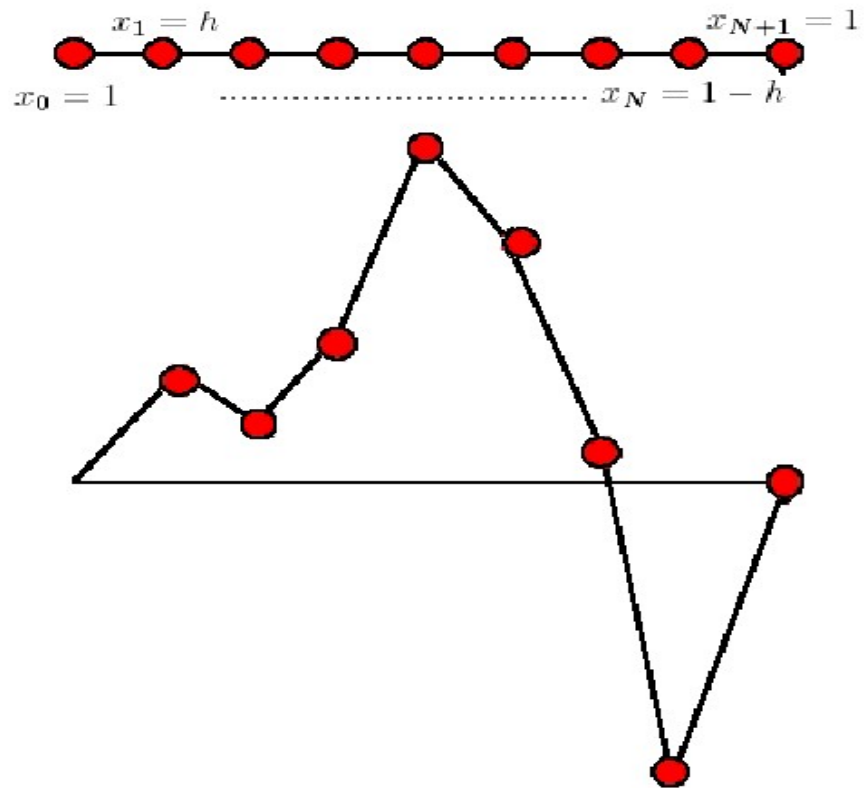
$$i \frac{du^h}{dt} + \Delta_h u^h = 0, t \neq 0, \quad u^h(0) = \varphi^h. \quad (2)$$

Here  $u^h \equiv \{u_j^h\}_{j \in \mathbf{Z}}$ ,  $u_j(t)$  being the approximation of the solution at the node  $x_j = jh$ , and  $\Delta_h \sim \partial_x^2$ :

$$\Delta_h u = \frac{1}{h^2} [u_{j+1} + u_{j-1} - 2u_j].$$

The scheme is consistent + stable in  $L^2(\mathbf{R})$  and, accordingly, it is also convergent, of order 2 (the error is of order  $O(h^2)$ ).

In fact,  $\|u^h(t)\|_{\ell^2} = \|\varphi\|_{\ell^2}$ , for all  $t \geq 0$ .



The same convergence result holds for semilinear equations

$$\begin{cases} iu_t + u_{xx} = f(u) & x \in \mathbf{R}, t > 0, \\ u(0, x) = \varphi & x \in \mathbf{R}, \end{cases} \quad (3)$$

provided the nonlinearity  $f : \mathbf{R} \rightarrow \mathbf{R}$  is **globally Lipschitz**.

The proof is completely standard and only requires the  $L^2$ -conservation property of the continuous and discrete equation.

BUT THIS ANALYSIS IS INSUFFICIENT TO DEAL WITH OTHER NONLINEARITIES, FOR INSTANCE:

$$f(u) = |u|^{p-1}u, \quad p > 1.$$

IT IS JUST A MATTER OF WORKING HARDER, OR DO WE NEED TO CHANGE THE NUMERICAL SCHEME?



The following is well-known for the NSE:

$$\begin{cases} iu_t + u_{xx} = |u|^p u & x \in \mathbf{R}, t > 0, \\ u(0, x) = \varphi(x) & x \in \mathbf{R}. \end{cases} \quad (4)$$

**Theorem 1** (*Global existence in  $L^2$ , Tsutsumi, 1987*). For  $0 \leq p < 4$  and  $\varphi \in L^2(\mathbf{R})$ , there exists a unique solution  $u$  in  $C(\mathbf{R}, L^2(\mathbf{R})) \cap L^q_{loc}(L^{p+2})$  with  $q = 4(p+1)/p$  that satisfies the  $L^2$ -norm conservation and depends continuously on the initial condition in  $L^2$ .

This result can not be proved by methods based purely on energy arguments.

## LACK OF DISPERSION OF THE NUMERICAL SCHEME

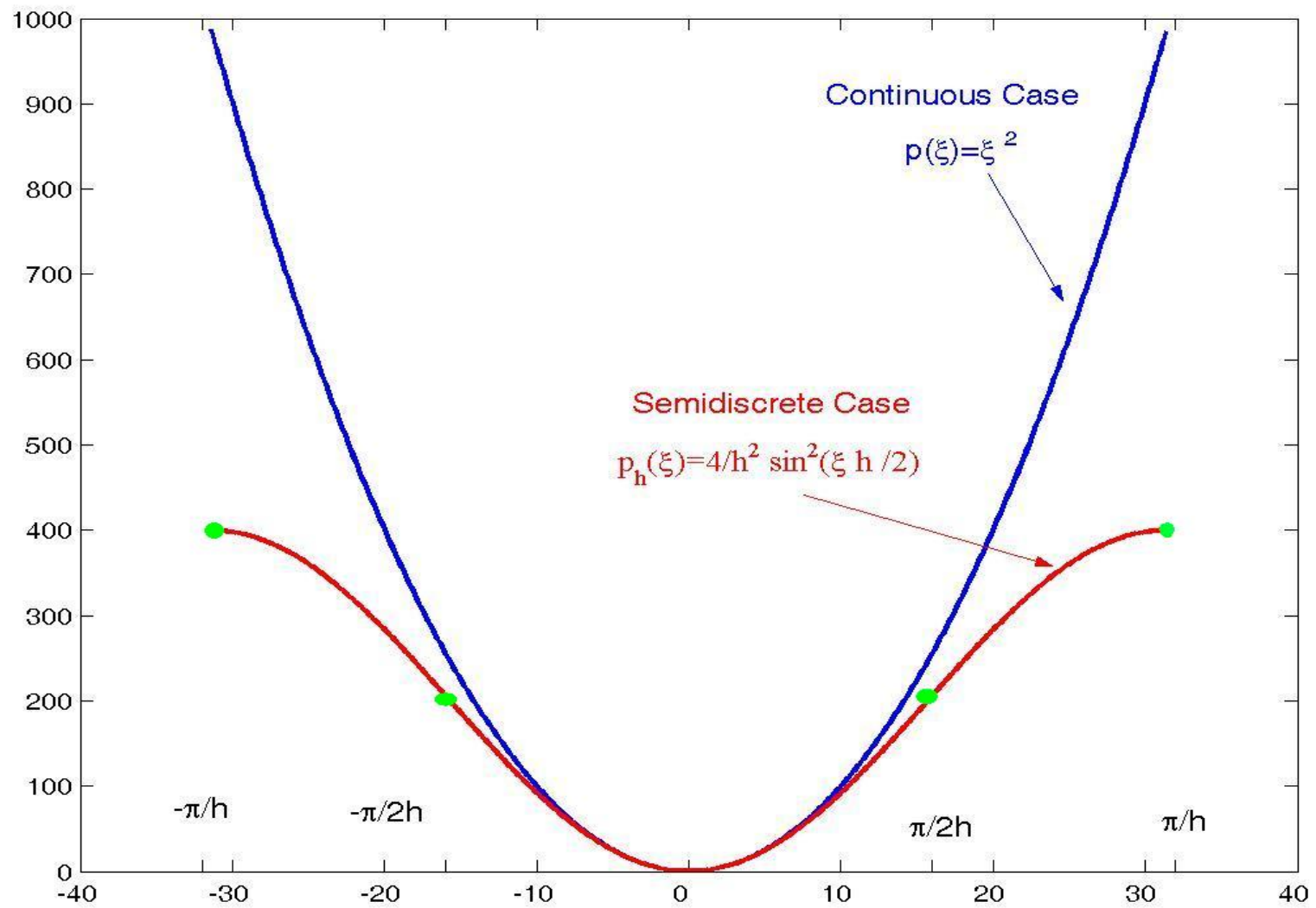
Consider the semi-discrete Fourier Transform

$$\hat{u} = h \sum_{j \in \mathbf{Z}} u_j e^{-ijh\xi}, \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right].$$

There are “slight” but important differences between the symbols of the operators  $\Delta$  and  $\Delta_h$ :

$$p(\xi) = -\xi^2, \quad \xi \in \mathbf{R}; \quad p_h(\xi) = -\frac{4}{h^2} \sin^2\left(\frac{\xi h}{2}\right), \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right].$$

For a fixed frequency  $\xi$ , obviously,  $p_h(\xi) \rightarrow p(\xi)$ , as  $h \rightarrow 0$ . This confirms the convergence of the scheme. But this is far from being sufficient for our goals.



The main differences are:

- $p(\xi)$  is a convex function;  $p_h(\xi)$  changes convexity at  $\pm\frac{\pi}{2h}$ .
- $p'(\xi)$  has a unique zero,  $\xi = 0$ ;  $p'_h(\xi)$  has the zeros at  $\xi = \pm\frac{\pi}{h}$  as well.

These “slight” changes on the shape of the symbol are not an obstacle for the convergence of the numerical scheme in the  $L^2(\mathbf{R})$  sense. But produce the lack of uniform (in  $h$ ) dispersion of the numerical scheme and consequently, makes the scheme useless for nonlinear problems.

LACK OF CONVEXITY = LACK OF INTEGRABILITY GAIN.

The symbol  $p_h(\xi)$  loses convexity near  $\pm\pi/2h$ . Applying the stationary phase lemma (G. Gigante, F. Soria, IMRN, 2002):

**Theorem 2** *Let  $1 \leq q_1 < q_2$ . Then, for all positive  $t$ ,*

$$\sup_{h>0, \varphi^h \in l_h^{q_1}(\mathbf{Z})} \frac{\|\exp(it\Delta_h)\varphi^h\|_{l_h^{q_2}(\mathbf{Z})}}{\|\varphi^h\|_{l_h^{q_1}(\mathbf{Z})}} = \infty. \quad (5)$$

Initial datum with Fourier transform concentrated on  $\pi/2h$ .

LACK OF CONVEXITY = LACK OF LAPLACIAN.

Independent work on the Schrödinger equation in lattices:

A. Stefanov & P. G. Kevrekidis, *Nonlinearity* 18 (2005) 1841-1857.

L. Giannoulis, M. Herrmann & A. Mielke, *Multiscale volume*, 2006.

It is shown that the fundamental solution on the discrete lattice decays in  $L^\infty$  as  $t^{-1/3}$  and not as  $t^{-1/2}$  as in the continuous frame.

**Lemma 1** (*Van der Corput*)

Suppose  $\phi$  is a real-valued and smooth function in  $(a, b)$  that

$$|\phi^{(k)}(\xi)| \geq 1$$

for all  $x \in (a, b)$ . Then

$$\left| \int_a^b e^{it\phi(\xi)} d\xi \right| \leq c_k t^{-1/k} \quad (6)$$

LACK OF SLOPE= LACK OF REGULARITY GAIN.

**Theorem 3** *Let  $q \in [1, 2]$  and  $s > 0$ . Then*

$$\sup_{h>0, \varphi^h \in l_h^q(\mathbf{Z})} \frac{\left| S^h(t) \varphi^h \right|_{\tilde{h}_{loc}^s(\mathbf{Z})}}{\left| \varphi^h \right|_{l_h^q(\mathbf{Z})}} = \infty. \quad (7)$$

Initial data whose Fourier transform is concentrated around  $\pi/h$ .

LACK OF SLOPE= VANISHING GROUP VELOCITY.

Trefethen, L. N. (1982). *SIAM Rev.*, **24** (2), pp. 113–136.



## A REMEDY: FOURIER FILTERING

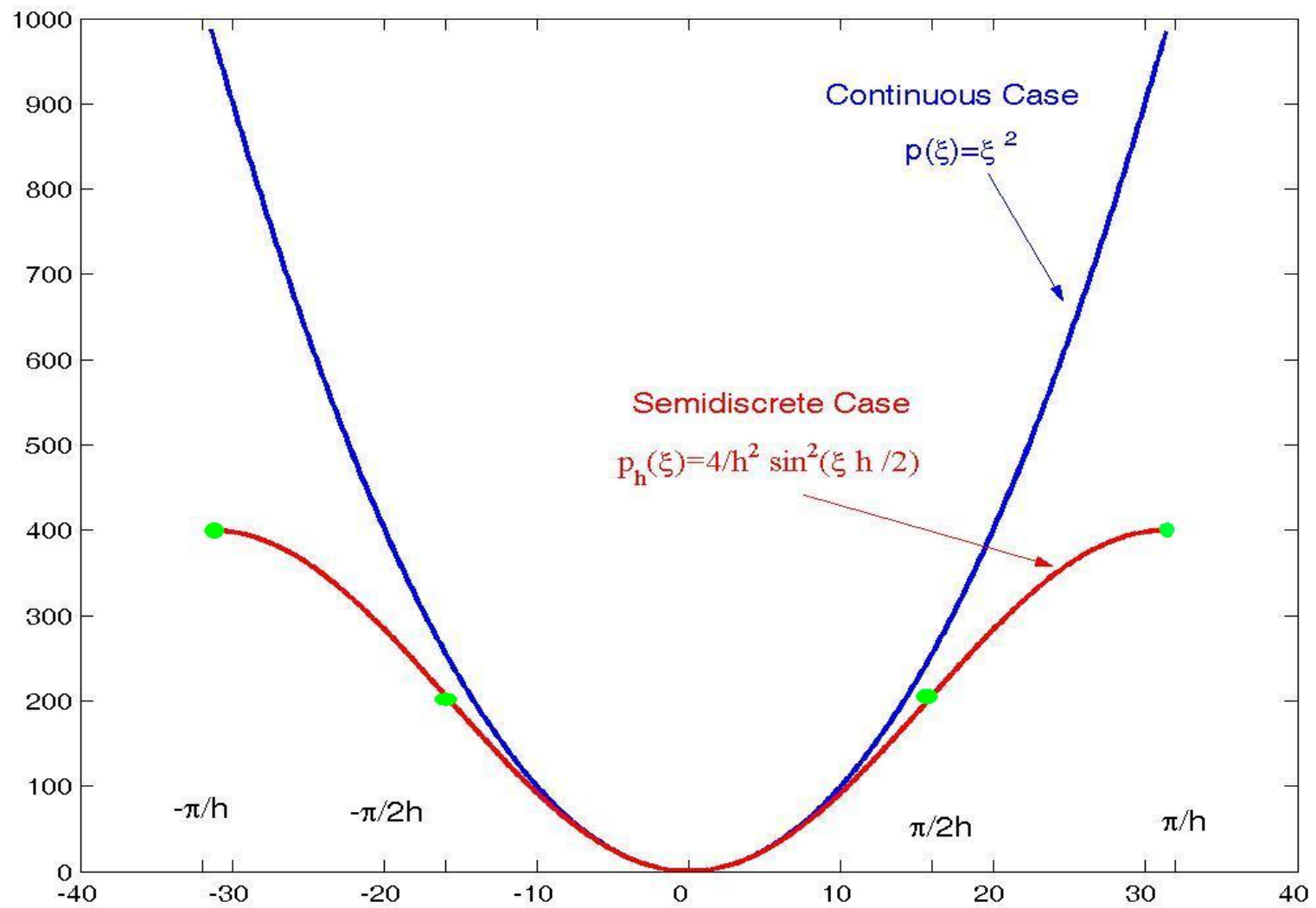
Eliminate the pathologies that are concentrated on the points  $\pm\pi/2h$  and  $\pm\pi/h$  of the spectrum, i. e. replace the complete solution

$$u_j(t) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ijh\xi} e^{ip_h(\xi)t} \varphi(\xi) d\xi, \quad j \in \mathbf{Z}.$$

by the filtered one

$$u_j^*(t) = \frac{1}{2\pi} \int_{-(1-\delta)\pi/2h}^{(1-\delta)\pi/2h} e^{ijh\xi} e^{ip_h(\xi)t} \varphi(\xi) d\xi, \quad j \in \mathbf{Z}.$$

This guarantees the same dispersion properties of the continuous Schrödinger equation to be uniformly (on  $h$ ) true together with the convergence of the filtered numerical scheme.



But **Fourier filtering**:

- Is **computationally expensive**: Compute the complete solution in the numerical mesh, compute its Fourier transform, filter and then go back to the physical space by applying the inverse Fourier transform;
- Is of **little use in nonlinear problems**.

**Other more efficient methods?**

## A VISCOUS FINITE-DIFFERENCE SCHEME

Consider:

$$\begin{cases} i\frac{du^h}{dt} + \Delta_h u^h = ia(h)\Delta_h u^h, & t > 0, \\ u^h(0) = \varphi^h, \end{cases} \quad (8)$$

where the numerical viscosity parameter  $a(h) > 0$  is such that

$$a(h) \rightarrow 0$$

as  $h \rightarrow 0$ .

This condition guarantess the consistency with the LSE.

This scheme generates a *dissipative semigroup*  $S_+^h(t)$ , for  $t > 0$ :

$$\|u(t)\|_{\ell^2}^2 = \|\varphi\|_{\ell^2}^2 - 2a(h) \int_0^t \|u(\tau)\|_{\tilde{h}^1}^2 d\tau.$$

Two dynamical systems are mixed in this scheme:

- the *purely conservative* one,  $i \frac{du^h}{dt} + \Delta_h u^h = 0$ ,
- the *heat equation*  $u_t^h - a(h) \Delta_h u^h = 0$  with viscosity  $a(h)$ .

- Viscous regularization is a typical mechanism to improve convergence of numerical schemes: (hyperbolic conservation laws and shocks).
- It is natural also from a mechanical point of view: elasticity  $\rightarrow$  viscoelasticity.

The main dispersive properties of this scheme are as follows:

**Theorem 4** ( *$L^p$ -decay*) *Let us fix  $p \in [2, \infty]$  and  $\alpha \in (1/2, 1]$ . Then for*

$$a(h) = h^{2-1/\alpha},$$

*$S_{\pm}^h(t)$  maps continuously  $l_h^{p'}(\mathbf{Z})$  to  $l_h^p(\mathbf{Z})$  and there exists some positive constants  $c(p)$  such that*

$$\|S_{\pm}^h(t)\varphi^h\|_{l_h^p(\mathbf{Z})} \leq c(p)(|t|^{-\alpha(1-\frac{2}{p})} + |t|^{-\frac{1}{2}(1-\frac{2}{p})})\|\varphi^h\|_{l_h^{p'}(\mathbf{Z})} \quad (9)$$

*holds for all  $|t| \neq 0$ ,  $\varphi \in l_h^{p'}(\mathbf{Z})$  and  $h > 0$ .*

**Theorem 5 (Smoothing)** *Let  $q \in [2\alpha, 2]$  and  $s \in [0, 1/2\alpha - 1/q]$ . Then for any bounded interval  $I$  and  $\psi \in C_c^\infty(\mathbf{R})$  there exists a constant  $C(I, \psi, q, s)$  such that*

$$\left| \psi E^h u^h(t) \right|_{L^2(I; H^s(\mathbf{R}))} \leq C(I, \psi, q, s) \left| \varphi^h \right|_{l_h^q(\mathbf{Z})}. \quad (10)$$

*for all  $\varphi^h \in l_h^q(\mathbf{Z})$  and all  $h < 1$ .*

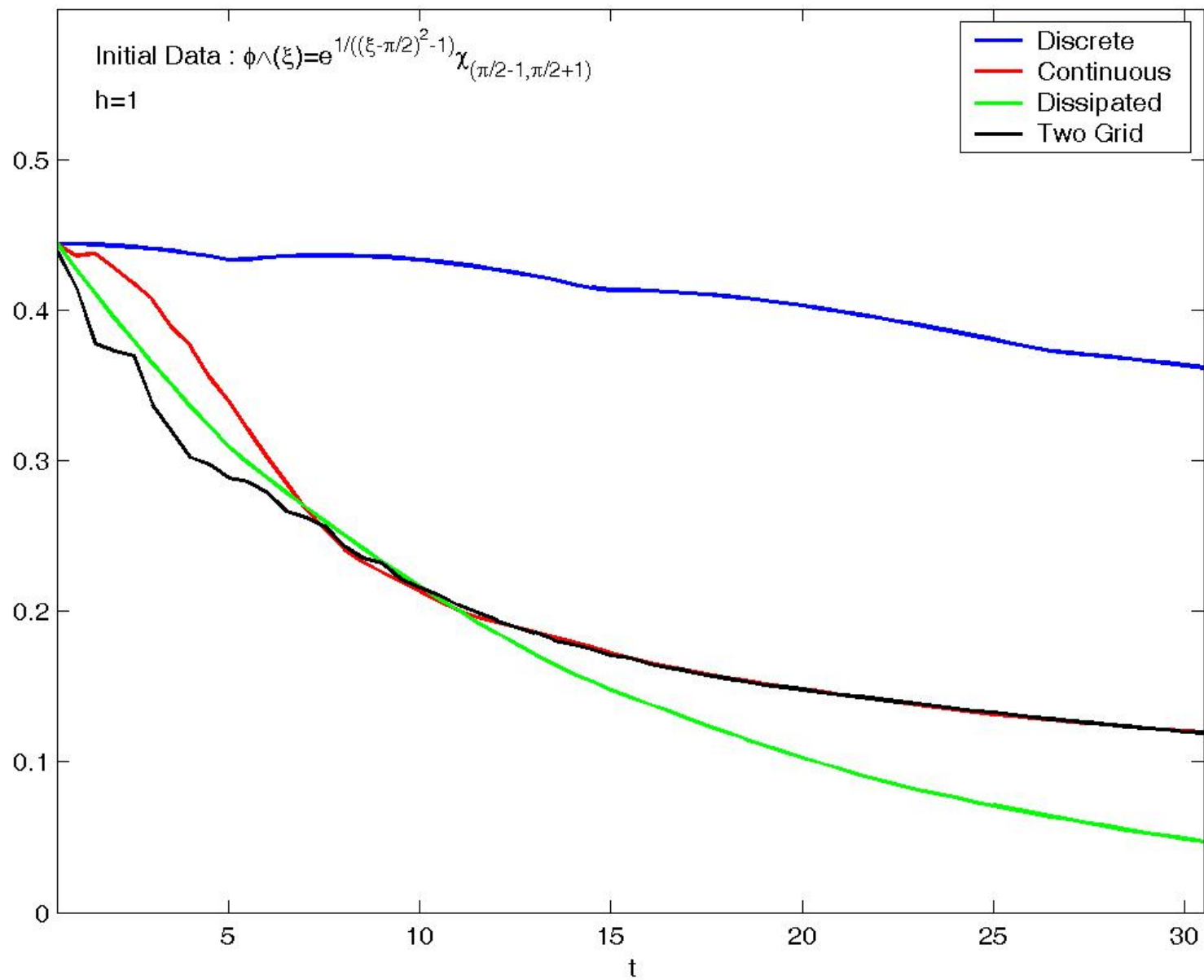
For  $q = 2$ ,  $s = \frac{1}{2} \left( \frac{1}{\alpha} - 1 \right)$ . Adding numerical viscosity at a suitable scale we can reach the  $H^s$ -regularization for all  $s < 1/2$ , but not for the optimal case  $s = 1/2$ . This will be a limitation to deal with critical nonlinearities. Indeed, when  $\alpha = 1/2$ ,  $a(h) = 1$  and the scheme is no longer an approximation of the Schrödinger equation itself.

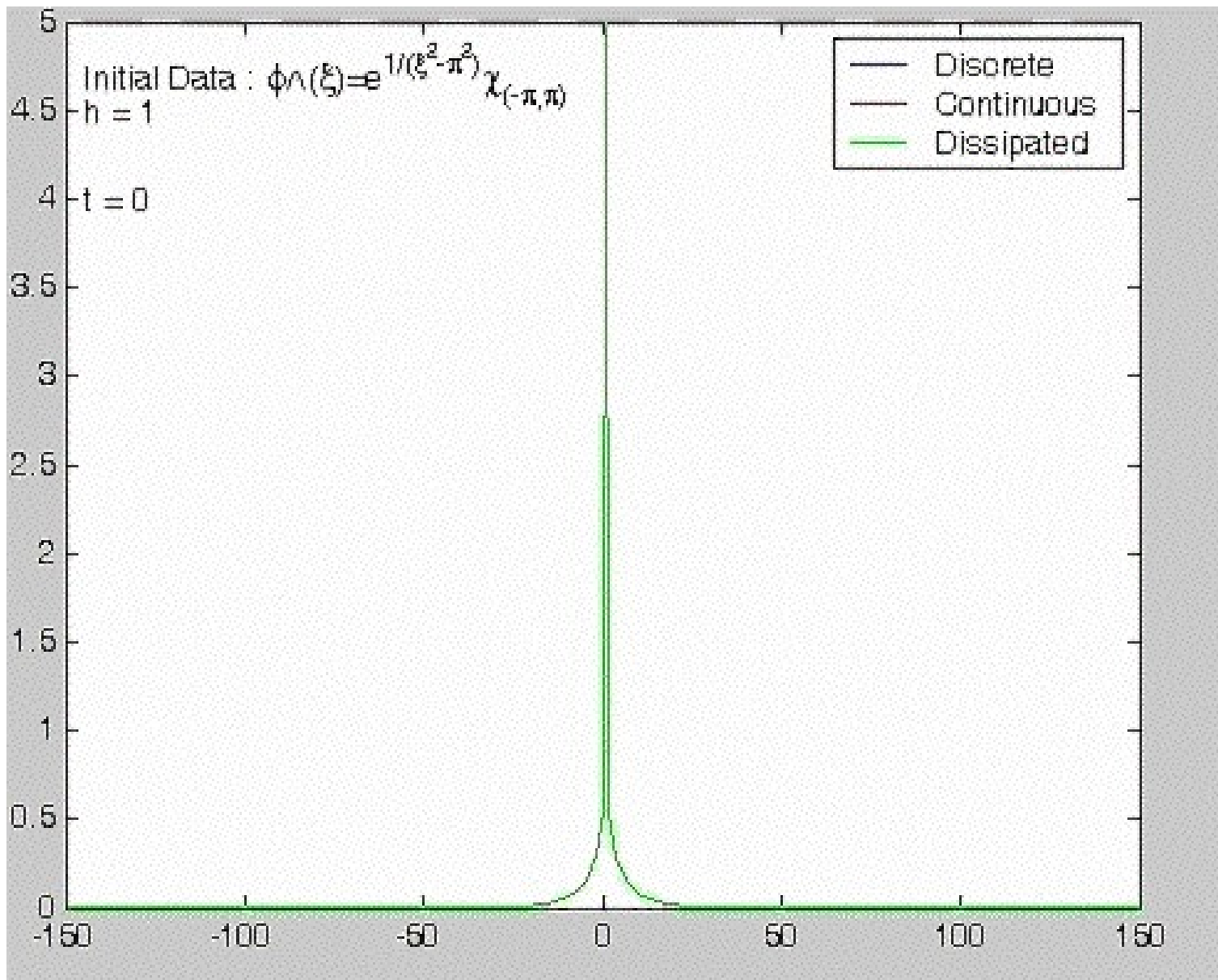


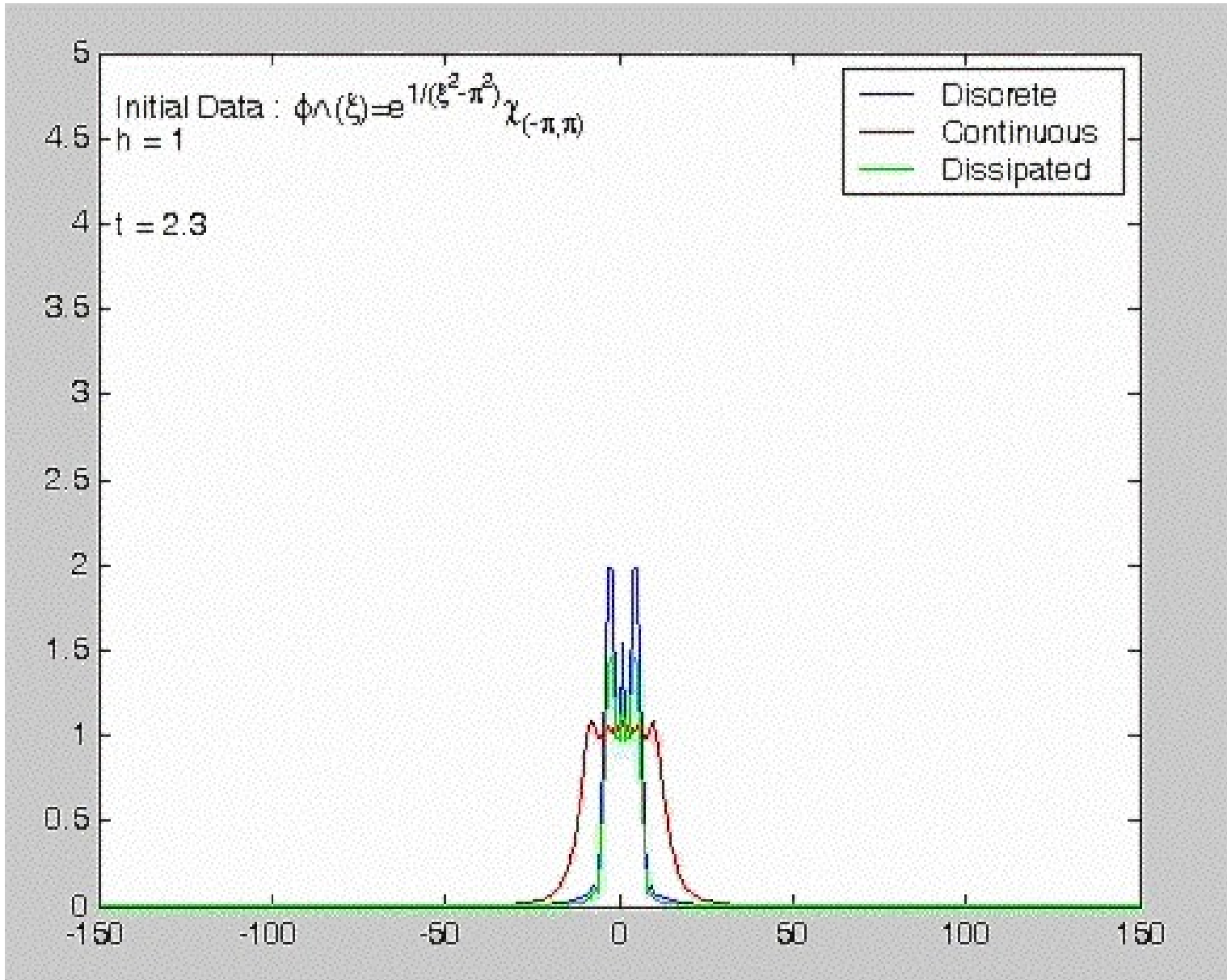
## Sketch of the proof:

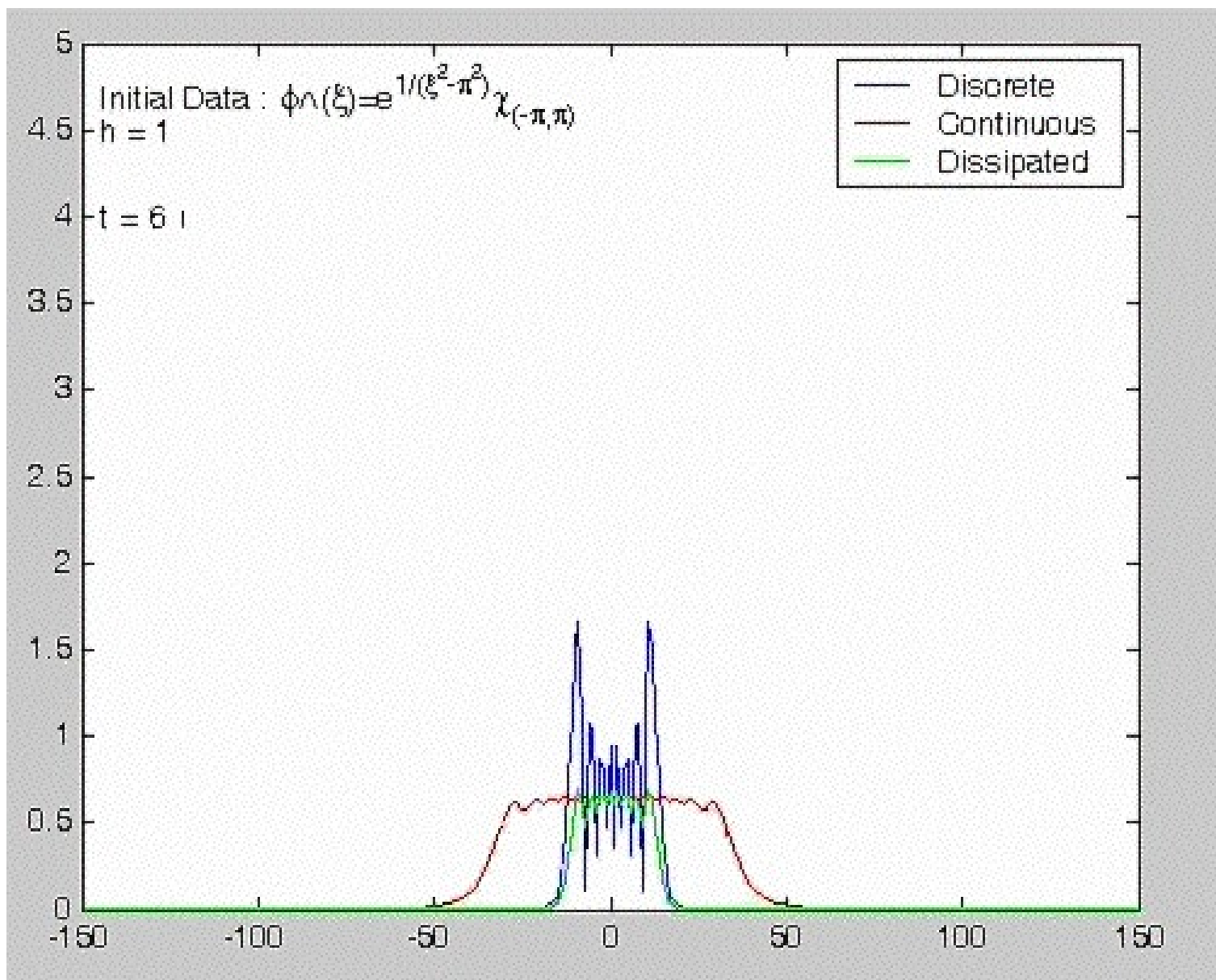
- Solutions are obtained as an iterated convolution of a discrete Schrödinger Kernel and a parabolic one. The heat kernel kills the high frequencies, while for the low ones the discrete Schrödinger kernel behaves very much the same as the continuous one.
- At a technical level, the proof combines the methods of Harmonic Analysis for continuous dispersive and sharp estimates of Bessel functions arising in the explicit form of the discrete heat kernel (Kenig-Ponce-Vega, Barceló-Córdoba,...).

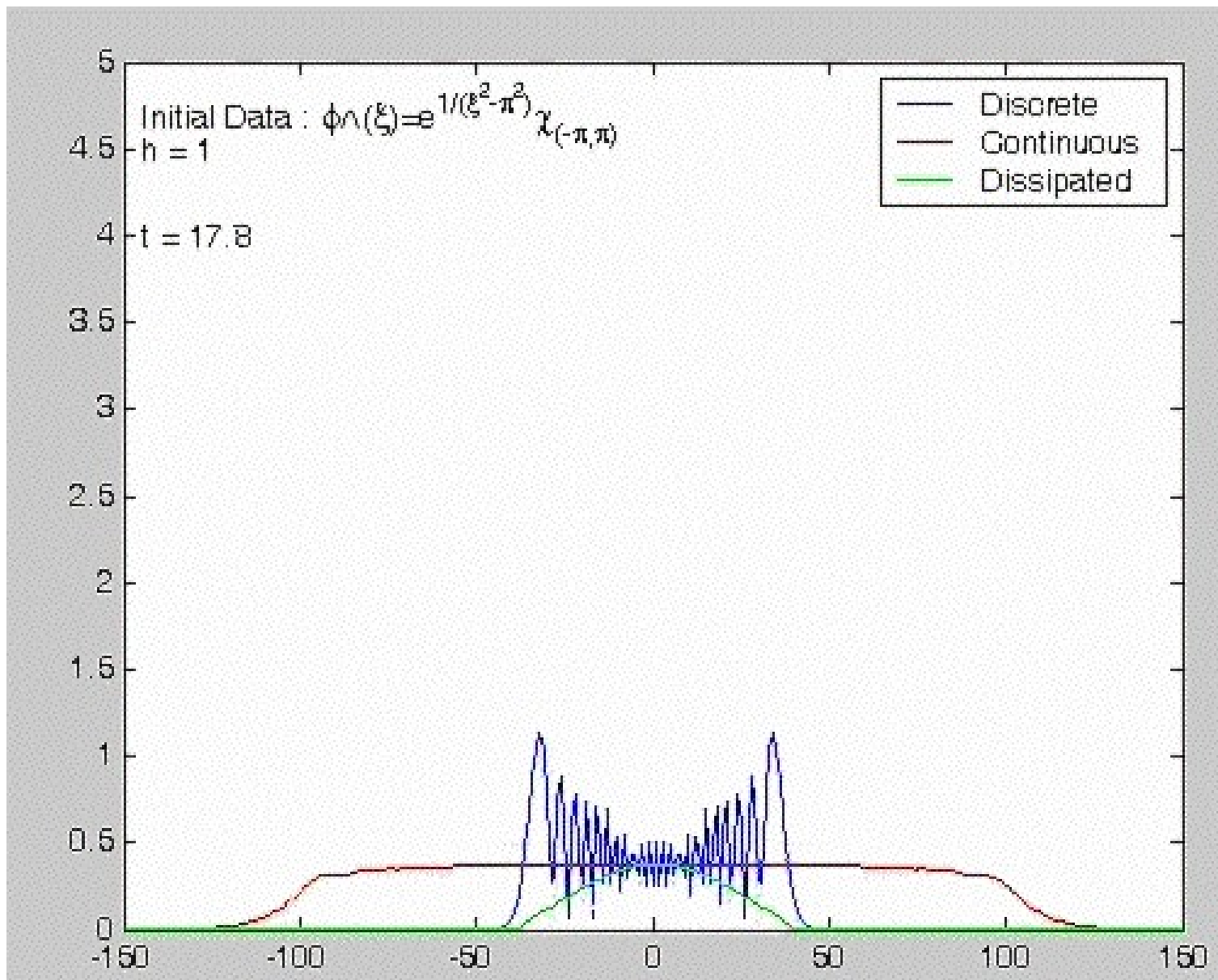
$L^\infty$  norm decay











## NUMERICAL APPROXIMATION OF THE NLSE

The lack of dispersive properties of the conservative linear scheme indicates it is hard to use for solving nonlinear problems. But, in fact, explicit travelling wave solutions for

$$i \frac{du^h}{dt} + \Delta_h u^h = |u_j^h|^2 (u_{j+1}^h + u_{j-1}^h),$$

show that this nonlinear discrete model does not have any further integrability property (uniformly on  $h$ ) other than the trivial  $L^2$ -estimate (M. J. Ablowitz & J. F. Ladik, J. Math. Phys., 1975.)

Consider now the NSE:

$$\begin{cases} iu_t + u_{xx} = |u|^p u & x \in \mathbf{R}, t > 0, \\ u(0, x) = \varphi(x) & x \in \mathbf{R}. \end{cases} \quad (11)$$

According to Tsutsumi's result (1987) the equation is well-posed in  $C(\mathbf{R}, L^2(\mathbf{R})) \cap L_{loc}^q(L^{p+2})$  with  $q = 4(p+1)/p$  for  $0 \leq p < 4$  and  $\varphi \in L^2(\mathbf{R})$ .

Consider now the semi-discretization:

$$\begin{cases} i \frac{du^h}{dt} + \Delta_h u^h = ia(h) \Delta_h u^h + |u^h|^p u^h, & t > 0 \\ u^h(0) = \varphi^h \end{cases} \quad (12)$$

with  $a(h) = h^{2 - \frac{1}{\alpha(h)}}$  such that

$$\alpha(h) \downarrow 1/2, \quad a(h) \rightarrow 0 \text{ as } h \downarrow 0.$$



Then:

- The viscous semi-discrete nonlinear Schrödinger equation is **globally in time well-posed**;
- The solutions of the semi-discrete system **converge** to those of the continuous Schrödinger equation as  $h \rightarrow 0$ .

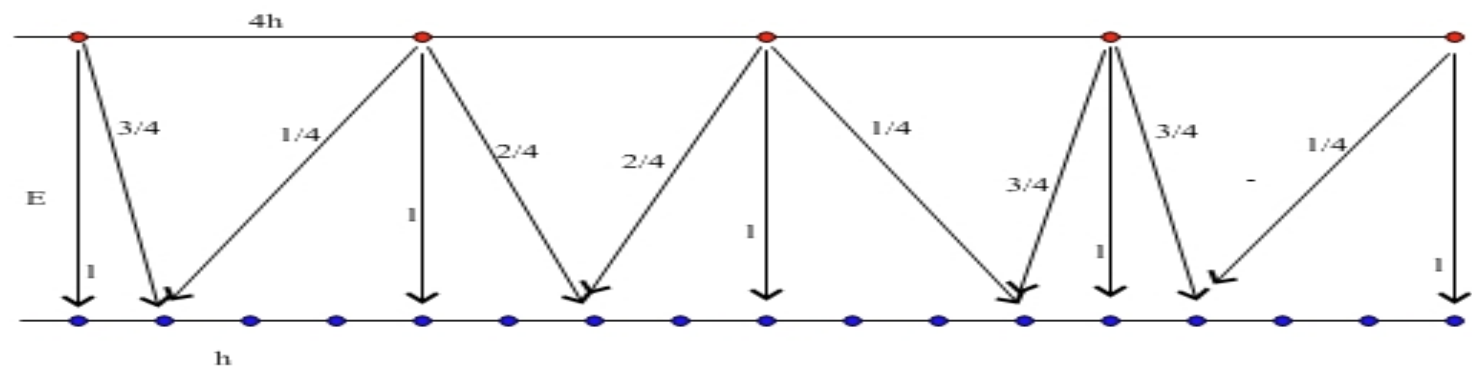
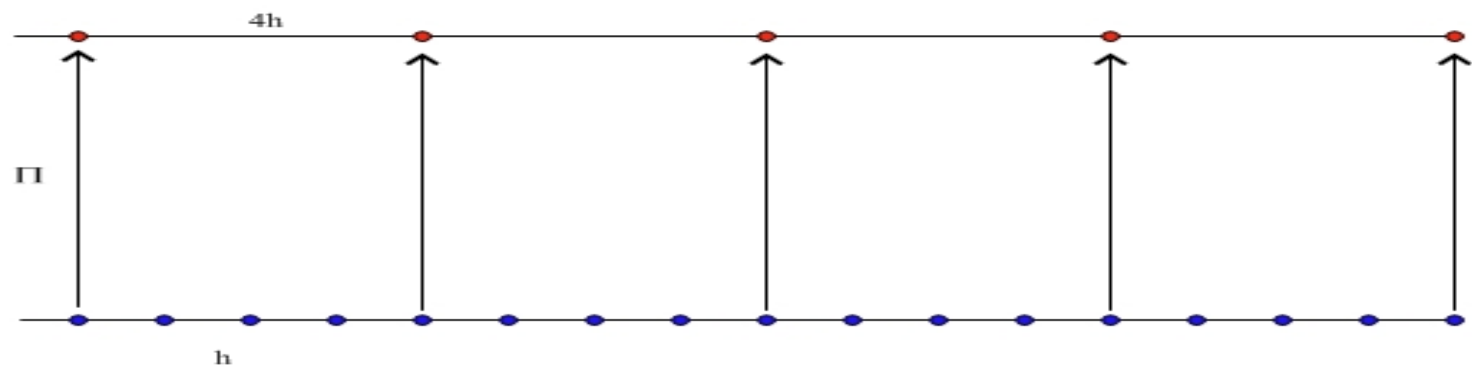
## A TWO-GRID ALGORITHM $\equiv$ A CONSERVATIVE SCHEME

Inspired on the method introduced by R. Glowinski (J. Compt. Phys., 1992) for the numerical approximation of controls for wave equations.

The idea: To work on the grid of mesh-size  $h$  with slowly oscillating data interpolated from a coarser grid of size  $4h$ . The ratio  $1/2$  of meshes does not suffice!

The space of discrete functions on the coarse mesh  $4h\mathbf{Z}$  can be embedded into the fine one  $h\mathbf{Z}$  as follows: Apply on  $\mathbb{C}_4^{h\mathbf{Z}} = \{\psi \in \mathbb{C}^{h\mathbf{Z}} : \text{supp } \psi \subset 4h\mathbf{Z}\}$ , the extension operator  $E$ :

$$(E\psi)((4j+r)h) = \frac{4-r}{4}\psi(4jh) + \frac{r}{4}\psi((4j+4)h), \quad \forall j \in \mathbf{Z}, r = \overline{0, 3}, \psi \in \mathbb{C}_4^{h\mathbf{Z}}.$$



Let  $V_4^h$  be the space of slowly oscillating sequences (SOS) on the fine grid

$$V_4^h = \{E\psi : \psi \in C_4^{h\mathbf{Z}}\},$$

and the projection operator  $\Pi : \mathbb{C}^{h\mathbf{Z}} \rightarrow \mathbb{C}_4^{h\mathbf{Z}}$  :

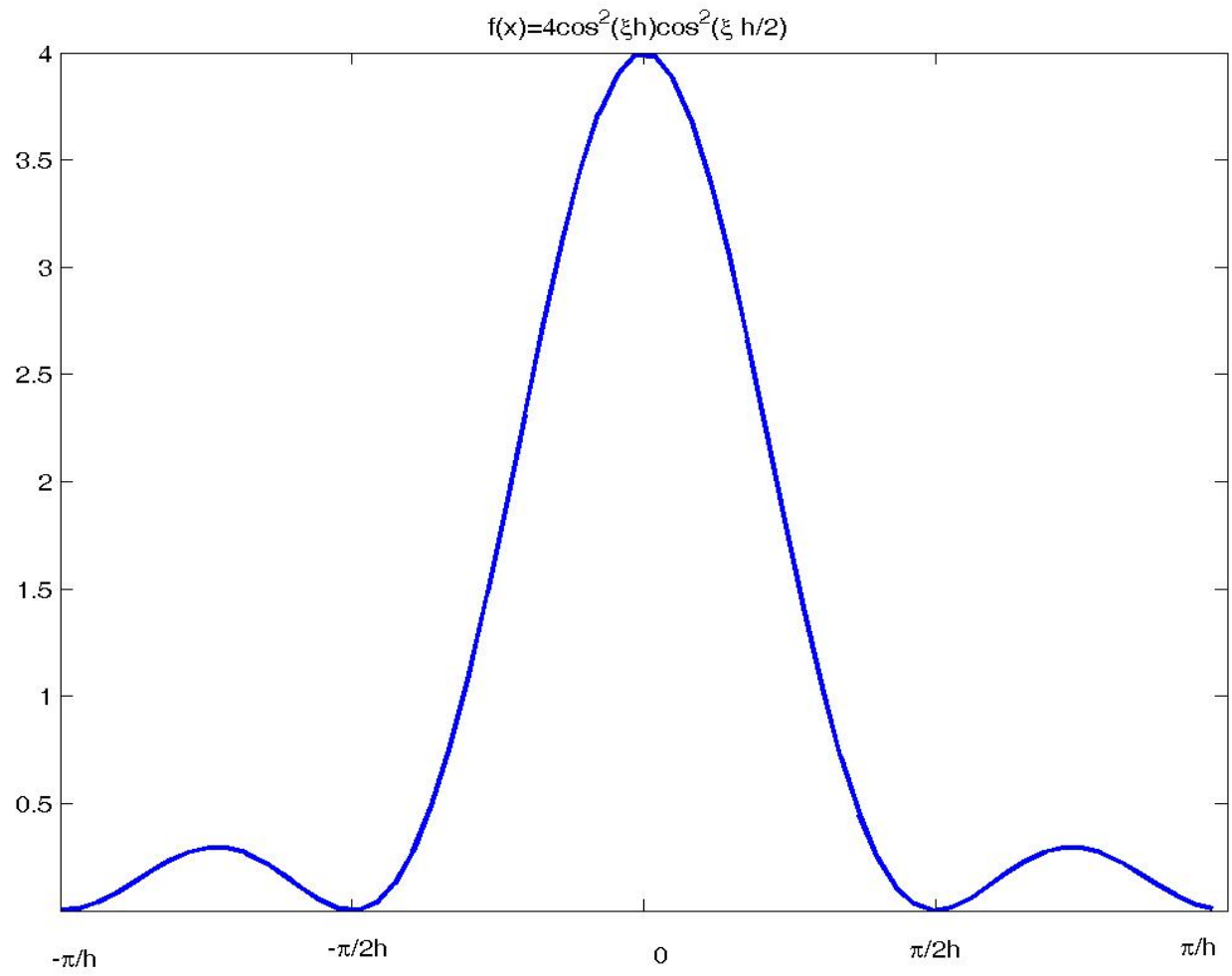
$$(\Pi\phi)((4j + r)h) = \phi((4j + r)h)\delta_{4r}, \forall j \in \mathbf{Z}, r = \overline{0, 3}, \phi \in \mathbb{C}^{h\mathbf{Z}}.$$

We now define the smoothing operator

$$\tilde{\Pi} = E\Pi : \mathbb{C}^{h\mathbf{Z}} \rightarrow V_4^h,$$

which acts as a filtering, associating to each sequence on the fine grid a slowly oscillating sequence. The discrete Fourier transform of a slowly oscillating sequence SOS is as follows:

$$\widehat{\tilde{\Pi}\phi}(\xi) = 4 \cos^2(\xi h) \cos^2(\xi h/2) \widehat{\Pi\phi}(\xi).$$



The semi-discrete Schrödinger semigroup when acting on SOS has the same properties as the continuous Schrödinger equation:

**Theorem 6** *i) For  $p \geq 2$ ,*

$$\left| e^{it\Delta_h} \tilde{\Pi} \varphi \right|_{l^p(h\mathbf{Z})} \lesssim |t|^{-1/2(1/p' - 1/p)} \left| \tilde{\Pi} \varphi \right|_{l^{p'}(h\mathbf{Z})}.$$

*ii) Furthermore, for every admissible pair  $(q, r)$ ,*

$$\left| e^{it\Delta_h} \tilde{\Pi} \varphi \right|_{L^q(\mathbb{R}, l^r(h\mathbf{Z}))} \lesssim \left| \tilde{\Pi} \varphi \right|_{l^2(h\mathbf{Z})}.$$

**Sketch of the Proof.** By scaling, we can assume that  $h = 1$ . We write  $T(t)$  as a convolution operator  $T(t)\psi = K^t * \psi$  where

$$\widehat{K^t}(\xi) = 4e^{-4it \sin^2 \xi/2} \cos^2 \xi \cos^2(\xi/2).$$

We need

$$\left| K^t \right|_{l^\infty(\mathbf{Z})} \lesssim 1/\sqrt{t}.$$

The fact that  $(4 \sin^2(\xi/2))'' = 2 \cos(\xi)$  allows applying the sharp results by **Kenig-Ponce-Vega** and **Keel-Tao** to derive the desired decay.

SOS vanish at the spectral points  $\pm\pi/2h$  implies gain of integrability.

This is consistent with the previous analysis of the viscosity method.

Concerning the **local smoothing** properties we can prove that

**Theorem 7** *Let  $r \in (1, 2]$ . Then*

$$\sup_{j \in \mathbf{Z}} \int_{-\infty}^{\infty} \left| (D^{1-1/r} e^{it\Delta_h} \tilde{\Pi} f)_j \right|^2 dt \lesssim \left| \tilde{\Pi} f \right|_{l^r(h\mathbf{Z})}^2 \quad (13)$$

for all  $f \in l^r(h\mathbf{Z})$ , uniformly in  $h > 0$ .

**Sketch of the Proof.** Applying results by Kenig-Ponce-Vega we have to  $T_1$  we get

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |(T_1(t)\varphi)(x)|^2 dt \lesssim \int_{-\pi}^{\pi} \frac{|\hat{f}(\xi)|^2 \cos^4 \xi \cos^4(\xi/2)}{|\sin \xi|} d\xi.$$

Then, using the fact that  $\cos^4 \xi \cos^4(\xi/2)$  vanishes at  $\xi = \pm\pi$ , we can compensate the singularity of  $\sin(\xi)$  in the denominator and guarantee



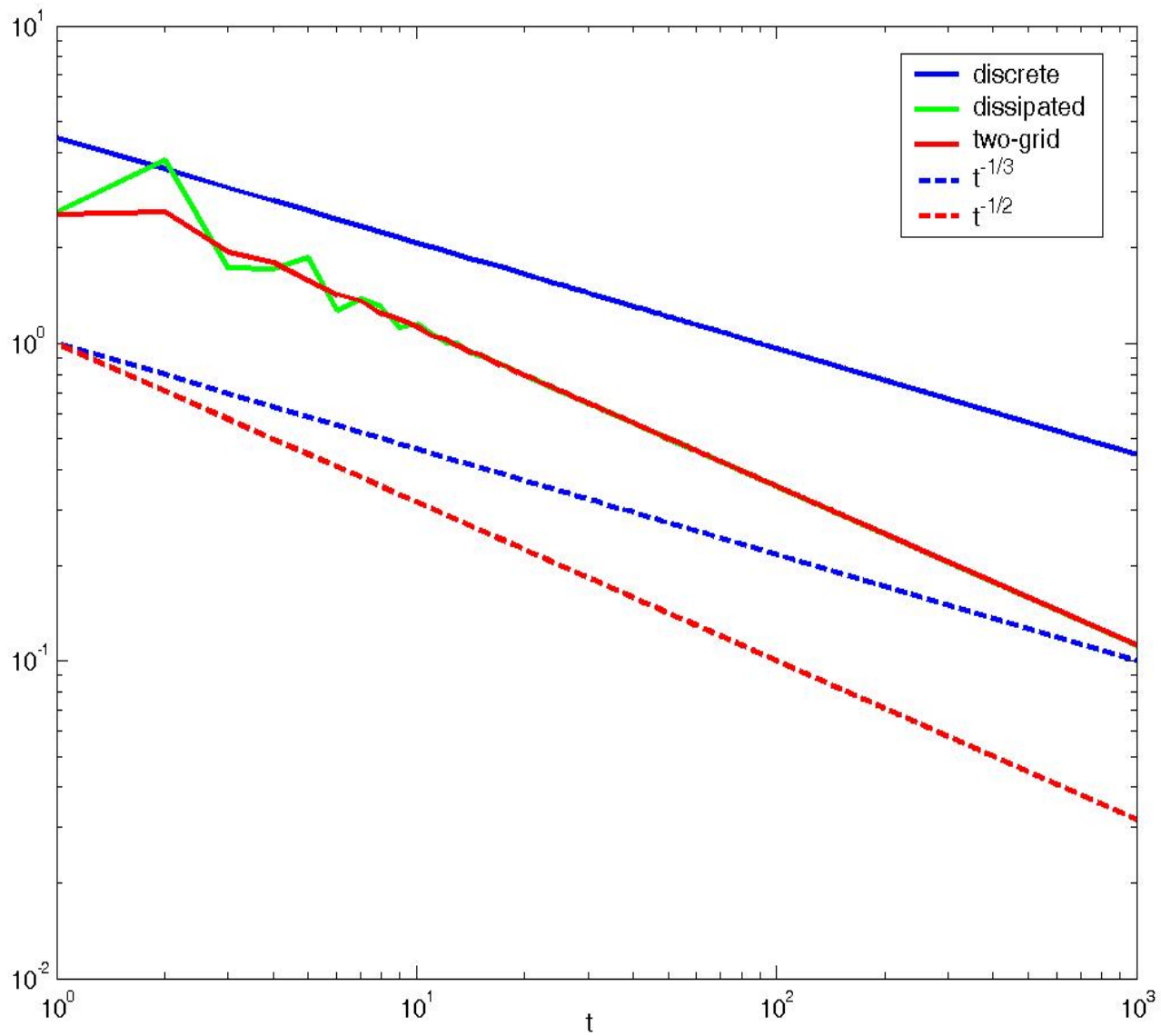
that

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |(T_1(t)\varphi)(x)|^2 dt \lesssim \int_{-\pi}^{\pi} \frac{|\hat{f}(\xi)|^2}{|\xi|} d\xi \lesssim \left| D^{-1/2} f \right|_{L^2(\mathbb{R})}^2.$$

SOS vanish at the spectral points  $= \pm\pi$ , implies gain of local regularity.

This is also consistent with the results obtained by means of the viscosity method.

Log-log plot of the temporal evolution of the  $l^\infty$  norm of the fundamental solutions



## A TWO-GRID CONSERVATIVE APPROXIMATION OF THE NLSE

Consider the semi-discretization

$$i \frac{du^h}{dt} + \Delta_h u^h = \tilde{\Pi} \left[ |\tilde{\Pi}^*(u^h)|^p \tilde{\Pi}^*(u^h) \right], \quad t \in \mathbb{R}; \quad u^h(0) = \tilde{\Pi} \varphi^h, \quad (14)$$

with  $0 < p < 4$ .

By using the two-grid filtering operator  $\tilde{\Pi}$  both in the nonlinearity and on the initial data we guarantee that the corresponding trajectories enjoy the properties above of **gain of local regularity and integrability**.

But to prove the stability of the scheme we need to guarantee the **conservation of the  $l^2(h\mathbf{Z})$  norm** of solutions, a property that the

solutions of NLSE satisfy. For that the nonlinear term  $f(u^h)$  has to be chosen such that

$$(\tilde{\Pi} f(u^h), u^h)_{l^2(h\mathbf{Z})} \in \mathbb{R}.$$

This property is guaranteed with the choice

$$f(u^h) = |\tilde{\Pi}^*(u^h)|^p \tilde{\Pi}^*(u^h)$$

i.e.

$$(f(u^h))_{4j} = g\left((u_{4j}^h + \sum_{r=1}^3 \frac{4-r}{4}(u_{4j+r}^h + u_{4j-r}^h))/4\right); \quad g(s) = |s|^p s.$$

The same arguments as in the viscosity method allow showing that the solutions of the two-grid numerical scheme converge as  $h \rightarrow 0$  to the solutions of the continuous NLSE.

## TWO GOOD NEWS:

- Lecture is ending.... ;
- Things improve when we also discretize in time.

Time discretization  $\sim$  time upwind  $\sim$  time viscosity  $\sim$  space-like viscosity.

L. Ignat, M3AS, to appear.

## CONCLUSIONS:

- FOURIER FILTERING (AND SOME OTHER VARIANTS LIKE NUMERICAL VISCOSITY, AND TWO-GRID FILTERING, ...) ALLOW BUILDING NUMERICAL SCHEMES FOR AN EFFICIENT APPROXIMATION OF LINEAR AND NONLINEAR SCHRÖDINGER EQUATIONS.
- THESE NEW SCHEMES ALLOW CAPTURING THE RIGHT DISPERSION PROPERTIES OF THE CONTINUOUS MODELS AND CONSEQUENTLY PROVIDE CONVERGENT APPROXIMATIONS FOR NONLINEAR EQUATIONS TOO.
- IN PRACTICE THE TWO-GRID METHOD IS EASIER TO APPLY. IT MAY ALSO BE EASIER TO ADAPT TO GENERAL NON-REGULAR MESHES.

- THE METHODS DEVELOPED IN THIS CONTEXT ARE STRONGLY INSPIRED ON OUR PREVIOUS WORK ON THE NUMERICAL APPROXIMATION OF CONTROLS FOR WAVE EQUATIONS.
- MUCH REMAINS TO BE DONE IN ORDER TO DEVELOP A COMPLETE THEORY (MULTIDIMENSIONAL PROBLEMS, BOUNDARY-VALUE PROBLEMS, NONREGULAR MESHES, OTHER PDE'S,...)
- A COMPLETE THEORY SHOULD COMBINE FINE HARMONIC ANALYSIS, NUMERICAL ANALYSIS AND PDE THEORY.

- THE SAME IDEAS SHOULD BE USEFUL TO DEAL WITH OTHER ISSUES SUCH AS TRANSPARENT BOUNDARY CONDITIONS, SCATTERING PROBLEMS, PML, INVERSE PROBLEMS, ...
- SIMILAR METHODS COULD HELP TO DEAL WITH THE NUMERICAL APPROXIMATION OF BILINEAR CONTROL PROBLEMS AS THOSE ADDRESSED BY K. BEAUCHARD & J. M. CORON.



## Refs.

- E. Z. Propagation, observation, and control of waves approximated by finite difference methods. *SIAM Review*, 47 (2) (2005), 197-243.
- L. IGNAT & E. Z., *Dispersive Properties of Numerical Schemes for Nonlinear Schrödinger Equations*, Foundations of Computational Mathematics, Santander 2005, London Mathematical Society Lecture Notes, 331, L. M. Pardo et al. eds, Cambridge University Press, 2006, pp. 181-207.