Optimal placement of sensors, actuators and dampers for waves

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Motivation

Often engineering and industrial processes as well as other areas of applications require designing controllers, actuators, sensors, dampers, and also often times the perturbations and signals to be faced are of wave-like nature:

- Noise reduction in cavities and vehicles.
- Laser control in Quantum mechanical and molecular systems.
- Seismic waves, earthquakes.
- Flexible structures.
- Environment: the Thames barrier.
- Optimal shape design in aeronautics.
- Human cardiovascular system: the bypass
- ....

The mathematical theory needed to understand these issues combines:

- Hyperbolic PDEs
- Control Theory
- Optimal Design
- Spectral analysis
- Microlocal analysis
- Numerical analysis
- ...

In this talk we aim to present some toy models and problems, together with some key results and research perspectives.

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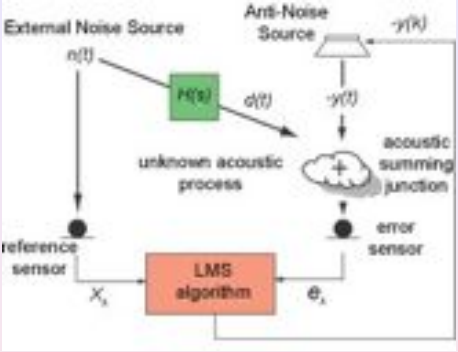
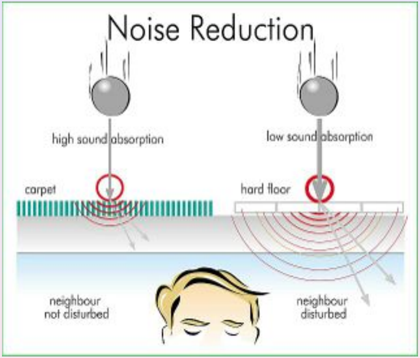
Outline





Acoustic noise reduction
Active versus passive controllers.

Outline

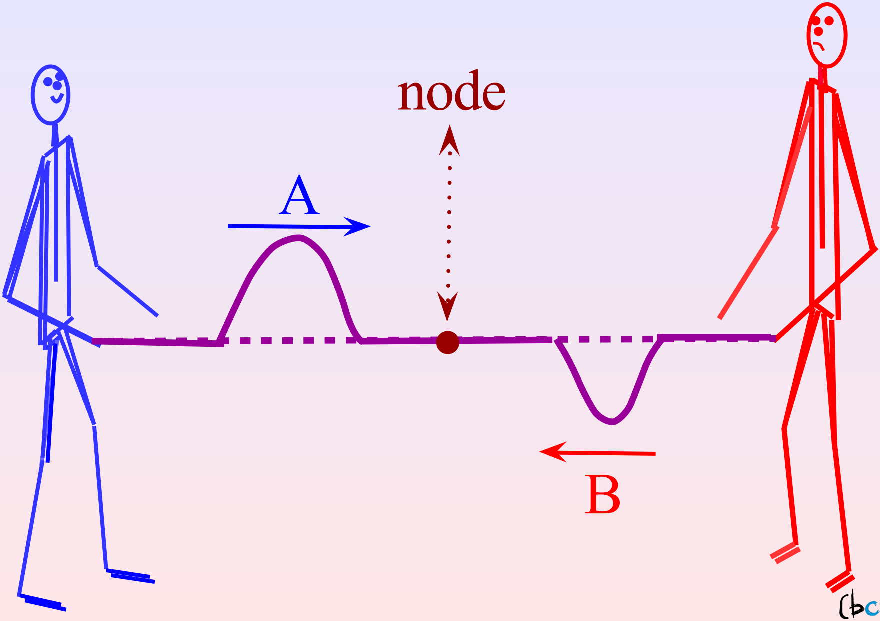
Control of $1-d$ vibrations of a string

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x), \; y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.
The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$


node
A
B

The dual observation problem

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\, 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\, \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

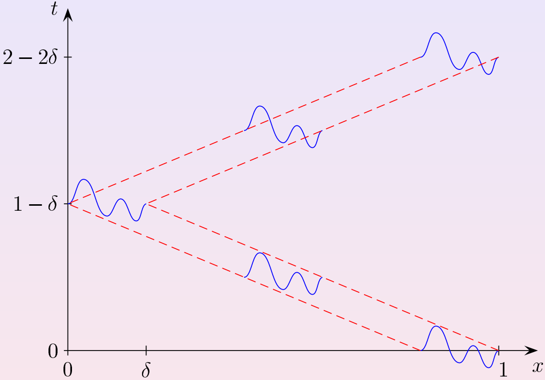
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2 \right] dx = E(0), \;\; \forall 0 \leq t \leq T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \leq C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \geq 2$.



$$E(0) \leq \int_0^T |\varphi_x(1,t)|^2 dt$$

Wave localized at $t = 0$ near the extreme $x = 1$ propagating with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

Construction of the Control

Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x(1, t),$$

where φ is the solution of the adjoint system corresponding to initial data $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1}\times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.
Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $\varphi_x(1,t) \in L^2(0,T)$ (hidden regularity). Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY. [1]

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[1] Norbert Wiener (1894–1964) defined Cybernetics as the science of control and communication in animals and machines

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Outline

Internal stabilization of waves:

Let ω be an open subset of Ω. Consider:

$$\begin{cases} y_{tt} - \Delta y = -y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega, \end{cases}$$

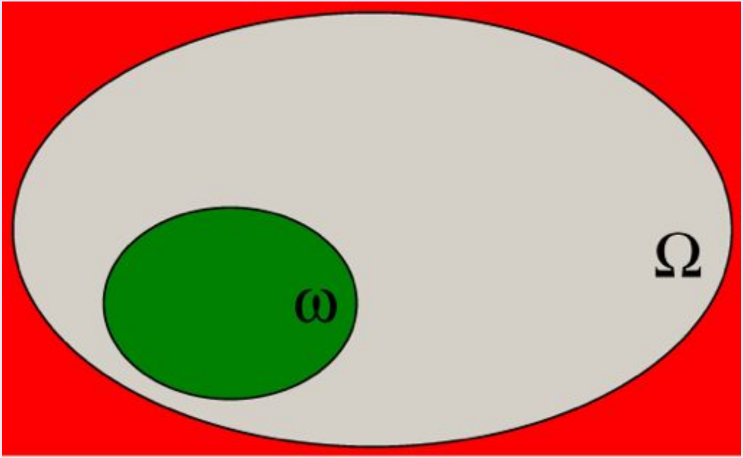
where 1_ω stands for the characteristic function of the subset ω.
The energy dissipation law is then

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx.$$

Question: Do they exist $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0), \quad \forall t \geq 0,$$

for all solution of the dissipative system?


ω
Ω

This is equivalent to an **observability property**: There exists $C > 0$ and $T > 0$ such that

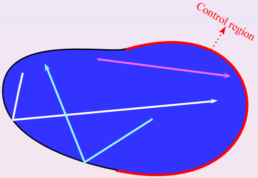
$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dxdt.$$

This estimate, together with the energy dissipation law, shows that

$$E(T) \leq \sigma E(0)$$

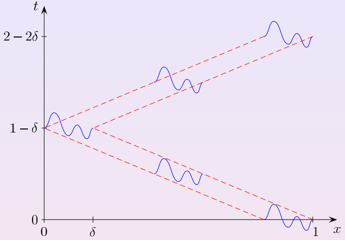
with $0 < \sigma < 1$. Accordingly the semigroup map $S(T)$ is a strict contraction. By the semigroup property one deduces immediately the exponential decay rate.

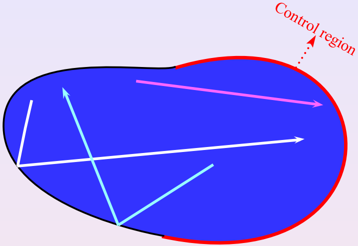
The observability inequality and, accordingly, the exponential decay property **holds if and only if the support of the dissipative mechanism, Γ_0 or ω, satisfies the so called the Geometric Control Condition (GCC)** (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch,...)

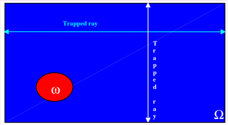


Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.

Qualitative change from $1-d$ to multi-d







A trapped ray scaping the damping region ω makes it impossible the decay rate to be exponential. Each trajectory tends to zero as $t \to \infty$ but the decay can be arbitrarily slow.

Outline

The optimal shape design problem

Optimize ω (or Γ_0) and the damping profile $a = a(x)$ to enhance the exponential decay rate:

$$\begin{cases} y_{tt} - \Delta y = -a(x) y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

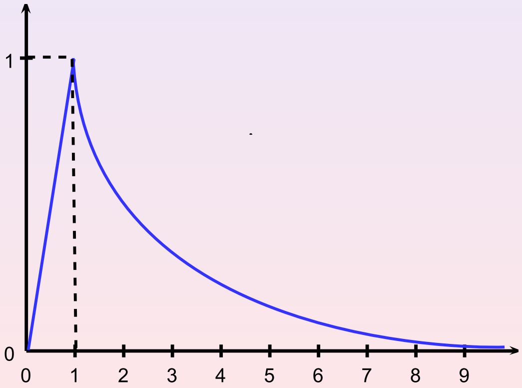
Within the class of ω and $a > 0$ such the exponential decay property holds:

$$E(t) \leq C e^{-\gamma_{a,\omega} t} E(0), \quad \forall t \geq 0$$

$$MAXIMIZE \; : \; (\omega, a) \to \gamma_{a,\omega}.$$

Overdamping!

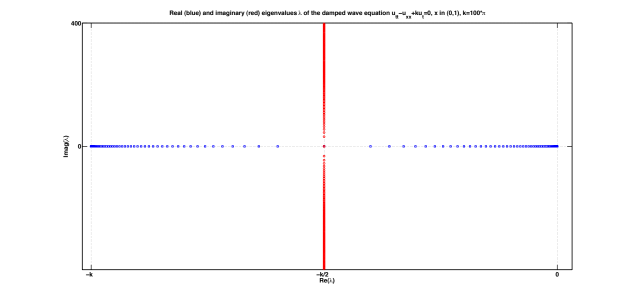
Obviously, the decay rate γ_a depends on the damping potential a. But, against the very first intuition, this map is not monotonic with respect to the size of the damping. A $1-d$ spectral computation for constant coefficients yields:



Some known results: $1-d$

- $1-d$: The exponential decay rate coincides with the spectral abscissa within the class of BV damping potentials. For large eigenvalues $Re(\lambda) \sim -\int_\omega a(x)dx/2$ (S. Cox & E. Z., CPDE, 1993). Thus:

$$\gamma_a \leq \int_\omega a(x)dx.$$

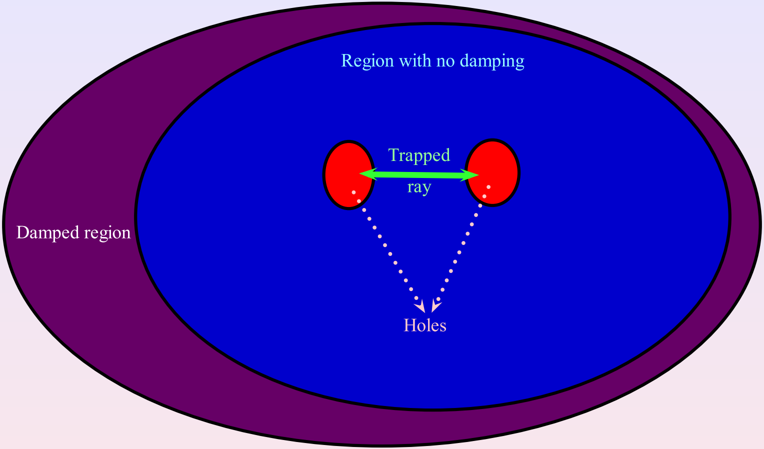


Multi$-d$

In the multidimensional case the situation is even more complex. The decay rate is determined as the minimum of two quantities (G. Lebeau, 1996):

- The spectral abscissa;
- The minimum asymptotic average (as $T \to \infty$) of the damping potential along rays of Geometric Optics.

  The later is in agreement with our intuition of waves traveling along rays of Geometric Optics.



Geometric configuration in which the spectral abscissa does not suffice to capture the decay rate. The decay rate vanishes due to a trapped ray, but the spectrum is uniformly shifted in the left complex half space.

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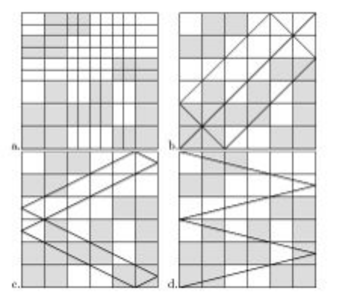
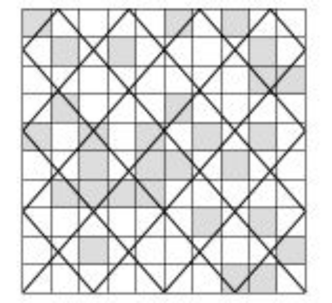
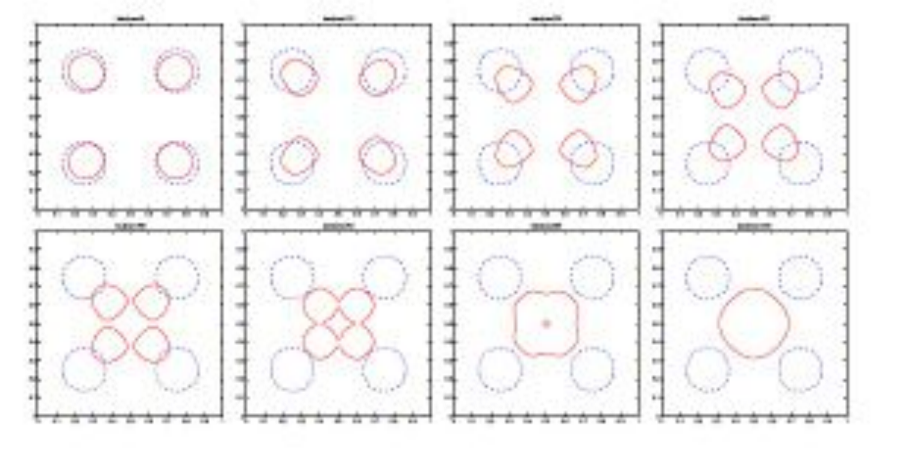
1. Motivation
2. The control of waves
   - An example: noise reduction
   - A toy model
3. Damped waves
   - Geometric conditions for exponential decay
   - Complexity of the decay rate
   - Optimal design
4. Optimal placement of sensors and actuators
   - Problem formulation
   - The spectral problem
   - Main results
5. Numerics
6. Conclusions and Perspectives





Optimal design of the billiard: Hébrard-Humbert, 2003



Optimal location of the sensor for the first eigenfunction by a level set approach: A. Munch, 2005.

Some analytical difficulties:

- There is no variational principle characterizing the decay rate.
- The complex way in which the eigenvalues depend on the damping potentials for low/high/intermediate frequencies, ...
- Different problems! The optimal damping for a given initial datum $\neq$ the optimal damping for all possible solutions...
- This is the case even for constant damping potentials k. The optimal damping for the ℓ-th eigenfunction is $k = 2\sqrt{\mu_\ell}$.

Most problems about the optimal location and distribution of damping to maximize the decay rate are still completely open.

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Outline

Joint work with Y. Privat and E. Trélat

Consider the conservative wave equation:

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ z = 0 & \text{for} \quad x \in \partial\Omega; \quad t \in (0, T) \\ z(x,0) = z^0(x), z_t(x,0) = z^1(x) & \text{in} \quad (0, \pi). \end{cases}$$

- Optimal placement problems are then of variational nature!
- It corresponds to the analysis of the behavior of the damped system as the damping is infinitesimally small.

Observability:

$$||z^0||^2_{L^2(\Omega)} + ||z^1||^2_{H^{-1}(\Omega)} \leq C(\omega, T) \int_0^T \int_\omega z^2 dxdt.$$

Fourier series shows that, in general, Fourier modes are mixed in quite an complicated manner thus making the understanding of these issues complex:

$$z(t) = \sum \hat{z}_k e^{i\sqrt{\lambda_k}t}\phi_k(x).$$

Thus,

$$\int_0^T \int_\omega |z|^2 dxdt = \sum\sum \hat{z}_k \hat{z}_j \int_\omega \phi_k(x)\phi_j(x)dx \int_0^T e^{[i\sqrt{\lambda_k} - i\sqrt{\lambda_j}]t}.$$

Outline

Reduction to a spectral problem

The problem becomes much simpler in several cases:

- The case $T = \infty$. We then look at the

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \int_{\omega} |z|^2 dxdt.$$

- Randomizing initial data and considering the expected observability constant (Zygmund lemma, recent works by N. Burq et al.)
- In 1-d, $\Omega = (0, \pi)$ in which case solutions are 2π-time periodic.

Cross terms vanish and we are led to the following observability problem:

$$\sum |\hat{z}_k|^2 \leq C(\omega) \sum |\hat{z}_k|^2 \int_{\omega} \phi_k^2(x) dx.$$

These issues can be considered, as mentioned above, in two different cases:

- Fixed initial data, and therefore fixed weights $|\hat{z}_k|^2$ in ℓ^1.
- All possible initial data of finite energy. Then, the problem becomes that of finding ω so that the following minimum is maximized:

$$\min_k \int_\omega \phi_k^2(x)dx.$$

Outline

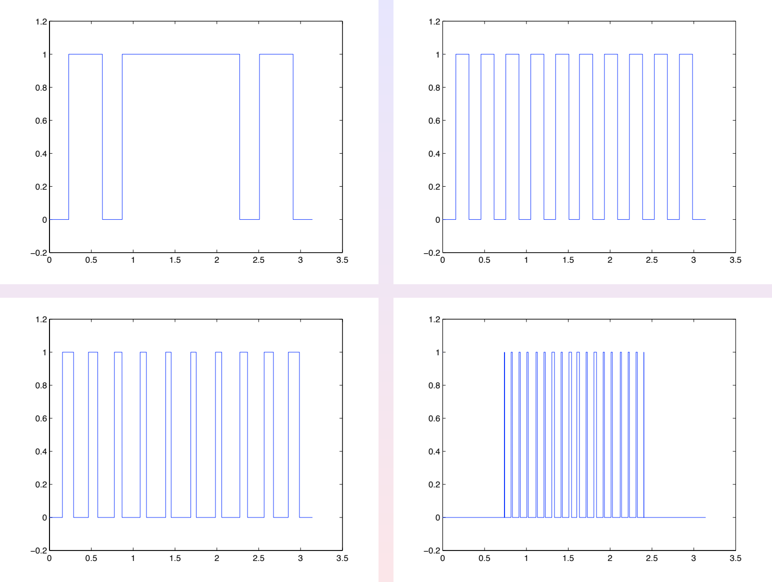
The following results are proved:

1.- *Optimal observation subdomain ω for fixed finite energy initial data (z^0, z^1).*

- For initial data that are analytic (exponential decay of Fourier coefficients), there is a unique minimizer with a finite number of connected components.
- The optimal set always exists but it can be a **Cantor set even for C^∞ smooth data**.

2.- *Spectral problem of optimally observing or controlling all the eigenfunctions uniformly.*

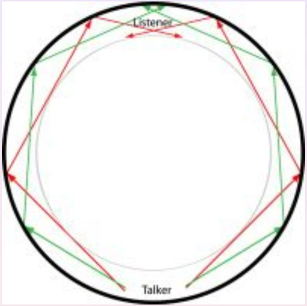
- Relaxation occurs (Hébrart-Henrot, 2005): the optimum is achieved by a density function $\rho(x)$ so that $\int_0^\pi \rho(x)dx = L$ and not by a measurable set with bang-bang densities (except for $L = \pi/2$).
- The infima coincide for both the relaxed and the classical problem (in the multi$-d$ case under the QUANTUM UNIQUE ERGODICITY (QUE) condition: $\phi_j^2 \rightharpoonup 1/|\Omega|$ as $j \to \infty$.)
- Spillover occurs $(1-d)$: The optimal design for the first N Fourier modes is the worst choice for the $N+1$-th one.

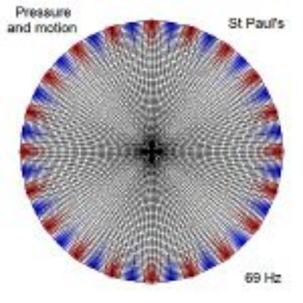


Simulations performed using AMPL + IPOPT

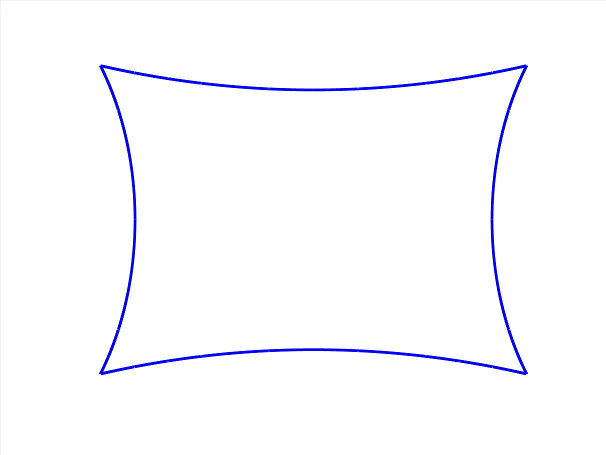
In the multi-dimensional case the problem is much more complex. Spectra do not behave according to our $1-d$ intuition

$$\sin^2(kx) \rightharpoonup 1/2 \quad \text{as} \quad k \to \infty.$$

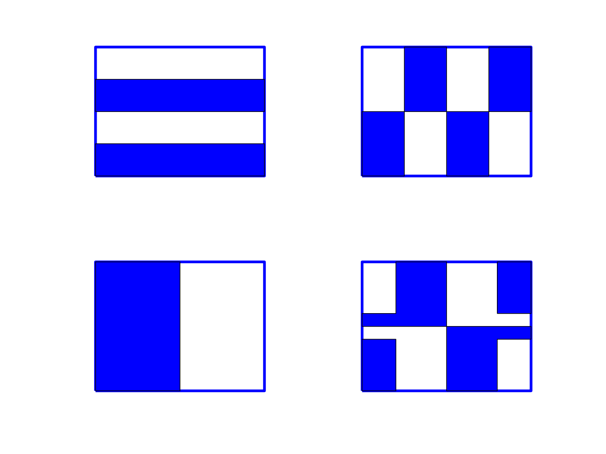




Our main contribution is to put in evidence the intimate relations between optimal design problems and ergodicity properties of the domain.



Domain conjectured to satisfy the QUE in A.H. Barnett, Asymptotic rate of quantum ergodicity in chaotic Euclidean billiards, Comm. Pure Appl. Math. 59 (2006), 1457Ð1488.



Optimal designs in $2-d$ for the square with volume fraction $1/2$ in the spectral criterium. This is an exceptional case where classical optimal domains exist.

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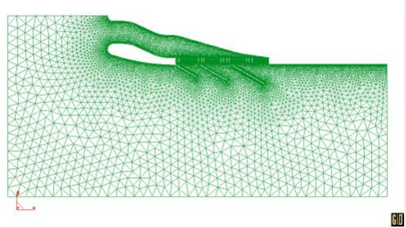
Joint work with A. Marica

- In practice, the actual computation of controllers, sensors and dampers needs to be done through numerical simulations.
- We describe the pathological behavior that numerical rays may present with respect to the continuous ones, due to the heterogeneity of the grids.
- Group velocity changes for high frequency numerical solutions.
- But, moreover, orientation of rays changes too due to the change of metric that the heterogeneity of the grid induces.



Problem formulation

The wave equation with variable coefficients on $\mathbb{R}$:

$$\rho(y)u_{tt} - (\sigma(y)u_y)_y = 0, t > 0,\ \ y \in \mathbb{R}. \tag{1}$$

Energy conserved in time:

$$\mathcal{E}_{\rho,\sigma}(u^0, u^1) := \frac{1}{2}\int_{\mathbb{R}} (\rho(y)|u_t(y,t)|^2 + \sigma(y)|u_y(y,t)|^2)\, dy.$$

Finite difference approximations

Let $h > 0$ be the mesh size parameter, $g : \mathbb{R} \to \mathbb{R}$ be an increasing function on $\mathbb{R}$, $\mathcal{G}^h := \{x_j := jh, j \in \mathbb{Z}\}$ and $\mathcal{G}^h_g := \{g_j := g(x_j), j \in \mathbb{Z}\}$ the *uniform grid* of size h of $\mathbb{R}$ and the *non-uniform* one obtained by transforming the uniform one through the map g.

Finite difference semi-discretization of (1) on the non-uniform grid $\mathcal{G}^h_g$:

$$\rho(g_j)u_{j,tt}(t) - \frac{\sigma(g_{j+1/2})\frac{u_{j+1}(t)-u_j(t)}{g_{j+1}-g_j} - \sigma(g_{j-1/2})\frac{u_j(t)-u_{j-1}(t)}{g_j-g_{j-1}}}{\frac{g_{j+1}-g_{j-1}}{2}} = 0. \qquad (2)$$

Energy is conserved in time

$$\mathcal{E}^h_{\rho,\sigma,g}(\mathbf{u}^{h,0}, \mathbf{u}^{h,1}) := \frac{h}{2}\sum_{j\in\mathbb{Z}} \left[\partial^h g_j \rho(g_j)|u_{j,t}(t)|^2 + \frac{\sigma(g_{j+1/2})}{\partial^{h,+} g_j}|\partial^{h,+} u_j(t)|^2\right].$$

This schemes provides a convergent numerical approximation.

The *principal symbol* :

$$\wp(x,t,\xi,\tau) := -g'(x)\rho(g(x))\tau^2 + 4\sin^2\left(\frac{\xi}{2}\right)\frac{\sigma(g(x))}{g'(x)}. \tag{3}$$

The *null bi-characteristic lines* associated to this *principal symbol* are the solutions of the Hamiltonian system:

$$\begin{cases} \dot{X}(s) = \partial_\xi \wp = 2\sin(\Xi(s))\frac{\sigma(g(X(s)))}{g'(X(s))}, \\ \dot{t}(s) = \partial_\tau \wp = -g'(X(s))\rho(g(X(s)))\tau, \\ \dot{\Xi}(s) = -\partial_x \wp = (g'(\cdot)\rho(g(\cdot)))'(X(s)) - 4\sin^2\left(\frac{\Xi(s)}{2}\right)\left(\frac{\sigma(g(\cdot))}{g'(\cdot)}\right)'(X(s)), \\ \dot{\tau}(s) = -\partial_t \wp = 0, \end{cases} \tag{4}$$

$(X(t), \Xi(t))$ solves the *Hamiltonian system*:

$$(X)'(t) = \mp c_g(X(t))\cos\left(\frac{\Xi(t)}{2}\right), \ (\Xi)'(t) = \pm c_g'(X(t)) 2\sin\left(\frac{\Xi(t)}{2}\right) \tag{5}$$

with

$$c_g(x) := \sqrt{\sigma(g(x))/\rho(g(x))}/g'(x).$$

Using Wigner transforms, high frequency solutions can be shown to propagate and concentrate along those rays provided $c_g \in C^{1,1}(\mathbb{R})$. This means that the coefficients σ and ρ need to be in $C^{1,1}(\mathbb{R})$ and the grid transformation $g \in C^{2,1}(\mathbb{R})$.

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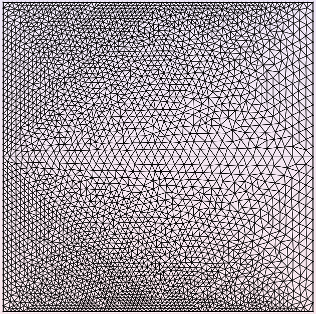
Numerical simulations

There is istill plenty to be done to understand the behavior of discrete waves over very irregular meshes:



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Conclusions

- There is a rich (and difficult!) field to be explored at the intersection of the following topics:
  - Damped wave equations
  - Partially damped hyperbolic systems
  - Control theory
  - Optimal design
  - Spectral theory
  - Microlocal analysis
  - Fine numerics
- At the continuous level there is a huge need of better spectral understanding.
- Correct answers can only be obtained by a fine combination of analytical and numerical tools. Unlike for elliptic problems, one can not expect numerical optimal designs to converge to continuous ones.
- Heterogeneous grids may lead to novel unexpected phenomena of discrete wave propagation. This makes it even more complex the numerical computation of optimal shapes or locations for dampers, sensors, controllers.

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