Control and Numerics

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Propagation, dispersion, control and numerical approximation of waves

Outline

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Motivation

IS THE CONTROL OF WAVES AND, MORE PARTICULARLY, OF THE WAVE EQUATION RELEVANT?

The answer is, definitely, YES.

- Noise reduction in cavities and vehicles.
- Laser control in Quantum mechanical and molecular systems.
- Seismic waves, earthquakes.
- Flexible structures.
- Environment: the Thames barrier.
- Optimal shape design in aeronautics.
- Human cardiovascular system: the bypass
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Similar problems arise in Control, Optimal Design and in Inverse Problems Theory and common techniques need to be developed.

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To develop efficient (consistent + stable) numerical algorithms allowing to compute the controls.

One needs to make sure that:

- The schemes are stable when solving the PDE's involved;
- The optimization procedures converge.
- Putting together Control + Numerics does not generate unexpected instabilities.

Warning!

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THE 1-D CONTROL PROBLEM

The $1 - d$ wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control one one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control. The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

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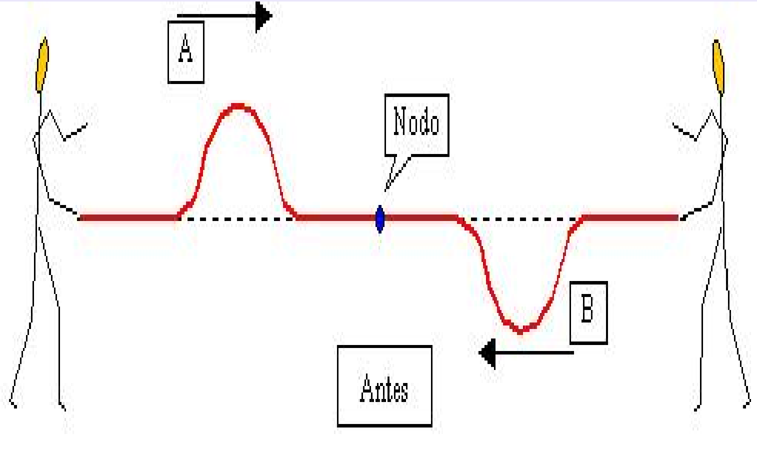
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THE $1-d$ OBSERVATION PROBLEM

The control problem above is equivalent to the observability problem on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\ \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

Namely:

$$E(0) \le C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt.$$

The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2} \int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2 \right] dx = E(0),\ \ \forall 0 \le t \le T.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \ge 2$.

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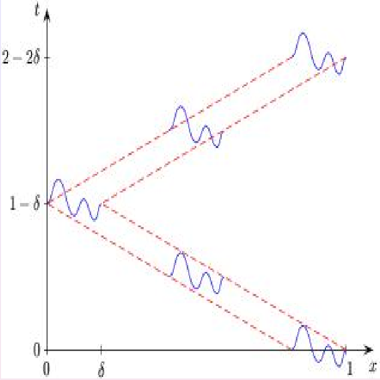

$$E(0) \leq C(T) \int_0^T |\varphi_x(1,t)|^2 dt$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

CONSTRUCTION OF THE CONTROL

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x^\star(1,t),$$

where φ is the solution of the adjoint system corresponding to initial data $(\varphi^{0,\star}, \varphi^{1,\star}) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(\varphi^0,\varphi^1) = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0\varphi^1 dx - < y^1, \varphi^0 >_{H^{-1}\times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.
Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $\varphi_x(1,t) \in L^2(0,T)$ (hidden regularity).

Thus:

- Controllability holds for all $T \geq 2$;
- The control is characterized by a minimization problem involving the adjoint system (duality).

Note that the fact that controllability holds only for $T \geq 2$ is typically a phenomenon related to the infinite-dimensional character of the model under consideration.

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THE MOST NATURAL NUMERICAL APPROXIMATION SCHEME

Set $h = 1/(N+1) > 0$ and consider the mesh

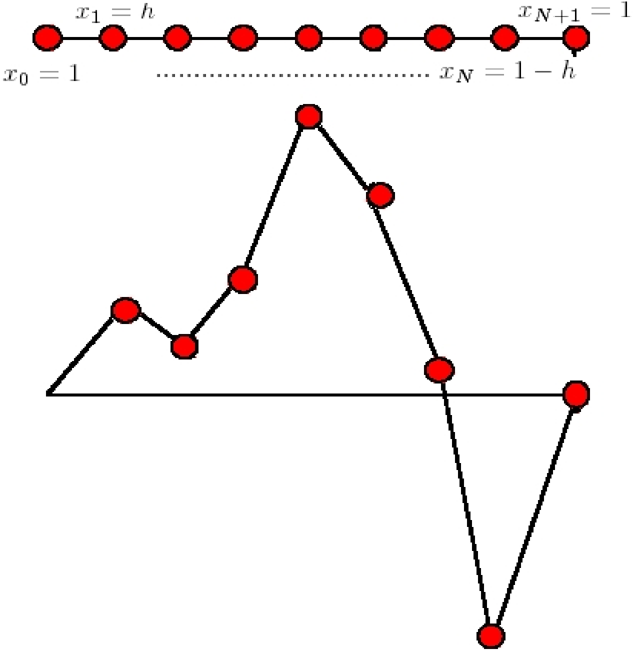
$$x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$$

which divides $[0, 1]$ into $N + 1$ subintervals

$$I_j = [x_j, x_{j+1}], \, j = 0, ..., N.$$

Finite difference semi-discrete approximation of the wave equation:

$$\begin{cases} \varphi_j'' - \frac{1}{h^2}\left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right] = 0, & 0 < t < T, \, j = 1, \ldots, N \\ \varphi_j(t) = 0, & j = 0, \, N+1, \, 0 < t < T \\ \varphi_j(0) = \varphi_j^0, \, \varphi_j'(0) = \varphi_j^1, & j = 1, \ldots, N. \end{cases}$$


$x_1 = h$
$x_{N+1} = 1$
$x_0 = 1$
$x_N = 1 - h$

Then it should be sufficient to minimize the discrete functional

$$J_h(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \frac{|\varphi_N(1,t)|^2}{h^2} dt + h \sum_{j=1}^{N} \varphi_j^1 y_j^0 - h \sum_{j=1}^{N} \varphi_j^0 y_j^1,$$

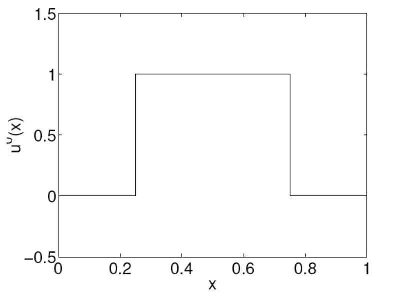
which is a discrete version of the functional J of the continuous wave equation since

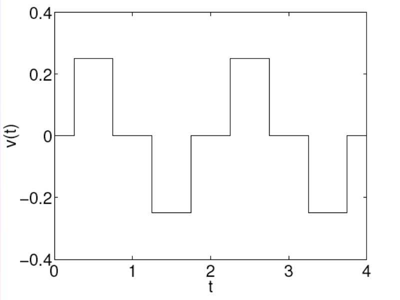
$$-\frac{\varphi_N(t)}{h} = \frac{\varphi_{N+1} - \varphi_N(t)}{h} \sim \varphi_x(1,t).$$

Then

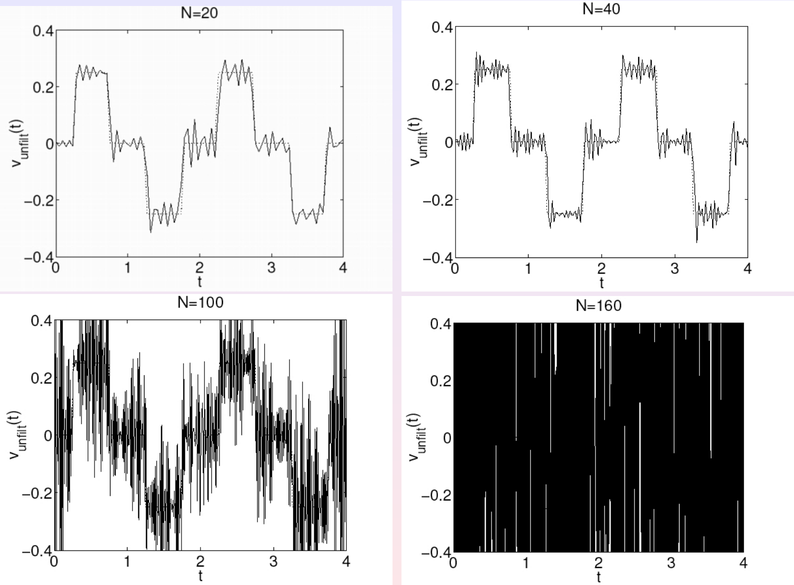
$$v_h(t) = -\frac{\varphi_N^\star(t)}{h}.$$

A NUMERICAL EXPERIMENT





Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.
Plot of the time evolution of the exact control for the wave equation in time $T = 4$.



The control diverges as $h \to 0$.

WHY?

The Fourier series expansion shows the analogy between continuous and discrete dynamics. Discrete solution:

$$\vec{\varphi} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h}\, t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h}\, t\right) \right) \vec{w}_k^h.$$

Continuous solution:

$$\varphi = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

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Recall that the discrete spectrum is as follows and converges to the continuous one:

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0$$

$$w_k^h = (w_{k,1}, \ldots, w_{k,N})^T : w_{k,j} = \sin(k\pi jh), \ k, j = 1, \ldots, N.$$

The only relevant differences arise at the level of the dispersion properties and the group velocity. High frequency waves do not propagate, remain captured within the grid, without never reaching the boundary. This makes it impossible the uniform boundary control and observation of the discrete schemes as $h \to 0$.

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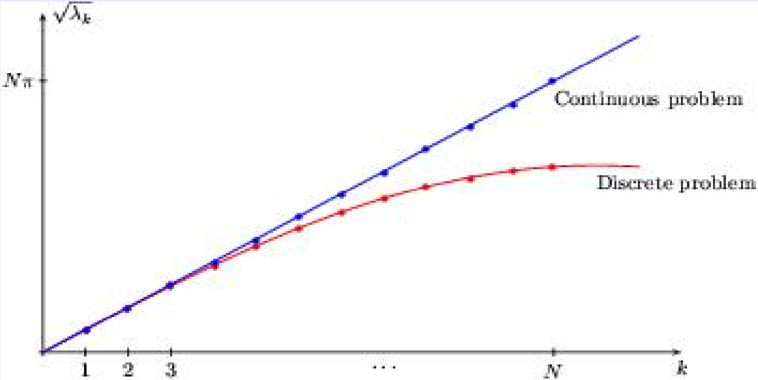
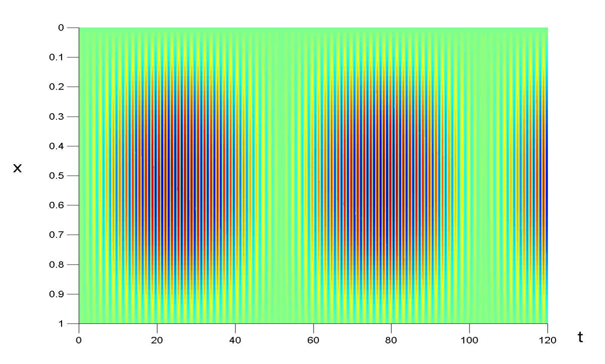


Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

A NUMERICAL PHAMTOM

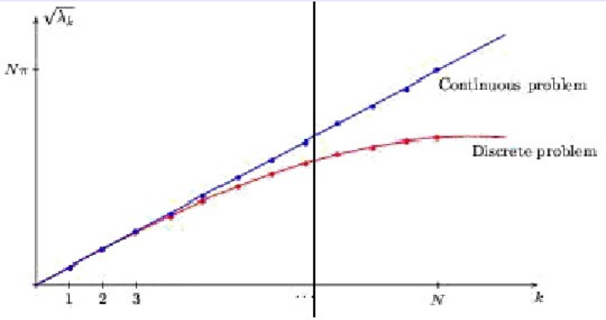
$$\vec{\varphi} = \exp\left(i\sqrt{\lambda_N(h)}\,t\right)\vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\,t\right)\vec{w}_{N-1}.$$

Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap: $\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h$.



$h = 1/61$, $(N = 60)$, $0 \leq t \leq 120$.

THE FIRST REMEDY: FOURIER FILTERING



filtering parameter:
$k \leq \delta N$, $0 < \delta < 1$

To filter the high frequencies, i.e. keep only the components of the solution corresponding to indexes: $k \leq \delta/h$ with $0 < \delta < 1$. This guarantees that the group velocity remains uniformly bounded below and allows observing uniformly filtered solutions in time $T(\delta) > 2$ such that $T(\delta) \to 2$ as $\delta \to 0$.

Then, the filtering algorithm can be implemented as follows:

- Minimize J_h over the class of filtered solutions with filtering parameter $0 < \delta < 1$ and $T > T(\delta)$;
- This yields controls v_h^δ such that
  - $v_h^\delta \to v$ as $h \to 0$;
  - The corresponding states $\vec{y}_h$ satisfiy:

$$\pi_\delta(\vec{y}_h) = \pi_\delta(\vec{y}_h{}') = 0.$$

This is a relaxed version of the controllability condition.

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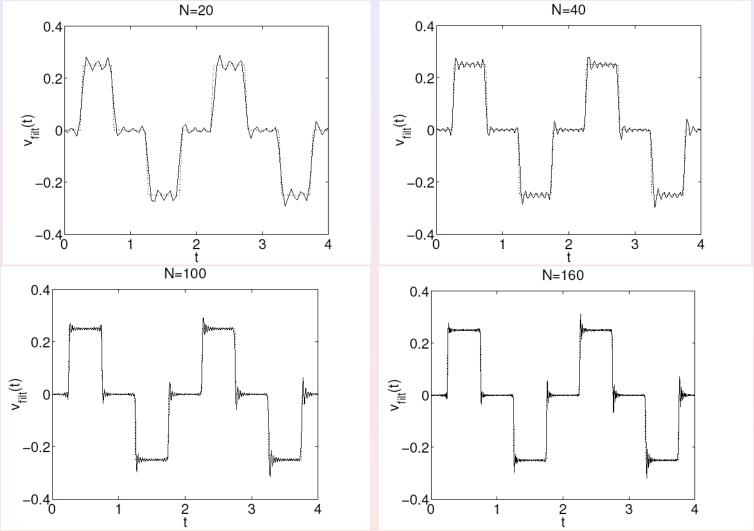
NUMERICAL EXPERIMENT WITH RELAXED CONTROLS:



CONCLUSION

- The minima of J_h diverge because its coercivity is vanishing as $h \to 0$;
- This is intimately related to the blow-up of the discrete observability constant $C_h(T) \to \infty$, for all $T > 0$ as $h \to 0$:

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{\varphi_N(t)}{h} \right|^2 dt$$

- This is due to the lack of propagation of high frequency numerical waves due to the dispersion that the numerical grid produces.
- Actually it is known that $C_h(T)$ diverges exponentially: S. Micu, Numerische Math., 2002.

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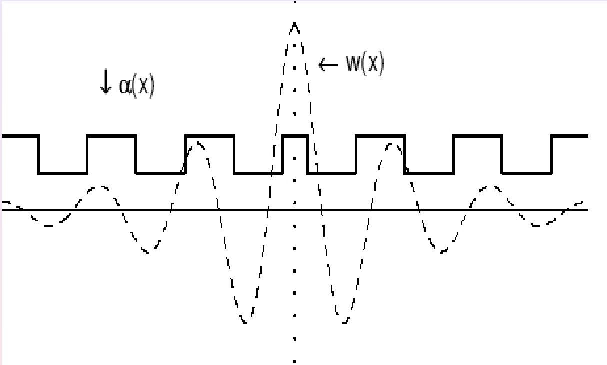
WELL KNOWN PHENOMENA FOR WAVES IN HIGHLY OSCILLATORY MEDIA



$$\varphi_{tt} - (\alpha(x)\varphi_x)_x = 0.$$

- F. Colombini & S. Spagnolo, Ann. Sci. ENS, 1989
- M. Avellaneda, C. Bardos & J. Rauch, Asymptotic Analysis, 1992.
- C. Castro & E. Z. Archive Rational Mechanics and Analysis, 2002.

DISCRETE MULTIPLIERS

The proof of uniform observability of discrete filtered solutions can developed in various ways:

- Using Ingham inequality in their Fourier series representation since filtering guarantess an uniform gap condition;
- Discrete multipliers:

The multiplier $x\varphi_x$ for the wave equation yields:

$$TE(0) + \int_0^1 x\varphi_x\varphi_t \, dx\Big|_0^T = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 \, dt.$$

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The multiplier $j(\varphi_{j+1} - \varphi_{j-1})$ for the discrete wave equation gives:

$$TE_h(0) + X_h(t)\Big|_0^T = \frac{1}{2}\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 dt + \frac{h}{2}\sum_{j=0}^{N}\int_0^T |\ \varphi_j' - \varphi_{j+1}'\ |^2\ dt,$$

Note that

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T |\ \varphi_j' - \varphi_{j+1}'\ |^2\ dt \sim \frac{h^2}{2}\int_0^T\int_0^1 |\varphi_{xt}|^2 dxdt.$$

Filtering is needed to absorb this higher order term: For $1 \leq j \leq \delta N$

$$\left|\frac{h}{2}\sum_{j=0}^{N}\int_0^T |\ \varphi_j' - \varphi_{j+1}'\ |^2\ dt\right| \leq \gamma(\delta)\, TE(0),$$

with $0 < \gamma(\delta) < 1$.

TWO-GRID ALGORITHM (R. Glowinski, M. Asch-G. Lebeau, M. Negreanu, L. Ignat, E. Z.)

To develop on the physical space a different remedy to Fourier filtering.

High frequencies producing lack of gap and spurious numerical solutions correspond to large eigenvalues

$$\sqrt{\lambda_N^h} \sim 2/h.$$

When refining the mesh

$$h \to h/2, \qquad \sqrt{\lambda_{2N}^{h/2}} \sim 4/h.$$

Refining the mesh $h \to h/2$ produces the same effect as filtering with parameter $1/2$.

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Solutions on the fine grid of size h corresponding to slowly oscillating data given in the coarse mesh $(2h)$ are no longer pathological:

$$\varphi = \varphi_l + \varphi_h, \varphi_l = \sum_{k=1}^{(N-1)/2} c_k \vec{w}_k, \varphi_h = \sum_{k=1}^{(N-1)/2} c_k \frac{\lambda_k}{\lambda_{N+1-k}} \vec{w}_{N+1-k},$$

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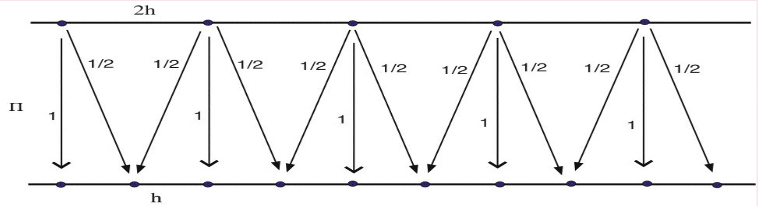

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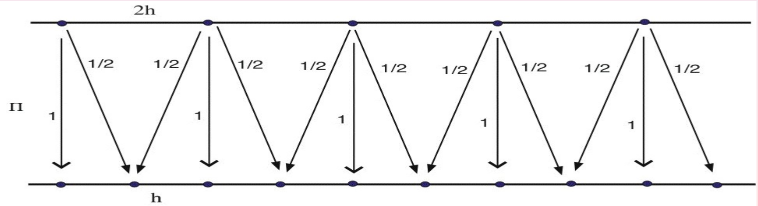

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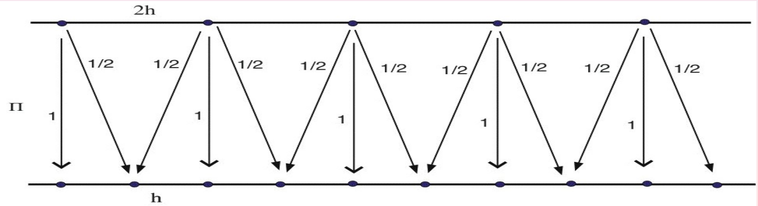

THE MULTI-DIMENSIONAL CASE

Similar results are true in several space dimensions. The region in which the observation/control applies needs to be large enough to capture all rays of Geometric Optics. This is the so-called Geometric Control Condition introduced by Ralston (1982) and Bardos-Lebeau-Rauch (1992).

Let Ω be a bounded domain of $\mathbf{R}^n$, $n \geq 1$, with boundary Γ of class C^2. Let Γ_0 be an open and non-empty subset of Γ and $T > 0$.

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ y = v(x,t) 1_{\Gamma_0} & \text{on} \quad \Sigma = \Gamma \times (0, T) \\ (x, 0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

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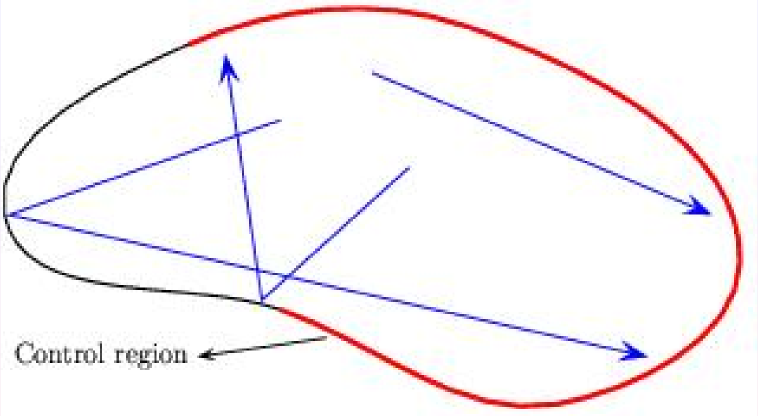


Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools form Microlocal Analysis.

In all cases the control is equivalent to an observation problem for the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ \varphi = 0 & \text{on} \quad \Sigma = \Gamma \times \times (0, T) \\ \varphi(x, 0) = \varphi^0(x), \varphi_t(x, 0) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases}$$

Is it true that:

$$E_0 \leq C(\Gamma_0, T) \int_{\Gamma_0} \int_0^T \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt \quad ?$$

And a sharp discussion of this inequality requires of Microlocal analysis. Partial results may be obtained by means of multipliers: $x \cdot \nabla\varphi$, φ_t, φ,...

THE 5-POINT FINITE-DIFFERENCE SCHEME

$$\varphi''_{j,k} - \frac{1}{h^2}\left[\varphi_{j+1,k} + \varphi_{j-1,k} - 4\varphi_{j,k} + \varphi_{j,k+1} + \varphi_{j,k-1}\right] = 0.$$

The energy of solutions is constant in time:

$$\begin{aligned} E_h(t) &= \frac{h^2}{2}\sum_{j=0}^{N}\sum_{k=0}^{N}\Big[\mid \varphi'_{jk}(t) \mid^2 \\ &\quad + \left|\frac{\varphi_{j+1,k}(t) - \varphi_{j,k}(t)}{h}\right|^2 + \left|\frac{\varphi_{j,k+1}(t) - \varphi_{j,k}(t)}{h}\right|^2\Big]. \end{aligned}$$

Without filtering observability inequalities fail in this case too.

Understanding how filtering should be used requires of a microlocal analysis of the propagation of numerical waves combining von Neumann analysis and Wigner measures developments (N. Trefethen, P. Gérard, P. L. Lions & Th. Paul, G. Lebeau, F. Macià, ...).

The von Neumann analysis.

Symbol of the semi-discrete system for solutions of wavelength h

$$p_h(\xi,\tau) = \tau^2 - 4\left(\sin^2(\xi_1/2) + \sin^2(\xi_2/2)\right),$$

versus $p(\xi,\tau) = \tau^2 - [|\xi_1|^2 + |\xi_2|^2]$.
Both symbols coincide for $(\xi_1,\xi_2) \sim (0,0)$.
Solving the bicharacteristic flow we get the discrete rays:

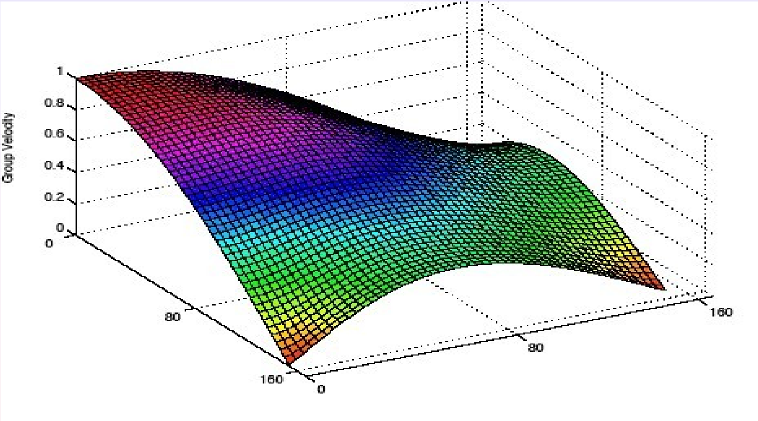
$$x_j(t) = -\frac{\sin(\xi_j)}{\tau} t + x_{j,0}, \quad (\text{versus } x_j(t) = -\frac{\xi_j}{\tau} t + x_{j,0}.)$$

RAYS ARE STILL STRAIGHT LINES. BUT! The velocity is

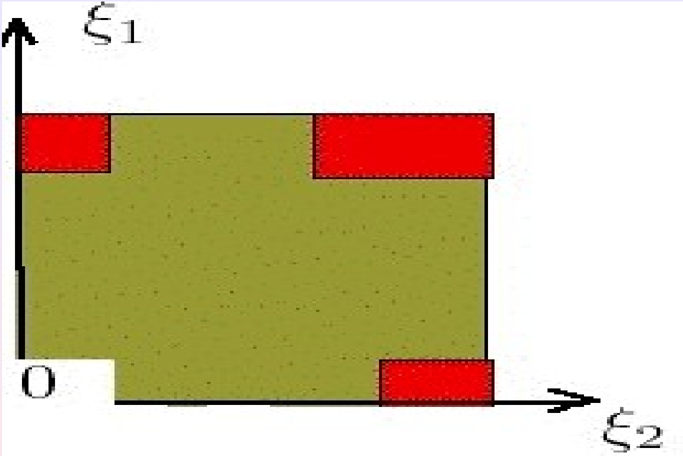
$$|x'(t)| \equiv \left[\left|\frac{\sin(\xi_1)}{\tau}\right|^2 + \left|\frac{\sin(\xi_2)}{\tau}\right|^2\right]^{1/2}$$

THE VELOCITY OF PROPAGATION VANISHES !!!!!!! in the following eight points

$$\xi_1 = 0, \pm\pi, \xi_2 = 0, \pm\pi, \quad (\xi_1,\xi_2) \neq (0,0).$$



Group velocity in dimension two, $h = 1/50$



The red areas stand for those that need to be filtered out in order to guarantee a uniform velocity of propagation in the semi-discrete models.

THE TWO-GRID ALGORITHM L. Ignat & E. Z., 2006

Theorem

Let Ω be the square and consider controls on all its boundary or on two consecutive sides. Then, the two-grid algorithm with mesh-ratio $1/4$ converges for T sufficiently large.

The proof uses:

- Previous results on the control of the solutions under Fourier filtering (E. Z. JMPA, 99')
- Fourier analysis showing that the total energy of the slowly oscillating discrete functions can be bounded above in terms of the low frequency components.
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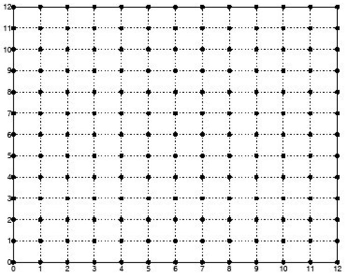
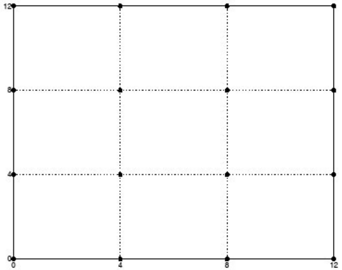
Grids: h & 4h



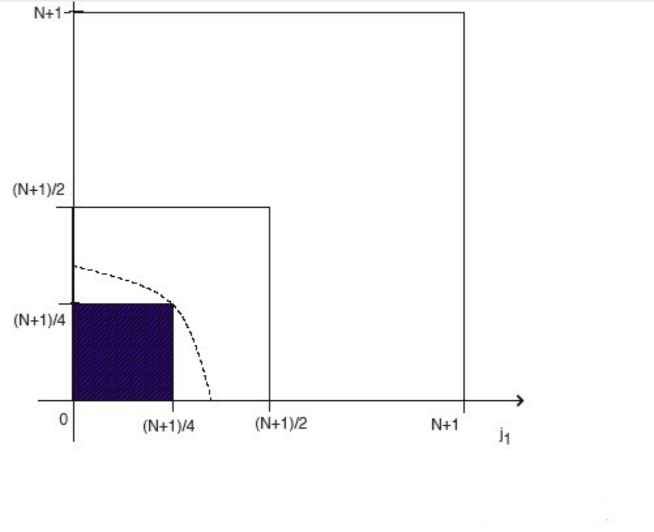
The fine grid G^h; N=11



The coarse grid G^{4h}; N=11

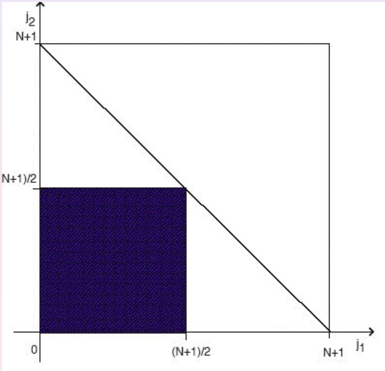
Grids: h & 4h

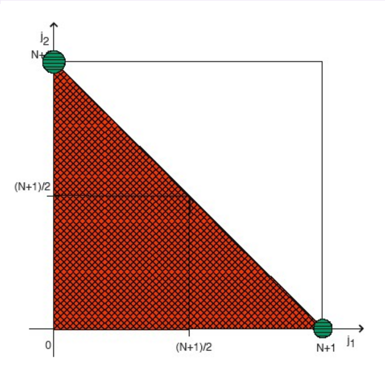
Low frequency subset concentrating the energy of solutions:



Why not using ratio 1/2 for the two-grids?

The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.





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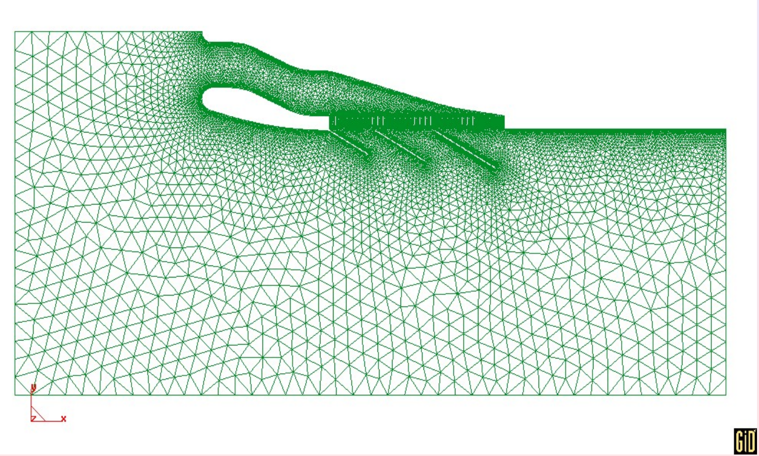
OPEN PROBLEMS

Complex geometries, variable and irregular coefficients, irregular meshes, the system of elasticity, nonlinear state equations, ...



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