UIMP, Santander, July 04

Hybrid systems of PDE's arising in fluid-structure interaction and multi-structures

Enrique Zuazua
Universidad Autónoma
28049 Madrid, Spain
enrique.zuazua@uam.es

The goal of this RSME-UIMP summer school is to present some of the many advances that take place nowadays in the interphase between nonlinear Partial Differential Equations (PDEs) and its applications to several scientific disciplines.

The courses deal with problems and techniques that belong to classical PDE theory ... but that have been successfully explored rather recently, and hence that open new research topics.

All of them are of interest for their applications to other scientific and technological disciplines, such as the design of optimal strategies, the physics of particle interactions, the study of periodic patterns, and the topic of structure-fluid interaction.

THE CLASSICAL CLASSIFICATION OF PDE's

Elliptic (Δ) , parabolic $(\partial_t - \Delta)$ and hyperbolic $(\partial^2_{tt} - \Delta)$

There is a long list of celebrated equations that, more or less, do indeed fit in this classification:

The wave equation (motion of $1-d$ vibrating strings, $2-d$ membranes, acoustic waves...)

$$u_{tt} - \Delta u = 0, \tag{1}$$

involving the Laplace operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{2}$$

The Helmholtz equation

$$-\Delta u = \lambda u. \tag{3}$$

The transport equation

$$u_t + \sum_{i=1}^{n} b_i u_{x_i} = 0 \tag{4}$$

and the adjoint one (Liouville)

$$u_t - \sum_{i=1}^{n} (b_i u)_{x_i} = 0 \tag{5}$$

In vector notation:

$$u_t + \vec{b} \cdot \nabla u = 0 \tag{6}$$

$$u_t - \text{ div } \left(\vec{b}u\right) = 0. \tag{7}$$

Note that:

$$u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x)\, u. \tag{8}$$

The Schrödinger equation in Quantum Mechanics:

$$iu_t + \Delta u = 0. \tag{9}$$

(u tis complex valued)

The plate or beam equation

$$u_{tt} + \Delta^2 u = 0 \tag{10}$$

$$\partial_t^2 + \Delta^2 = -\left(i\partial_t + \Delta\right)\left(i\partial_t - \Delta\right). \tag{11}$$

$$u_{tt} - u_{xx} + d\, u_t = 0 \qquad \text{(telegraph equation)} \tag{12}$$

$$du_t - u_{xx} = -u_{xx} + d\, u_t = 0 \qquad \text{(the heat equation)} \tag{13}$$

$$u_t + u_{xxx} = 0 \qquad \text{(Airy equation)}, \tag{14}$$

$$u_{tt} - \Delta u + u = 0 \qquad \text{(Klein-Gordon equation)}, \tag{15}$$

The Lamé system in $3-d$ elasticity

$$u_{tt} - \lambda \Delta u - (\lambda + \mu) \nabla \mathrm{div} u = 0. \tag{16}$$

$(u = (u_1, u_2, u_3))$

All previous equations were linear. In practice they are derived as linearizations (valid for small amplitude solutions) from nonlinear equations in Continuum Mechanics.

The Maxwell system for electromagnetic waves:

$$\begin{cases} E_t = \mathrm{curl} B \\ B_t = -\mathrm{curl} E \\ \mathrm{div} B = \mathrm{div} E = 0. \end{cases} \tag{17}$$

The eikonal equation

$$\mid \nabla u \mid = 1 \tag{18}$$

The Hamilton-Jacobi equation :

$$u_t + H(\nabla u, x) = 0. \tag{19}$$

The Korteweg-de Vries (KdV) for solitons :

$$u_t + uu_x + u_{xxx} = 0. \tag{20}$$

The Navier-Stokes equations

$$\begin{cases} u_t - \Delta u + u \cdot \nabla u = \nabla p \\ \text{div} u = 0 \end{cases} \tag{21}$$

and the Euler equations for perfect inviscid fluids

$$\begin{cases} u_t + u \cdot \nabla u = \nabla p \\ \text{div} u = 0. \end{cases} \tag{22}$$

The $1-d$ "toy-model", the Burgers equation :

$$u_t + uu_x - u_{xx} = 0, \tag{23}$$

$$u_t + uu_x = 0. \tag{24}$$

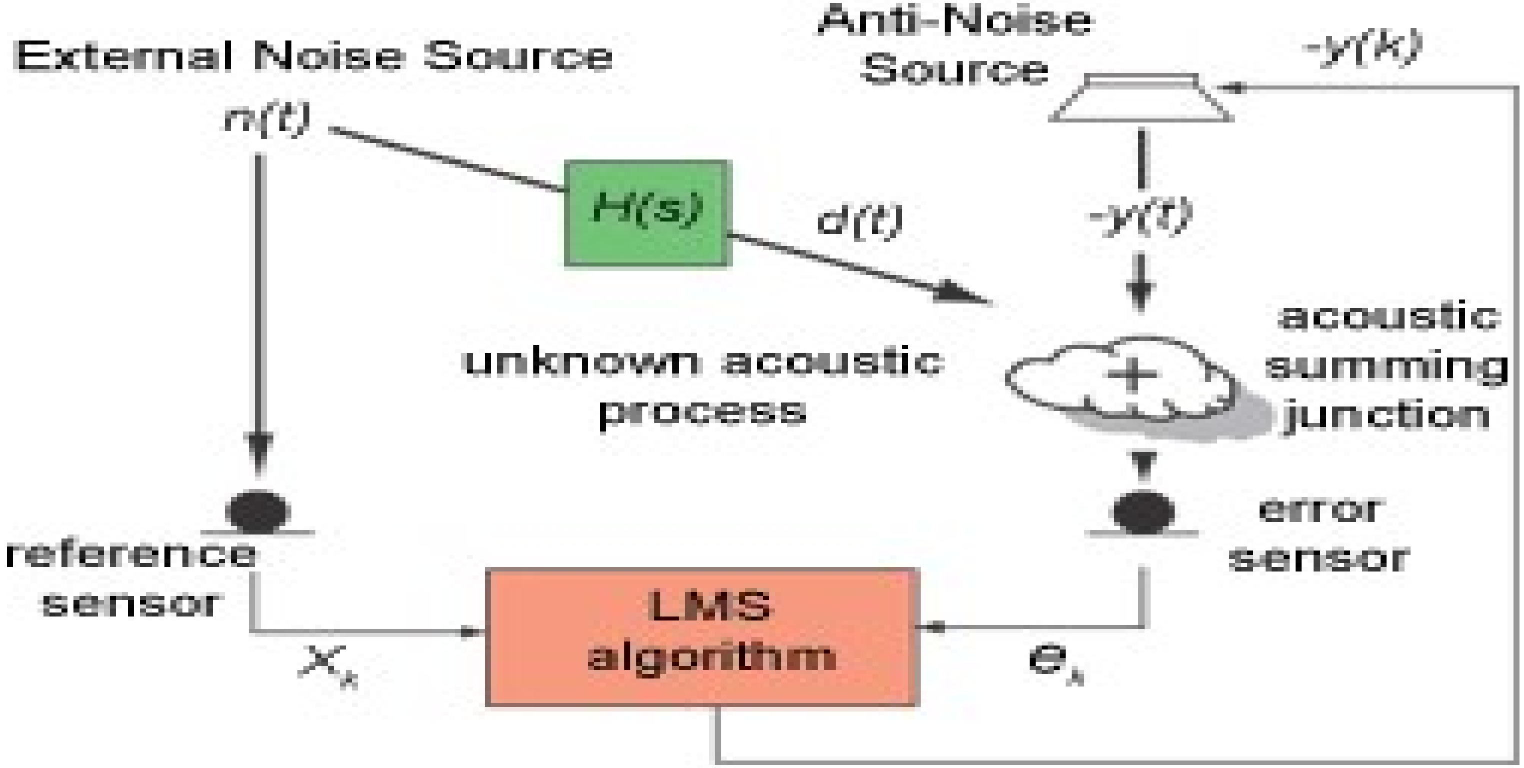
But REAL MODELS are often MORE COMPLEX

Often, reality is of a multi-physics nature and consequently, more realistic models are also of HYBRID type.

A FEW EXAMPLES:

- Noise reduction in cavities and vehicles.

  Typically, the models involve the wave equation for the acoustic waves coupled with some other equations modelling the dynamics of the boundary structure, the action of actuators, possibly through smart mechanisms and materials.


External Noise Source
n(t)
H(s)
d(t)
Anti-Noise
Source
-y(k)
-y(t)
acoustic
summing
junction
unknown acoustic
process
reference
sensor
error
sensor
LMS
algorithm
X_k
e_k

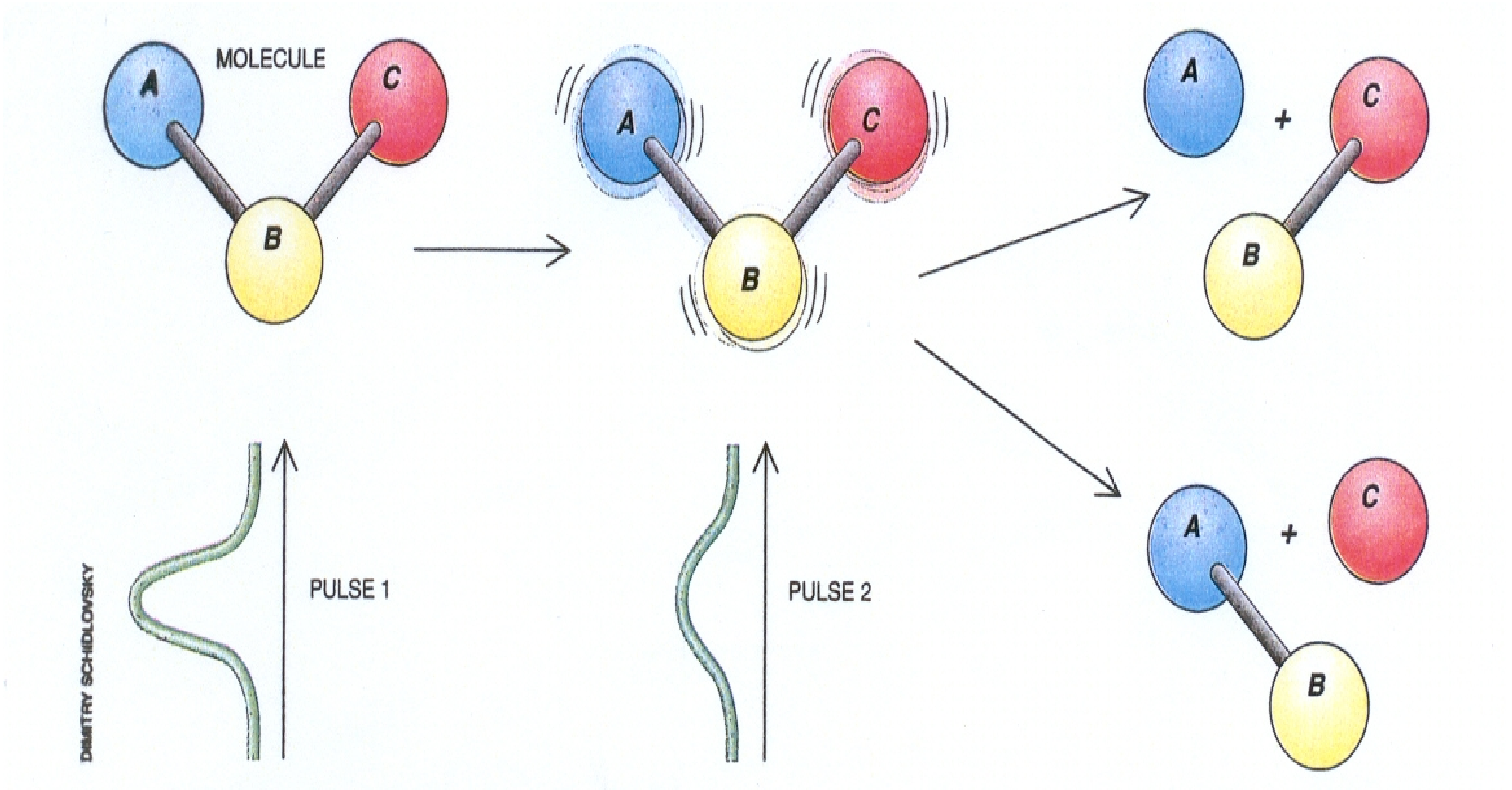
http://www.ind.rwth-aachen.de/research/noise_reduction.html

Noise reduction is a subject to research in many different fields. Depending on the environment, the application, the source signals, the noise, and so on, the solutions look very different. Here we consider noise reduction for audio signals, especially speech signals, and concentrate on common acoustic environments such an office room or inside a car. The goal of the noise reduction is to reduce the noise level without distorting the speech, thus reduce the stress on the listener and - ideally - increase intelligibility.

- Quantum control and Computing.

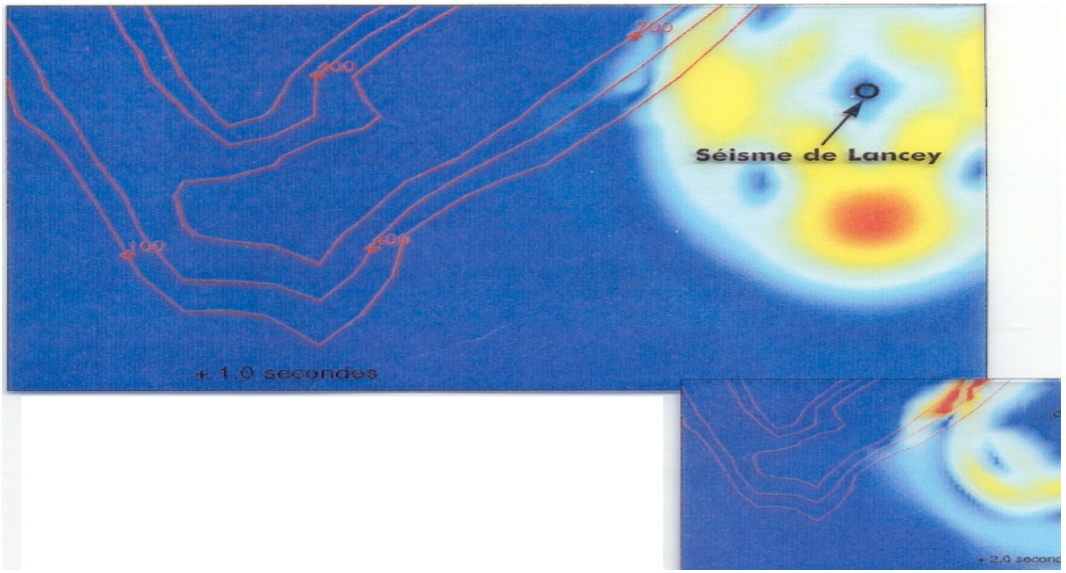
  Laser control in Quantum mechanical and molecular systems to design coherent vibrational states.

  In this case the fundamental equation is the Schrödinger equation that may be viewed as a wave equation with inifnite speed of propagation. The laser beam interacts with the Schrödinger free dynamics:



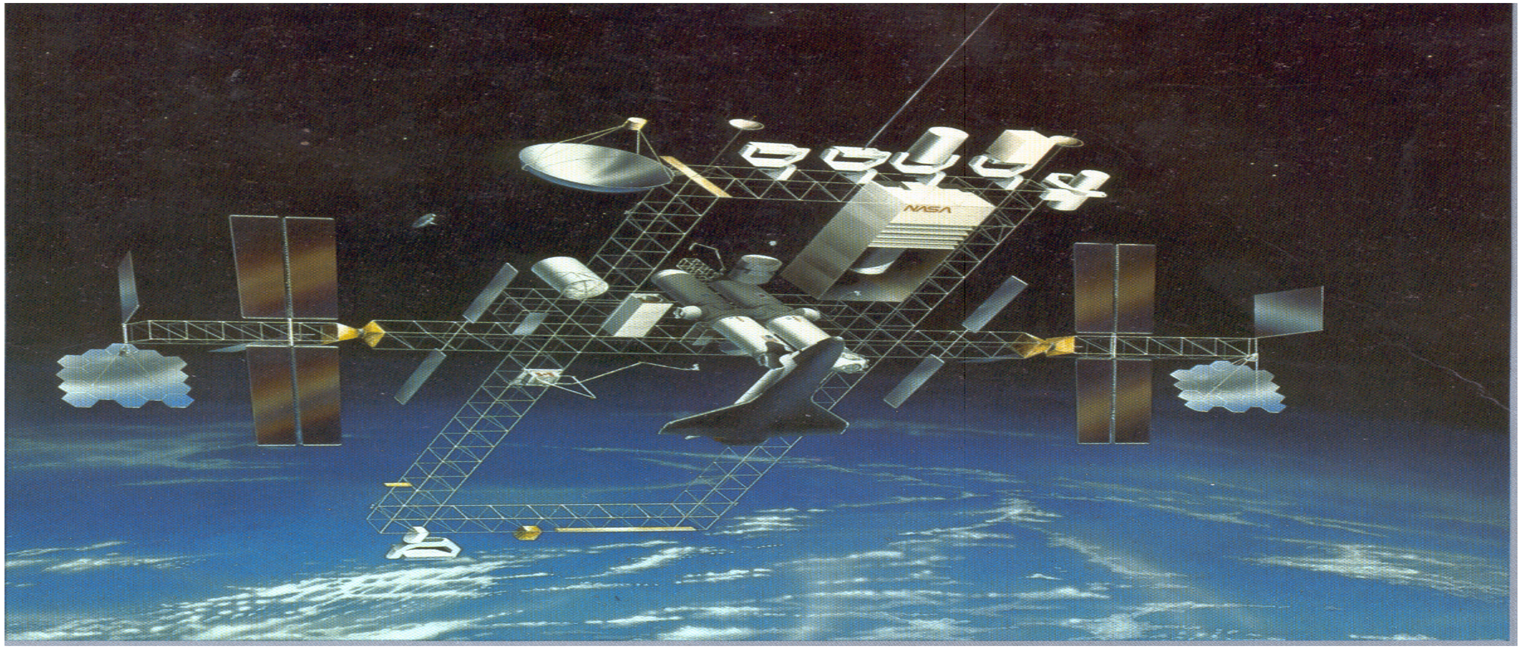
P. Brumer and M. Shapiro, Laser Control of Chemical reactions, Scientific American, March, 1995, pp.34-39.

- Seismic waves, earthquakes: The interaction of seismic waves with buildings...



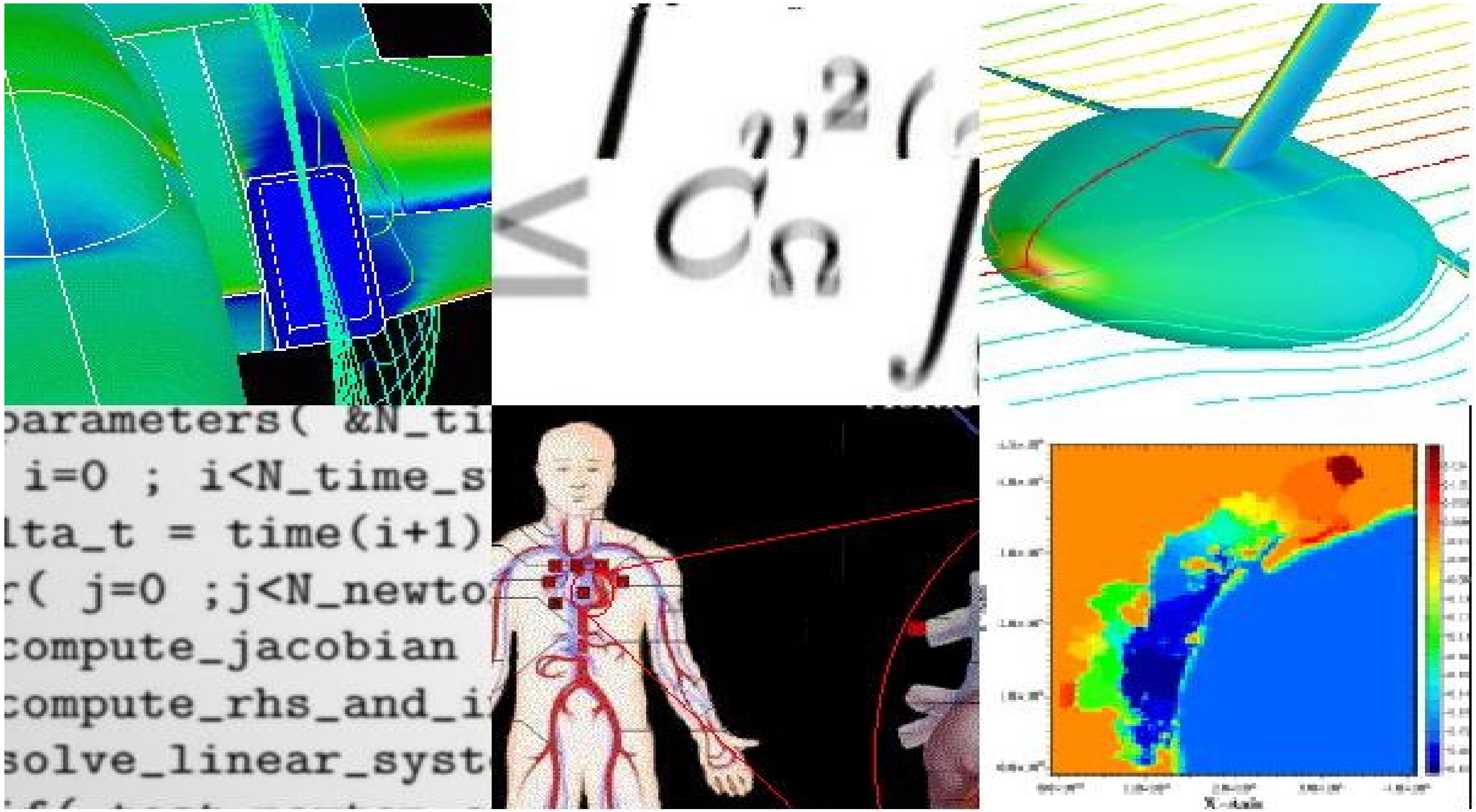
F. Cotton, P.-Y. Bard, C. Berge et D. Hatzfeld, Qu'est-ce qui fait vibrer Grenoble?, La Recherche, 320, Mai, 1999, 39-43.

- Flexible structures.



Cover page of the SIAM Report on "Future Directions in Control Theory. A Mathematical Perspective", W. H. Fleming, Chairman, 1988.

And many others...



Alfio Quarteroni's web page

The Tames barrier

Any explanation is needed....?.

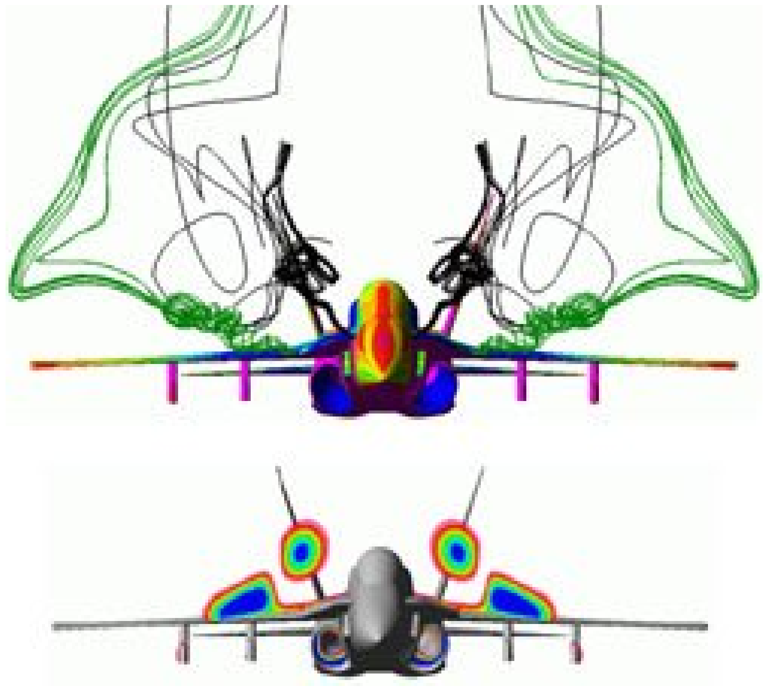
TWO KEY WORDS:

- MULTI-STRUCTURES.

  interconnected structures; structures of different dimensions.

- FLUID-STRUCTURE INTERACTION.

  Aeroelasticity: the influence of an exterior flow on a structure (aircraft in flight);



http://www.mscsoftware.com.au/products/software/cfdrc/cfd-fastran/fsi.

Aeroacoustics: the vibrations (noise) induced by a turbulent flow originated by a boundary layer on a solid wall (flight vehicles);

Hydroelasticity: small vibrations coupling an elastic structure and a liquid (gravity waves in the reservoir of a space vehicle);

Elastoacoustics: noise in a cavity produced by the vibrations of the surounding structure.

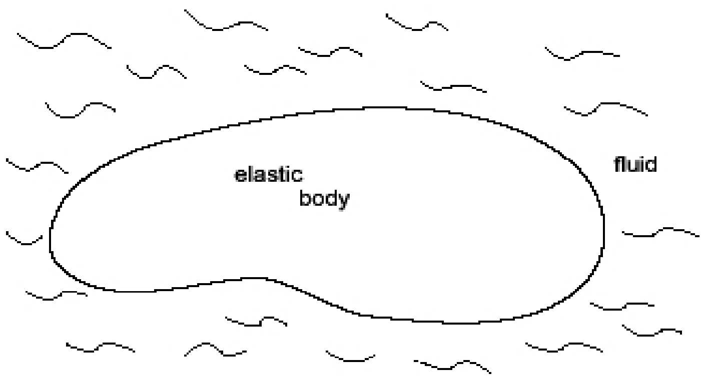
From a mathematical point of view:

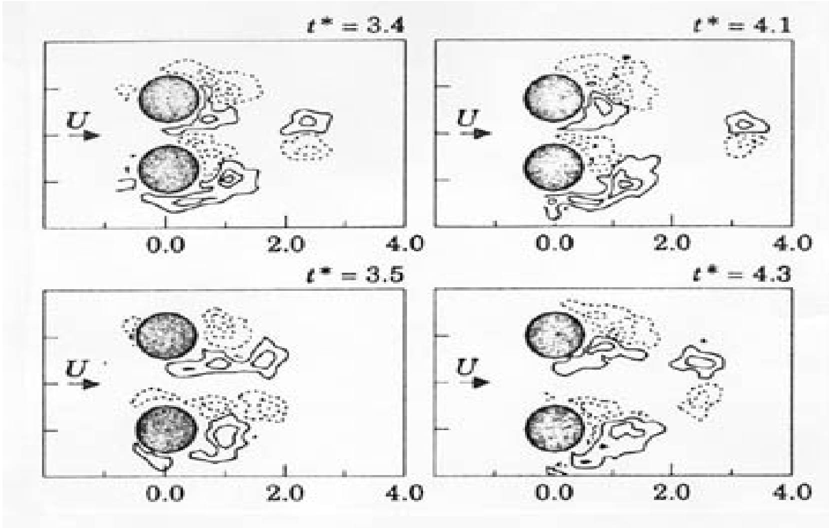
Membrane+String= 2-d wave equation + 1-d wave equation

Fluid+Structure: Navier-Stokes equations+elasticity equations

or, for a toy-model,

heat equation +wave equation.





The first figure describes the simplified geometrical setting in which problems are often studied analytically. The second Figure, taken from the web-page of Profs. S.J. Price and M.P. Paidoussis, shows the vorticity contours of wake behind two circular cylinders impulsively set into motion for different spacing between cylinders and different elapsed times obtained using the DPIV system.

THE MAIN DIFFICULTY:

Solving (analytically or numerically) these interaction systems is not simply superposing two separate theorems for the corresponding type of PDE.

Interactions do produce new (unexpected?) phenomena and new analytical (and numerical tools) are required.

THE RECEIPT: To decompose the original system into simpler subsystems of specific forms. Treat each subsystem by an ad-hoc method, and put or assamble together the results to get the global one.

THE BASIC MATHEMATICAL INGREDIENT: LIE'S THEOREM

Consider the system of ODE's:

$$\begin{cases} \dot{x}(t) = (A+B)x(t) \\ x(0) = x_0, \end{cases} \tag{25}$$

where $x = x(t) \in \mathbb{R}^N$, A and B are square $N \times N$-matrices with time-independent coefficients.

For $x_0 \in \mathbb{R}^N$ there exists a unique global solution $x(t) \in C^\omega\left(\mathbb{R}; \mathbb{R}^N\right)$. Moreover,

$$x(t) = e^{(A+B)t}x_0 = \sum_{k=1}^{\infty} \frac{(A+B)^k t^k}{k!} x_0. \tag{26}$$

Theorem 1 *(Lie's Theorem)*

$$e^{A+B} = \lim_{n \to \infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n . \tag{27}$$

Note that, in general, A and B do not commute and therefore $e^{(A+B)t}$ IS NOT THE PRODUCT OF e^{At} and e^{Bt}. Indeed, (for $t = 1$, for instance):

$$\begin{aligned} e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} &= I + [A+B] + \frac{[A+B]^2}{2} + \cdots \\ &= I + [A+B] + \frac{A^2 + AB + BA + B^2}{2} + \cdots \end{aligned} \tag{28}$$

But

$$
\begin{aligned}
e^A e^B &= \left[\sum_{k=0}^{\infty} \frac{A^k}{k!}\right] \left[\sum_{j=0}^{\infty} \frac{B^j}{j!}\right] \\
&= \left[I + A + \frac{A^2}{2} + \cdots\right] \cdot \left[I + B + \frac{B^2}{2} + \cdots\right] \\
&= I + [A + B] + AB + \frac{A^2}{2} + \frac{B^2}{2} + \cdots
\end{aligned} \tag{29}
$$

There is a gap or error on the quadratic terms: $\left[A^2 + B^2 + AB + BA\right] \Big/ 2$ and $\left[A^2 + B^2 + 2AB\right] \Big/ 2$. The gap is

$$[BA - AB] \Big/ 2.$$

The difference only vanishes when the commutator

$$[A, B] = BA - AB$$

vanishes.

But, according to Lie's Theorem, even when the commutation does not occur:

$$e^{A+B} \sim \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n,$$

and, for n fixed,

$$\left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n = e^{\frac{A}{n}} e^{\frac{B}{n}} \cdots e^{\frac{A}{n}} e^{\frac{B}{n}}.$$

It is the iterated product n times, of $e^{A/n} e^{B/n}$.

We have

$$\left[e^{A/n}e^{B/n}\right]x_0 = e^{A/n}\left[e^{B/n}x_0\right].$$

But $e^{B/n}x_0 = y_0$ is the value at time $t = 1/n$ of the solution of

$$\dot{y} = By;\ y(0) = x_0,$$

while $e^{A/n}y_0 = z_0$ is the value at time $t = 1/n$ of the solution of

$$\dot{z} = Az,\ z(0) = y_0.$$

Then, in view of,

$$x(t) = e^{(A+B)t}x_0 = \lim_{n\to\infty}\left(e^{\frac{At}{n}}e^{\frac{Bt}{n}}\right)^n x_0, \tag{30}$$

we approximate $x(t)$ by

$$x_n(t) = \left(e^{\frac{At}{n}}e^{\frac{Bt}{n}}\right)^n x_0$$

which represents the iteration (n times) of a procedure in which we solve in every subinterval the two isolated equations

$$x' = Ax \tag{31}$$

and

$$x' = Bx \tag{32}$$

SOLVING EACH SYSTEM SEPARATELY AND ITERATING WE EVENTUALLY GET THE SOLUTION OF THE GLOBAL SYSTEM.

PROOF=EXERCISE.

This idea can be applied in great generality:

- In the context of the Theory of Semigroups when A and B are generators os semigroups of contractions in Banach spaces. Thus, with many applications to PDE's.

- In the nonlinear context.

  In this way, for instance, the viscous Burgers equation

  $$u_t - \nu u_{xx} + u u_x = 0,$$

  can be viewd as the superposition of the linear heat equation

  $$u_t - \nu u_{xx} = 0,$$

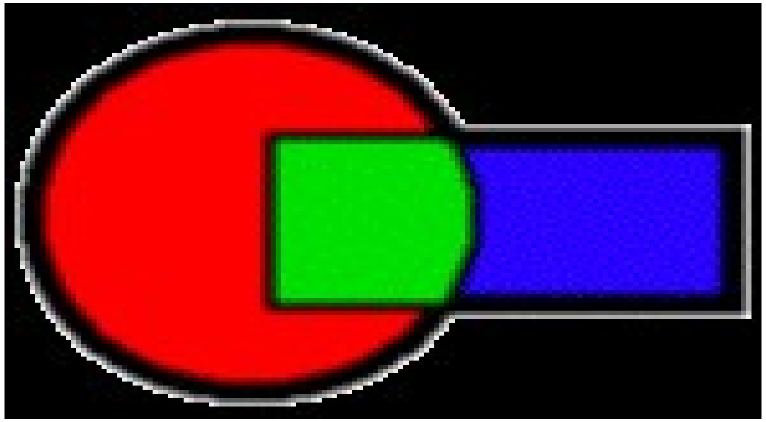
and the hyperbolic equation

$$u_t + uu_x = 0.$$

Viscous Burgers eqn. $=$ Heat eqn. $+$ inviscid Burgers eqn..

From a purely analytical point of view this does not seem to be a very good idea since solving hyperbolic equations is more difficult than solving parabolic ones. But it may be useful in numerics where we can exploit the properties of each specific system: characteristic methods for the hyperbolic component $+$ finite elements for the heat component, for instance.

- Domain decomposition.



DD was introduced by H.A. Schwarz in the end of the XIX-th century to compute the Green function for the laplacian in the domain above, superposition of a rectangle and a disk. Knowing the Green function on the subdomains (disk and rectangle), can one compute it in the whole domain?

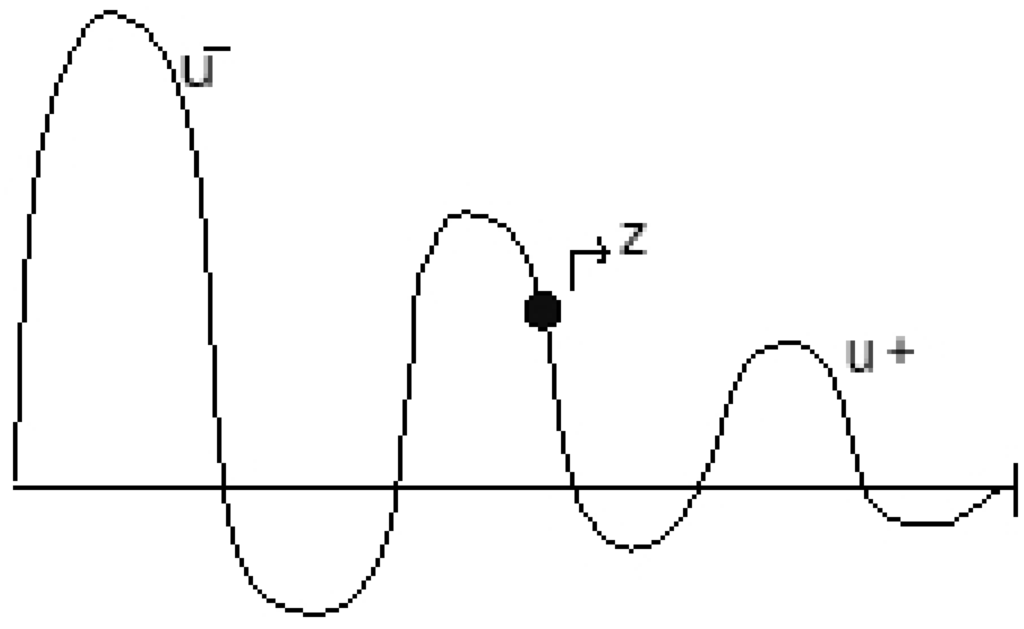
THE DRAWBACK:

These ideas (Lie's iteration, domain decomposition, etc.) hardly allow obtaining sharp qualitative properties for evolution problems. This is particularly true in the context of asymptotic behavior as t tends to ∞, and even more when the dynamical systems involved are of hyperbolic nature with weak damping, finite speed of propagation, etc.

VIBRATIONS OF MULTI-D STRUCTURES: $0-D+1-D$

(S. Hansen & E. Z, 1995, C. Castro & E. Z., 1996)

string+point mass + string


u⁻
z
u +

$$
\begin{cases}
u^-_{tt} - u^-_{xx} = 0, & -1 < x < 0, \quad t > 0 \\
u^+_{tt} - u^+_{xx} = 0, & 0 < x < 1, \quad\;\; t > 0 \\
z(t) = u^-(0,\, t) = u^+(0,\, t), & t > 0 \\
z''(t) = \left[u^+_x(0,\, t) - u^-_x(0,\, t)\right], & t > 0 \\
u^-(-1,\, t) = u^+(1,\, t) = 0, & t > 0 \\
+ \text{ initial conditions}
\end{cases}
$$

$\Leftrightarrow$

$$
\begin{aligned}
\rho(x) U_{tt} - U_{xx} = 0; \quad & U = u^- 1_{(-1,0)} + u^+ 1_{(0,1)}; \\
& \rho(x) = 1 + \delta_0
\end{aligned}
$$

The energy of the system is conserved:

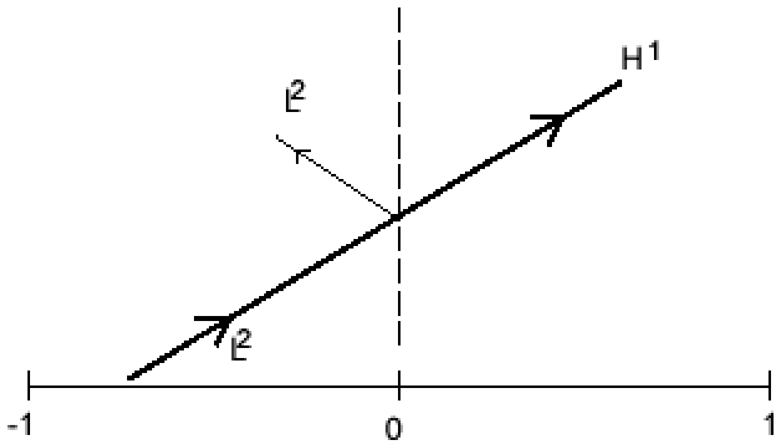
$$
\begin{aligned}
E(t) &= E_-(t) + E_+(t) + E_m(t) \\
E_- &= \frac{1}{2}\int_{-1}^{0}\left[\mid u_t^- \mid^2 + \mid u_x^- \mid^2\right] dx \\
E_+ &= \frac{1}{2}\int_{0}^{1}\left[\mid u_t^+ \mid^2 + \mid u_x^+ \mid^2\right] dx \\
E_m &= \frac{1}{2}\mid z' \mid^2 = \frac{1}{2}\left|\frac{dz}{dt}\right|^2 .
\end{aligned}
$$

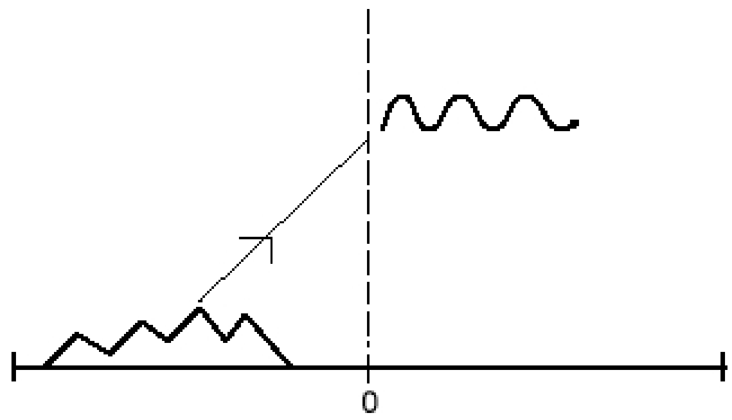
Accordingly the system is welposed in the energy space:

$$H = H_0^1(-1, 1) \times L^2(-1, 1) \times \mathbb{R}$$

This can be seen by all standard methods: semigroups, Galerkin,...

The first two spaces indicate where the overall solution, glueing both the motion of the left and right strings, is. The last space $\mathbb{R}$ stands for the velocity of the point mass. A careful (but simple) analysis of the propagation of waves when crossing the point-mass (using d'Alembert's formula) shows that a unexpected property holds: Waves are regularized by one derivative when crossing the mass.





As a consequence of this fact: The system is well-posed in an asymmetric space: finite energy solutions to one side, plus, one more derivative in L^2 in the other one:

$$\left[H^1(-1,\, 0) \times L^2(-1,\, 0)\right] \times \left[H^2(0,\, 1) \times H^1(0,\, 1)\right].$$

OF COURSE, THIS PROPERTY IS NOT TRUE FOR THE CLASSICAL CONSTANT COEFFICIENT WAVE EQUATION BECAUSE OF THE PROPAGATION OF SINGULARITIES.

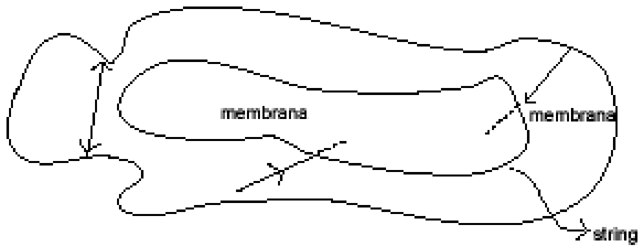
CONCLUSION:

- The system is more singular (δ_0 enters in the density).
- But this ends up affecting the system in a positive way since it can be solved in a richer class of spaces: the symmetric and the anti-symmetric ones.

A FEW QUESTIONS:

- Can this atypical asymmetric well-posedness property be predicted by classical methods?

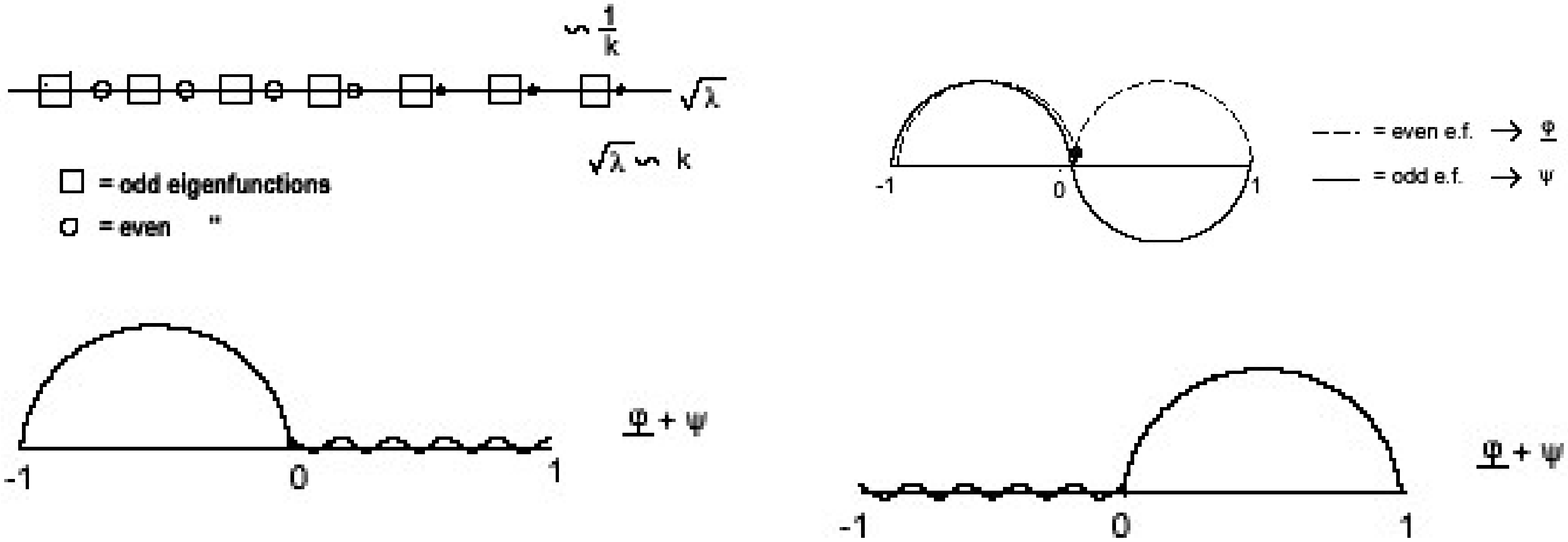
- What about several $d-s$?



Asymmetric microlocal spaces?

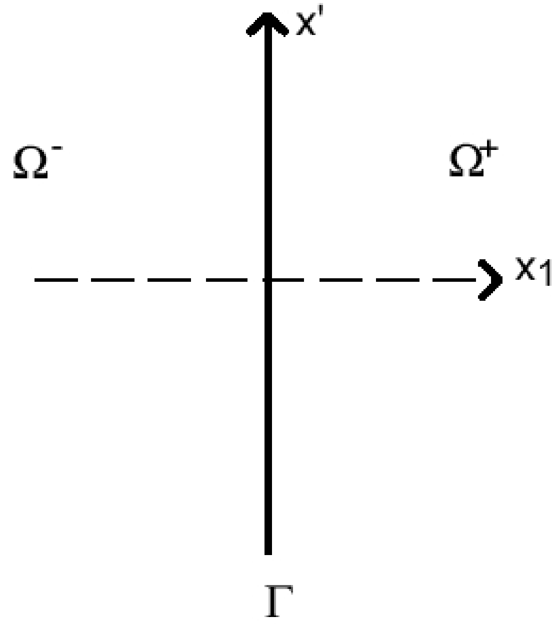
- What about other equations? (beams, heat,. . .)
- What's the impact on nonlinear problems?
- Are there numerical algorithms that reproduce asymmetry?

In the present simple case the asymmetry can be seen on the spectrum in which the classical asymptotic high frequency expansions for Sturm-Lionville problems with smooth coefficients fails.



EXTENSION TO $2-D$

(H. Koch & E. Z., in progress)


x'
Ω⁻
Ω⁺
x1
Γ

$$\begin{cases} u_{tt}^- - \Delta u^- = 0 & \Omega^- \times (0, \infty) \\ v_{tt} - c^2 \Delta' v = [u_{x_1}] & \Gamma \times (0, \infty) \\ u_{tt}^+ - \Delta u^+ = 0 & \Omega^+ \times (0, \infty) \end{cases}$$

The energy

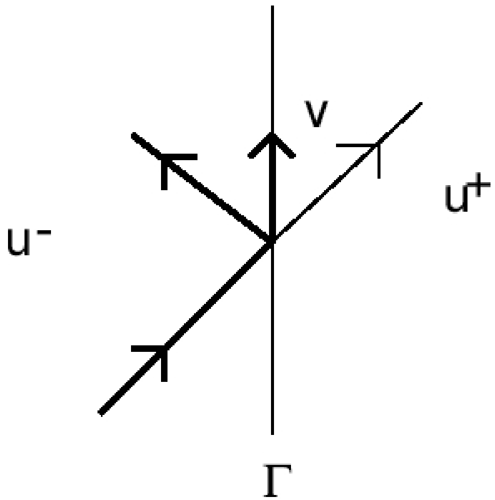
$$E(t) = \frac{1}{2} \int_{\Omega^-} \left(| u_t^- |^2 + | \nabla u^- |^2 \right) dx + \frac{1}{2} \int_{\Omega^+} \left(| u_t^+ |^2 + | \nabla u^+ |^2 \right) dx$$
$$+ \frac{1}{2} \int_{\Gamma} \left(| v_t |^2 + | \nabla' v |^2 \right) dx'$$

is constant in time and the system is well-posed in the natural energy space.

$$\left[H^1(\Omega^-) \times L^2(\Omega^-) \right] \times \left[H^1(\Gamma) \times L^2(\Gamma) \right] \times \left[H^1(\Omega^+) \times L^2(\Omega^+) \right]$$

IS THE SYSTEM WELL-POSED IN ANY ASYMMETRIC SPACE?

PLANE WAVE ANALYSIS ALLOWS EXCLUDING THIS PROPERTY IN A NUMBER OF SITUATIONS



THE GENERAL PRINCIPLE: Incoming waves (say in the domain Ω^-), when reaching the boundary are partly reflected. But part of the wave may also be absorbed by the interface. Moreover, the incoming wave may also produce by transmission a new wave in the second medium Ω^+.

A careful reading of the resulting diagram may lead to important conclusions about the possibility of being well-posed in asymmetric spaces.

$$\begin{aligned} u^- &= e^{i(x\cdot\xi+\tau t)/\varepsilon} + a e^{i(x\cdot\tilde{\xi}+\tau t)/\varepsilon} = \text{incoming} + \text{reflected wave} \\ u^+ &= b e^{i(x\cdot\xi+\tau t)/\varepsilon} = \text{ transmission wave} \\ v &= b e^{i(x'\cdot\xi'+\tau t)/\varepsilon} = \text{absorved wave} \end{aligned}$$

$$\xi = (\xi_1, \xi'); \, \tilde{\xi} = (-\xi_1, \xi'); \, \tau^2 = \mid \xi \mid^2 .$$

$a =$ reflection coefficient.

$b =$ transmission coefficient = absorption coefficient.

Continuity condition on the interface:

$$1 + a = b$$

Note that the restriction of all waves to the interface $x_1 = 0$ is an oscillatory tangential wave of the form $e^{i(x' \cdot \xi' + \tau t)/\varepsilon}$ up to a constant multiplicative factor.

$$\Box_c' v = \left(\partial_t^2 - c^2\Delta'\right) v = -\frac{b}{\varepsilon^2}\left(\tau^2 - c^2 \mid \xi' \mid^2\right) e^{i(x'\xi' + \tau t)/\varepsilon}$$

$$[u_{x_1}] = u_{x_1}^+ - u_{x_1}^- = \frac{2a\xi_1 i}{\varepsilon} e^{i(x'\xi' + \tau t)/\varepsilon}$$

Consequently, in view of the equation $v_{tt} - c^2\Delta' v = [u_{x_1}]$, and the fact that $\tau^2 = |\xi_1|^2 + |\xi'|^2$,

$$-\frac{b}{\varepsilon^2}\left(\tau^2 - c^2 \mid \xi' \mid^2\right) = -\frac{b}{\varepsilon^2}\left(\xi_1^2 + (1 - c^2) \mid \xi' \mid^2\right) = \frac{2a\xi_1 i}{\varepsilon}$$

$$a = -\frac{\xi_1^2 + (1 - c^2) \mid \xi' \mid^2}{2i\varepsilon\xi_1 + \left(\xi_1^2 + (1 - c^2) \mid \xi' \mid^2\right)}$$

$$b = \frac{2i\varepsilon\xi_1}{2i\varepsilon\xi_1 + \xi_1^2 + (1 - c^2) \mid \xi' \mid^2}$$

We have to distinguish the following three cases, depending on the velocity of propagation of the waves on the interface: $c = 1$, $c < 1$ or $c > 1$.

The case $\boxed{c = 1}$

$$b = \frac{2i\varepsilon}{\xi_1 + 2i\varepsilon}$$

$$\xi_1 \sim \varepsilon \Rightarrow b \sim \frac{2i}{1 + 2i} \quad \textbf{EFFECTIVE TRANSMISSION.}$$

This indeed shows that, nearly tangent high frequency waves, when reaching the interface, produce by transmission a new wave, the amplitude being of the order of the incoming one.

This is incompatible with any well-posedness property in asymmetric spaces. But it is compatible with the result we found in $1-D$. Indeed, in $1-D$ all waves reaching the interface are perpendicular to it. This corresponds to the case where $\xi' = 0$, i.e. $\tau = \pm\xi_1$. Then,

$$b = \frac{2i\varepsilon}{\xi_1 + 2i\varepsilon} \sim \varepsilon \sim 0,$$

which is compatible with a gain of one derivative in the transmission.

Indeed, $[\varepsilon]^{-1} \sim \partial$, in view of the structure of the waves under consideration: $e^{i(x\cdot\xi+\tau t)/\varepsilon}$.

$\boxed{c < 1}$

$$\xi_1^2 + (1 - c^2) \mid \xi' \mid^2 \geqslant \alpha \mid \xi \mid^2$$

$$b = \frac{2i\varepsilon\xi_1}{2i\varepsilon\xi_1 + \xi_1^2 + (1 - c^2) \mid \xi' \mid^2} \sim \varepsilon \to 0.$$

NO TRANSMISSION.

This is compatible with the well-posedness of the system in an asymmetric space, with a gap of one derivative from one side to another.

$\boxed{c > 1}$

In this case there are directions in which the quadratic denominator of the formulas we found vanishes.

$$\xi_1^2 + (1 - c^2) \mid \xi' \mid^2 = 0$$

$$\mid \xi_1 \mid = \sqrt{\frac{c^2 - 1}{1 + c^2}}$$

In that direction

$$b = 1$$

COMPLETE TRANSMISSION.

Once more, this is incompatible with well-posedness on asymmetric spaces.

Theorem (H, Koch & E. Z., 2004):

- The problem may not be well-posed in asymmetric spaces when $c \geqslant 1$.

- The problem is indeed well-posed in asymmetric spaces when $c < 1$:

$$\underbrace{H^1(\Omega^-) \times L^2(\Omega^-)}_{u^-} \times \underbrace{H^2(\Gamma) \times H^1(\Gamma)}_{v} \times \underbrace{H^2(\Omega^+) \times H^1(\Omega^+)}_{u^+}$$

PROOF:

- The negative result is a consequence of the plane wave analysis above;

- The positive one requires of a more careful microlocal analysis.