Dissipative wave equations: theory, numerics & optimal design

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1. Introduction
2. Geometric issues on the stabilization of waves
3. Characterizing the exponential decay rate
4. Optimal design of dampers
5. Numerical approximation
6. Numerical viscosity

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Motivation

Feedback or closed-loop **stabilization** of wave like equations is a classical problem in Control Theory with important applications in: noise reduction, complex flexible structures, etc.

The wave equation is the simplest model among many other conservative PDE's arising in Mechanics for which this issue is relevant.

The property of a system of being stabilizable is closely related to other control theoretical concepts as that of **observability** which concerns the possibility of getting estimates on the total energy of vibrations in terms of partial measurements done through the stabilizing mechanism.

The topic is also very close to the existing literature on the qualitative properties of infinite dimensional dynamical systems (attractors, inertial manifolds,....)

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An example: Boundary stabilization of the wave equation

Let Ω be a bounded domain of $\mathbf{R}^n$, $n \geq 1$, with boundary Γ of class C^2 and Γ_0 be an open and non-empty subset of Γ.

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma_1 = (\Gamma \setminus \Gamma_0) \times (0, \infty) \\ \frac{\partial y}{\partial \nu} = -y_t & \text{on} \quad \Sigma_0 = \Gamma_0 \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

The energy is then of the form

$$E(t) = \frac{1}{2} \int_\Omega \left[|y_t|^2 + |\nabla y|^2 \right] dx$$

and satisfies the energy dissipation law

$$\frac{dE(t)}{dt} = - \int_{\Gamma_0} |y_t|^2 d\Gamma.$$

A variant: Internal stabilization. Let ω be an open subset of Ω. Consider:

$$\begin{cases} y_{tt} - \Delta y = -y_t 1_\omega & \text{in} \quad Q = \Omega \times (0,\infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0,\infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega, \end{cases}$$

where 1_ω stands for the characteristic function of the subset ω. The energy dissipation law is then

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx.$$

Question: Do they exist $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0), \quad \forall t \geq 0,$$

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for all solution of the dissipative system?

This is equivalent to an observability property: There exists $C > 0$ and $T > 0$ such that

$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dxdt.$$

In other words, the exponential decay property is equivalent to showing that the dissipated energy within an interval $[0, T]$ contains a fraction of the initial energy, uniformly for all solutions. This estimate, together with the energy dissipation law, shows that

$$E(T) \leq \sigma E(0)$$

with $0 < \sigma < 1$. Accordingly the semigroup map $S(T)$ is a strict contraction. By the semigroup property one deduces immediately the exponential decay rate.

Note that, for dissipative semigroups, the following alternative holds: Either $||S(t)|| = 1$ for all $t \geq 0$ or $||S(T)|| < 1$ for some $T > 0$ and then the energy decays exponentially as $t \to \infty$.

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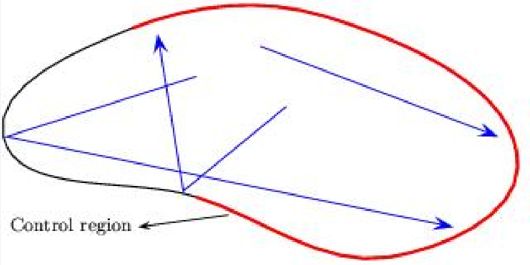
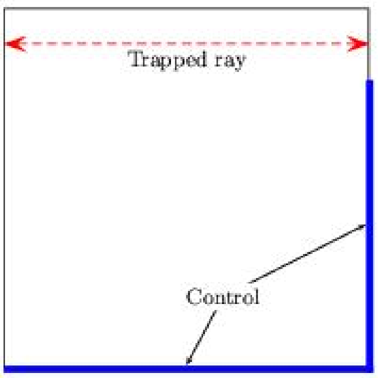
The observability inequality and, accordingly, the exponential decay property holds if and only if the support of the dissipative mechanism, Γ_0 or ω, satisfies the so called the Geometric Control Condition (GCC) (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch,...)



Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.



The Geometric Control Condition is not satisfied, whatever $T > 0$ is, in the square domain when the control is located on a subset of two consecutive sides of the boundary, leaving a subsegment uncontrolled. There is an horizontal a ray that bounces back and forth for all time perpendicularly on two points of the vertical boundaries where the control does not act.

When the GCC fails, the uniform exponential decay property does not hold. In that case one only gets a logarithmic decay rate for solutions with initial data in $H^2 \times H^1$. [1] This result is sharp in general and it is in agreement with the exponential rate of concentration of gaussian beams along non-observed rays.

[1] L. Robbiano, Fonction de coût et contrôle des solutions des équations hyperboliques, Asymptotic Anal., **10** (1995), 95–115.

Often, in particular, in the context of internal stabilization and essentally also in the boundary stabilization problem, it suffices to prove the observability property for the undamped system:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ \varphi = 0 & \text{on} \quad \Sigma = \Gamma \times (0, T) \\ \varphi(x,0) = \varphi^0(x), \varphi_t(x,0) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases}$$

The problem is then reduced to showing that:

$$E_0 \leq C(\Gamma_0, T) \int_{\Gamma_0 \times (0,T)} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\sigma dt \quad ?$$

A sharp discussion of this inequality requires of Microlocal analysis. Partial results may be obtained by means of multipliers: $x \cdot \nabla\varphi$, φ_t, φ,...

Similar problems arise in Control, Optimal Design and in Inverse Problems Theory and common techniques need to be developed.

The simplest and most systematic way for addressing this problem is the use of multipliers. More precisely, when multiplying the wave equation by $(x - x_0) \cdot \nabla\varphi$, φ and φ_t, the following identity is obtained:

$$TE_0 + \int_\Omega \left[\varphi_t (x - x_0) \cdot \nabla\varphi + \frac{n-1}{2}\varphi\right] dx \Big|_0^T = \frac{1}{2}\int_0^T \int_\Gamma (x - x_0) \cdot \nu \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt.$$

Out of it, we deduce that

$$(T - 2R)E_0 \leq \frac{R}{2}\int_0^T \int_{\Gamma(x_0)} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt.$$

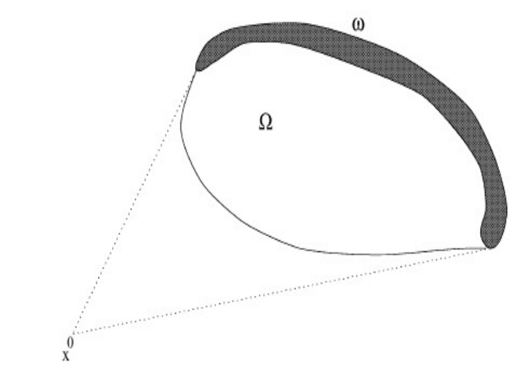
where

$$\Gamma(x_0) = \{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}; \; R = ||x - x_0||_{L^\infty(\Omega)}.$$

Note that, in view of the previous identity, by taking limits as T tends to ∞ it follows that

$$E_0 = \lim_{T \to \infty} \frac{1}{2T} \int_0^T \int_\Gamma ((x - x_0) \cdot \nu) \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt.$$

It would be interesting to see whether this has some interpretation in microlocal terms. Note however that the weight $(x - x_0) \cdot \nu$ changes sign.



Subset ω of Ω for which stabilization holds. ω is the intersection of Ω with a neighborhood of a subset of the boundary of the form $\Gamma(x_0)$.

Given a subdomain ω (or Γ_0) for which the stabilization problem holds, it is natural to address the problem of optimizing the profile of the damping potential $a = a(x)$ to enhance the exponential decay rate. Consider

$$\begin{cases} y_{tt} - \Delta y = -a(x) y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

Then, for any $a > 0$ the exponential decay property holds:

$$E(t) \le C e^{-\gamma_a t} E(0), \quad \forall t \ge 0.$$

Obviously, the exponential decay rate γ_a depends on the damping potential a.

It is therefore natural to analyze the nature of the mapping $a \to \gamma_a$. [2]

[2] At this point we should not address the, also very interesting, problem of the dependence of the decay rate on the geometry of the subdomain ω.

The first intuition is that this map should be monotonic: Is it really true that larger damping potentials a lead to greater exponential decay rates? The answer is negative: overdamping.

This can be easily checked when the damping acts everywhere in the domain

$$y_{tt} - \Delta y + k y_t = 0.$$

Then, as $k \to \infty$, γ_k, the exponential decay rate, tends to zero. Indeed, the eigenvalues of the system are

$$\lambda_{\pm}(\mu) = \frac{-k \pm \sqrt{k^2 - 4\mu}}{2},$$

μ being any eigenvalue of the Dirichlet laplacian in Ω. It is easy to see that, for any μ fixed, as $k \to \infty$, $Re(\lambda_+(\mu)) \to 0$.

Moreover, within the class of constant damping potentials, the optimal one is $k = 2\sqrt{\mu_1}$, for which the exponential decay rate is

$$\gamma = 2\sqrt{\mu_1}.$$

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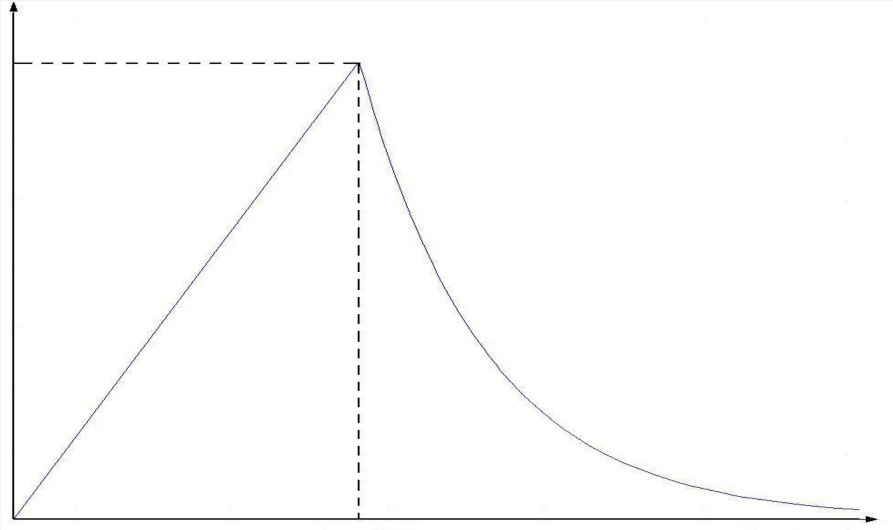

What about the case of variable damping potentials?

- $1-d$.
  In one space dimension the exponential decay rate coincides with the spectral abscissa within the class of BV damping potentials. For large eigenvalues $Re(\lambda) \sim -\int_\omega a(x)dx/2$. Consequently, the exponential decay rate is then limited (Cox-Zuazua, CPDE, 1993):

$$\gamma_a \leq \int_\omega a(x)dx.$$

  But, as we have seen, in the frame of constant potentials, there is another limitation due to **overdamping**. Despite of this, the following surprising result was proved (Castro-Cox, SICON, 2001): The decay rate may be made arbitrarily large by approximating singular potentials of the form $a(x) = 2/x$ for the space interval $\Omega = (0,1)$.

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This potential plays the role of transparent boundary conditions for which solutions achieve the equilibrium in finite time:

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } (0,1) \times (0,\infty) \\ y = 0 & \text{at } x = 1, t \geq 0 \\ y_x - y_t = 0 & \text{at } x = 0, t \geq 0 \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in } (0,1), \end{cases}$$

In this case:

$$y \equiv 0, \qquad \text{for } \ t \geq 2,$$

and the exponential decay rate is arbitrarily large.

In the multidimensional case the situation is even more complex. In this case it is not longer true that the spectral absicssa characterizes the exponential decay rate. There are actually two quantities that enter in such characterization (G. Lebeau, 1996):

- The spectral abscissa;
- The minimum asymptotic average (as $T \to \infty$) of the damping potential along rays of Geometric Optics.

  The later is in agreement with the intuition we gain through the analysis of the GCC: If a ray escapes the damping region the exponential decay property fails. Similarly, if a ray stays for a very short time interval within the decay region, then the dissipative mechanism is very weak on the solutions localized on it.

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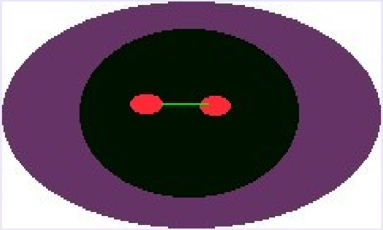

This is a typical situation in which the spectral abscissa does not suffice to capture the decay rate. The damping mechanism is active on the outer neighborhood of the exterior boundary. When the domain is the ellipsoid this produces the exponential decay. But, in the presence of the two holes, the exponential decay rate is lost, due to the existence of a trapped ray that never meets the damping region. In this case the decay rate is zero but the spectrum is not essentially affected if the holes are small enough. Thus the spectrum is unable to characterize the null decay rate.

The optimal design of the damping potential with constraints (size, shape, etc.) is still widely open.

- Hébrard-Henrot, SCL, (2003) show the complexity of the problem in the $1-d$ case for small amplitude damping potentials located on the union of a finite number of intervals.
- Hébrard-Humbert, 2003): Optimization of the shape of ω in a square domain in view of the geometric optics quantity entering in the characterization of the decay rate.
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- A. Munch, 2005-2006: Numerical simulation of the optimal shape in multi-dimensional problems using level set methods.
- Asch-Lebeau, 2003[3] Numerical approximation using two-grid methods for capturing propagation along rays.

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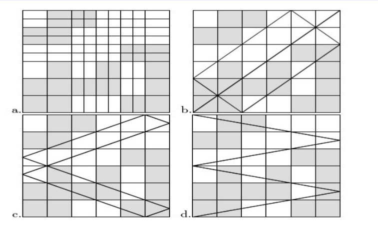

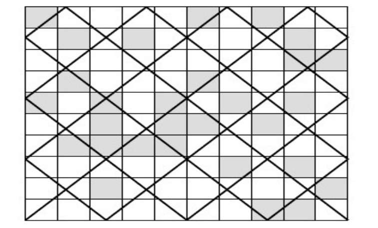

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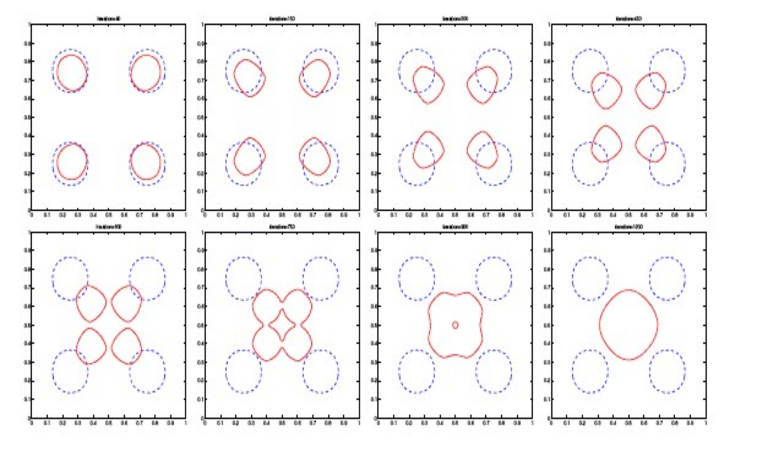

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Futhermore, not always all authors deal with the same problem. For instance, the optimal damping for a given initial datum may differ significantly from the optimal damping when considering globally all possible solutions...

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A different way of formulating the problem of the optimal damper consists in considering the total energy accumulated by the solutions for all $0 < t < \infty$, i. e. the quantity

$$F(y^0, y^1) = \int_0^\infty E(t)dt.$$

We can then measure the efficiency of the dasmper in terms of

$$e(a) = \max_{E(y^0, y^1) \leq 1} F(y^0, y^1).$$

The problem can be also formulated in terms of this function $e(a)$. How does the function $a \to e(a)$ behave? What is the minimizer of $e(a)$ in a given class of potentials a?
Similar issues can be raised in what concerns the dependence on the damping region.

A closely related problem that has been also investigated is that in which the damping potential changes sign. It is natural to analyze whether the exponential decay property holds for potentials with positive average.

The situation is quite complex. The following results are known:

- For damping potentials ka with k large and a changing sign there exist eigenfunctions for which the eigenvalue is real and positive. Thus, the system becomes unstable. [4]
- For εa with ε small enough, in one space dimension, the exponential decay property holds if and only if the following inequalities are satisfied:

$$\int a(x)\phi_k(x)dx > 0,$$

for all eigenfunction of the Dirichlet Laplacian ϕ_k. This assumption is much stronger than simply assuming that the average of a is positive. [5]

[4] J. López-Gómez, JDE.

[5] P. Freitas & E. Z. Stability results for the wave equation with indefinite

The extension of the decay result for small amplitude potentials to several space dimensions is an open problem. According the characterization above on the decay rate it is also natural to assume that the average of the potential along all rays of geometric optics is positive.

From a numerical point of view it is natural to address the following issues:

- To develop numerical schemes that reproduce the same decay properties of the original PDE;
- To analyze optimal design problems at the numerical level and see if the numerical optimal designs converge as the mesh-size tends to zero to the continuous optimal design.

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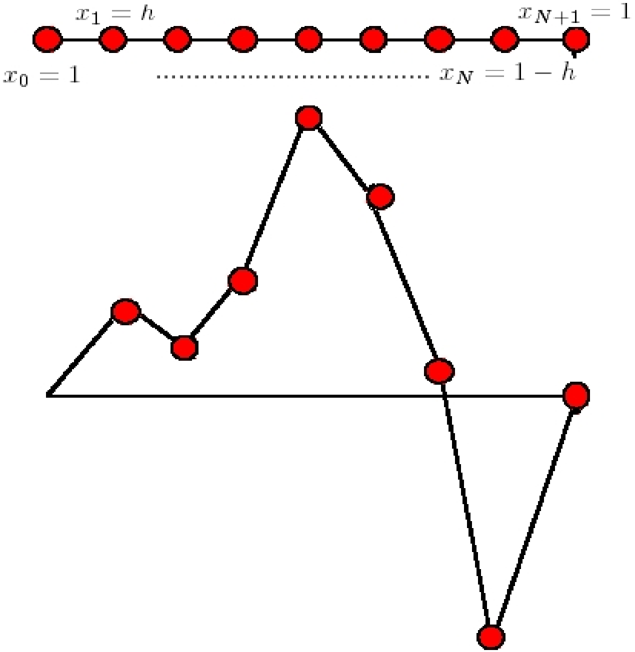
Let us analyze the finite difference semi-discrete approximation of the $1-d$ wave equation:

$$\begin{cases} y_j'' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] + a_j 1_{\omega_h} y_j' = 0, & t > 0,\ j = 1, \ldots, N \\ y_j(t) = 0, & j = 0,\ N+1,\ t > 0 \\ y_j(0) = \varphi_j^0,\ y_j'(0) = \varphi_j^1, & j = 1, \ldots, N. \end{cases}$$

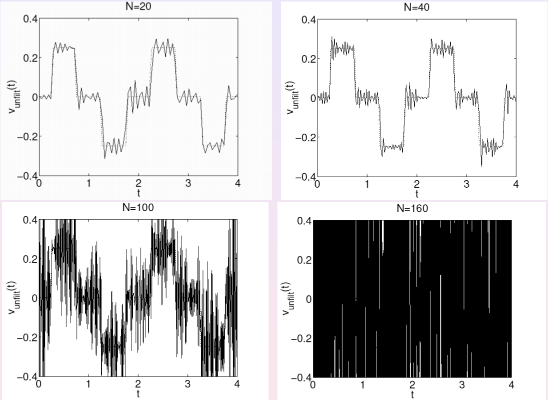
Here $h = 1/(N+1) > 0$ and consider the mesh $x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$ which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}],\ j = 0, ..., N.$



$x_1 = h$
$x_{N+1} = 1$
$x_0 = 1$
$x_N = 1 - h$

WARNING!!!!!!

E. Z., SIAM Review, 47 (2) (2005), 197-243.



Boundary controls diverge as the mesh size h tends to zero. This is a clear evidence that the finite-difference dynamics is not able to reproduce the behavior of the continuous wave equation.

WHY?

It suffices to analyze the behavior of the undamped equation: The Fourier series expansion shows the analogy between continuous and discrete dynamics.

Discrete solution:

$$\vec{\varphi} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h}\, t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h}\, t\right) \right) \vec{w}_k^h.$$

Continuous solution:

$$\varphi = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

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Recall that the discrete spectrum is as follows and converges to the continuous one:

$$\lambda_k^h = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0$$

$$w_k^h = (w_{k,1}, \ldots, w_{k,N})^T : w_{k,j} = \sin(k\pi jh), \ k, j = 1, \ldots, N.$$

The only relevant differences arise at the level of the dispersion properties and the group velocity. High frequency waves do not propagate, remain captured within the grid, without never reaching the boundary. This makes it impossible the uniform boundary control and observation of the discrete schemes as $h \to 0$.

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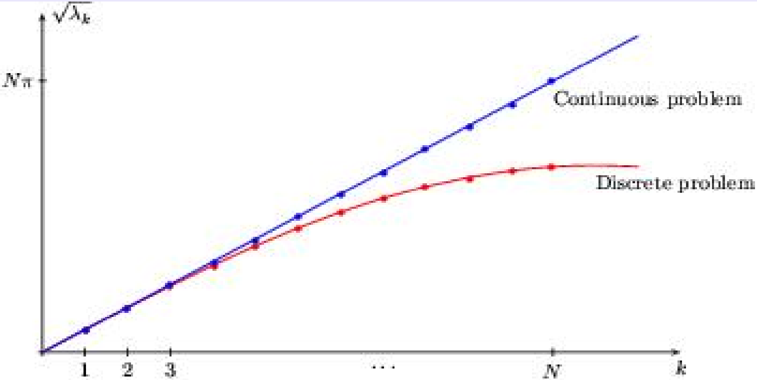
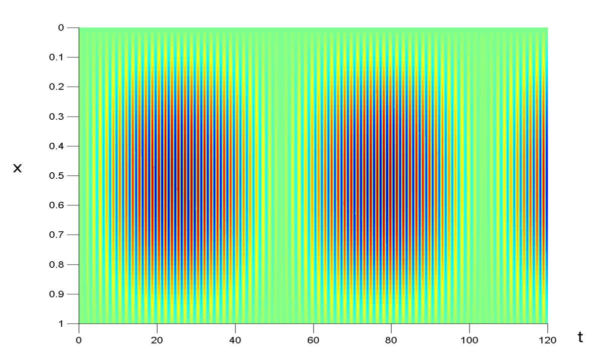


Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

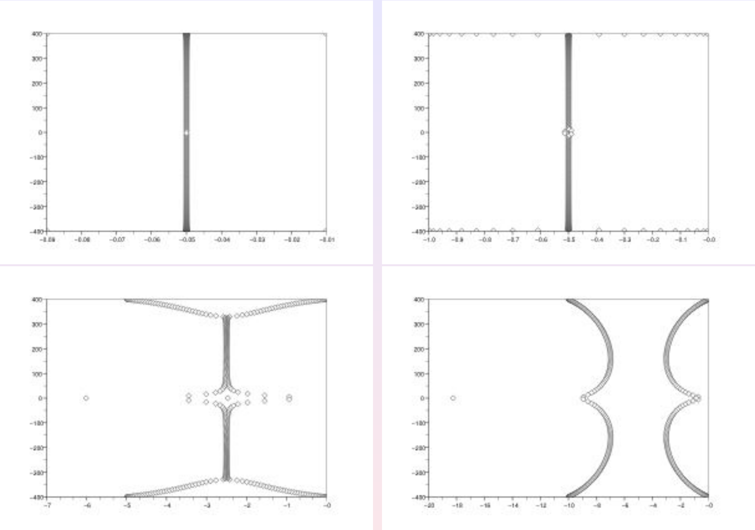
A NUMERICAL PHAMTOM

$$\vec{\varphi} = \exp\left(i\sqrt{\lambda_N(h)}\,t\right)\vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\,t\right)\vec{w}_{N-1}.$$

Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap: $\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h$.



$h = 1/61$, $(N = 60)$, $0 \leq t \leq 120$.



Semi-discrete spectrum with $N = 200$ *nodes and damping coefficients* 2.10^{-1}, 2, 10 *and* 20 *on the interval* $(1/2, 1)$.

As a consequences of this analysis we see that:

- There are high frequency solutions that propagate with a velocity which is of the order of h. This can be rigorously done using wave packets concentrated on the highest part of the spectrum in which the slope of the dispersion curve vanishes (Trefethen, SIAM Rev. 1982, Macià, 2004, Mielke, ARMA, 2006,...)
- These solutions only reach the boundary in a time of the order of $T_h \sim 1/h$. Thus, the numerical version of

$$E_0 \leq C(\Gamma_0, T) \int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial \nu}\right|^2 d\sigma dt$$

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- In view of this one may expect a very different behavior of the optimal design problem in the numerical and the continuous level. This has been observed and proved by Hébrard-Henrot, SICON, 2005.
- Note that, in view of the dispersion diagram, even when excluding the highest frequencies, one observes the existence of lots of numerical solutions for which the velocity of propagation is not the same as for the continuous wave eqation. Numerically the velocity of propagation can be 1/2, 1/4, 1/8,... depending on the points of the dispersion diagram in which the wave packets are concentrated.
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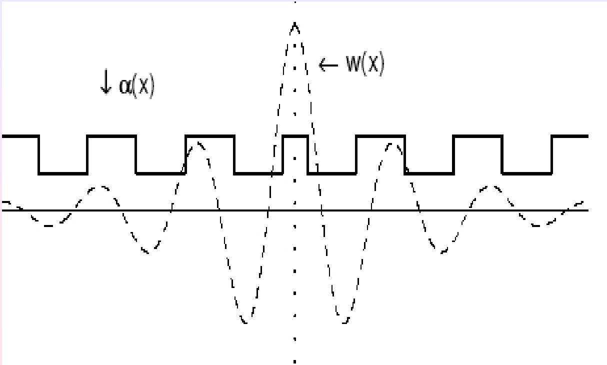
WELL KNOWN PHENOMENA FOR WAVES IN HIGHLY OSCILLATORY MEDIA



$$\varphi_{tt} - (\alpha(x)\varphi_x)_x = 0.$$

- F. Colombini & S. Spagnolo, Ann. Sci. ENS, 1989
- M. Avellaneda, C. Bardos & J. Rauch, Asymptotic Analysis, 1992.
- C. Castro & E. Z. Archive Rational Mechanics and Analysis, 2002.

DISCRETE MULTIPLIERS

For the continuous wave equation the key observability inequality was proved using multipliers. Let us do it at the disrete level: The multiplier $j(\varphi_{j+1} - \varphi_{j-1})$ (as a discrete version of $x\varphi_x$) for the discrete wave equation gives:

$$TE_h(0)+X_h(t)\Big|_0^T = \frac{1}{2}\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 dt + \frac{h}{2}\sum_{j=0}^{N}\int_0^T \mid \varphi_j' - \varphi_{j+1}' \mid^2 dt,$$

Note that

$$\frac{h}{2}\sum_{j=0}^{N}\int_0^T \mid \varphi_j' - \varphi_{j+1}' \mid^2 dt \sim \frac{h^2}{2}\int_0^T\int_0^1 |\varphi_{xt}|^2 dxdt.$$

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Recall that at the continuous level we got

$$E_0 = \lim_{T\to\infty} \frac{1}{2T} \int_0^T \int_\Gamma ((x - x_0)\cdot\nu) \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt.$$

However, at the discrete level the corresponding identity is:

$$E_h(0) = \lim_{T\to\infty} \frac{1}{2T} \left[\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 dt + \frac{h}{2} \sum_{j=0}^{N} \int_0^T \mid \varphi_j' - \varphi_{j+1}' \mid^2 dt. \right]$$

L. R. Tcheugoue-Tebou, E. Z., Numerische Math., 2003.

Consider the viscous numerical approximation scheme:

$$y_j'' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] - \left[y'_{j+1} + y'_{j-1} - 2y'_j\right] + a_j 1_{\omega_h} y'_j = 0.$$

This is the semi-discrete analog of

$$y_{tt} - \Delta y - h^2 \Delta y_t + a(x) 1_\omega y_t = 0.$$

The energy dissipation law is this time:

$$\frac{dE_h(t)}{dt} = -h \sum_{j \in \omega_h} a_j |y'_j|^2 - h^3 \sum_{j=0}^{N} \frac{|y'_{j+1} - y'_j|^2}{h^2}.$$

The right hand side terms reproduce the effect of the two damping terms in this scheme:

- The velocity damping, discrete version of $a(x)y_t$;
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Theorem: The decay rate of this viscous numerical scheme is uniform, independent of h. Furthermore, the scheme converges (of order 2) in the classical sense of numerical analysis.

This result has been later extended in various ways:

- The $1-d$ wave equation with boundary damping (L. R. Tcheugoue-Tebou, E. Z. 2003);
- Multi-dimensional problems (A. Munch-A. Pazoto. ESAIM:COCV, to appear.)
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- More general methods, including the finite element method on irregular meshes;
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- More general methods, including the finite element method on irregular meshes;
- The obtention, at the discrete level, sharp geometric conditions as the GCC;
- Wave equations with variable coefficients;
- Other models and systems: plate, beam and shell equations, Schrödinger, elasticity, Maxwell,...

But, even in the context of the constant coefficient wave equation, and with the finite-difference semi-discrete scheme above, several issues are still to be clarified:

- How to characterize the optimal decay rate. Can the results on the continuous wave equation be extended? The notion of bicharacteristic ray can be adapted to the present setting. But the damping is now made of two pieces: The discrete version of the velocity damping, and the viscous damping acting everywhere on the domain.
- Does the decay rate of the semi-discrete viscous scheme converge to the one of the continuous wave equation? *Even though this scheme provides a uniform exponential decay, it is unclear whether it yields the same rate of decay as $h \to 0$. Consequently, it is also highly unclear whether optimal dampers will converge as $h \to 0$. This is an interesting topic for further investigation.*

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- The possible consequences of this type of result in connection with the convergence of numerical optimal dampers towards continuous ones are also to be developed.
- Closely related also to other qualitative properties such as attractors, inertial manifolds, transversality,... (G. Raugel, R. Joly).
- This topic is closely linked with the theory of transparent boundary conditions and Perfectly Matching Layers (PML). Work in this direction, inspired in the ides presented here, is being developed: S. Ervedoza & E. Z.

  Note however that, in this case, the wave equation has to be written as a system of two equations of first order and that the damping term has to be added in both equations. In this way one gets a dissipative wave equation with a dispersive term for which overdamping phenomena do not occur.

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