Control and numerical simulation in large time horizons

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Motivation

In various fields of Science, Engineering and Industry control and design issues play often a key role.
Many of these issues have a great impact in our planet and quality of life:

- Seismic waves, earthquakes
- Environment: Floodings
- Optimal shape design in aeronautics
- Human cardiovascular system: the bypass
- Oil prospection and recovery
- Irrigation systems
- ....

From the perspective of the climate sciences, the following issues are particularly relevant:

- Accurate numerical simulations for large times
- Finite time horizon versus steady state control
- Robust control under systems uncertainties

And they can be only addressed combining a number of tools of Applied Mathematics:

- Partial Differential Equations: Models describing motion in the various fields of Mechanics: Elasticity, Fluids,...
- Numerical Analysis: Allowing to discretize these models so that solutions may be approximated algorithmically, with emphasis on long time accuracy.
- Control: Automatic and active control of processes to guarantee their best possible behavior and dynamics.
- Optimal Design: Design of shapes to enhance the desired properties (bridges, dams, airplanes,..)

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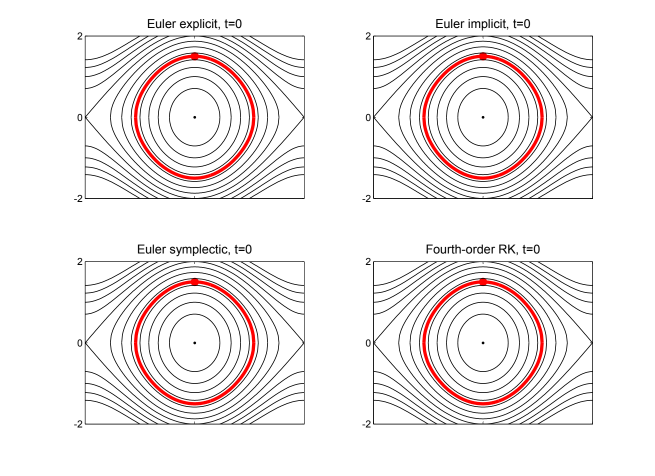
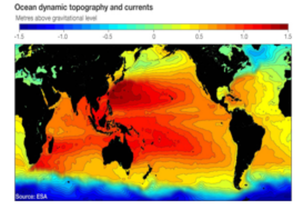
Geometric integration

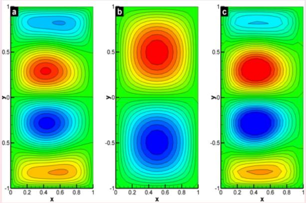
Numerical integration of the pendulum



Climate modelling

- Climate modeling is a grand challenge computational problem, a research topic at the frontier of computational science.
- Simplified models for geophysical flows have been developed aim to: capture the important geophysical structures, while keeping the computational cost at a minimum.
- Although successful in numerical weather prediction, these models have a prohibitively high computational cost in climate modeling.





Xu Wang, `www.ima.umn.edu/ wangzhu/`

Thames barrier

- The Thames Barrier's purpose is to prevent London from being flooded by exceptionally high tides and storm surges.
- A storm surge generated by low pressure in the Atlantic Ocean, past the north of Scotland may then be driven into the shallow waters of the North Sea. The surge tide is funnelled down the North Sea which narrows towards the English Channel and the Thames Estuary. If the storm surge coincides with a spring tide, dangerously high water levels can occur in the Thames Estuary. This situation combined with downstream flows in the Thames provides the triggers for flood defence operations.

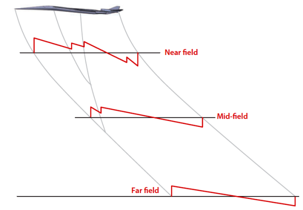
Tsunamis

- Some isolated waves (solitons) are large and travel without loss of energy.
- This is the case of tsunamis and rogue waves.

Warning: Hence, there is no use trying sending a counterwave to stop a tsunami!

Sonic boom

- Goal: the development of supersonic aircraft that are sufficiently quiet so that they can be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when it reaches the ground, (a) it can barely be perceived by the human ear, and (b) it results in disturbances to man-made structures that do not exceed the threshold of annoyance for a significant percentage of the population.



Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 26.

Joint work with L. Ignat & A. Pozo

Consider the 1-D conservation law with or without viscosity:

$$u_t + \left[u^2\right]_x = \varepsilon u_{xx}, x \in \mathbb{R}, t > 0.$$

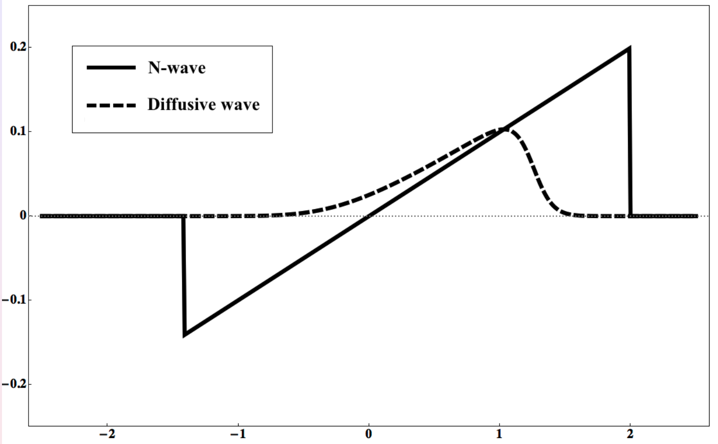
Then:

- If $\varepsilon = 0$, $u(\cdot, t) \sim N(\cdot, t)$ as $t \to \infty$;
- If $\varepsilon > 0$, $u(\cdot, t) \sim u_M(\cdot, t)$ as $t \to \infty$,

u_M is the constant sign self-similar solution of the viscous Burgers equation (defined by the mass M of u_0), while N is the so-called hyperbolic N-wave, defined as:

$$N(x,t) := \begin{cases} \frac{x}{t}, & \text{if } -2(pt)^{\frac{1}{2}} < x < (2qt)^{\frac{1}{2}} \\ 0 & \text{otherwise} \end{cases}$$

$$p := -2 \min_{y \in \mathbf{R}} \int_{\infty}^{y} u^0(x)dx, \quad q := 2 \max_{y \in \mathbf{R}} \int_{\infty}^{y} u^0(x)dx$$


N-wave
Diffusive wave
0.2
0.1
0
−0.1
−0.2
−2
−1
0
1
2

Conservative schemes

Let us consider now numerical approximation schemes

$$\begin{cases} u_j^{n+1} = u_n^j - \dfrac{\Delta t}{\Delta x}\left(g_{j+1/2}^n - g_{j-1/2}^n\right), & j \in \mathbf{Z}, \mathbf{n} > \mathbf{0}. \\ u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)dx, & j \in \mathbf{Z}, \end{cases}$$

The approximated solution u_Δ is given by

$$u_\Delta(t,x) = u_j^n, \quad x_{j-1/2} < x < x_{j+1/2}, \; t_n \le t < t_{n+1},$$

where $t_n = n\Delta t$ and $x_{j+1/2} = (j + \frac{1}{2})\Delta x$.

Is the large tine dynamics of these discrete systems, a discrete version of the continuous one?

3-point conservative schemes

1. Lax-Friedrichs

$$g^{LF}(u,v) = \frac{u^2+v^2}{4} - \frac{\Delta x}{\Delta t}\left(\frac{v-u}{2}\right),$$

2. Engquist-Osher

$$g^{EO}(u,v) = \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4},$$

3. Godunov

$$g^{G}(u,v) = \begin{cases} \min\limits_{w\in[u,v]} \frac{w^2}{2}, & \text{if } u \le v, \\ \max\limits_{w\in[v,u]} \frac{w^2}{2}, & \text{if } v \le u. \end{cases}$$

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Numerical viscosity

We can rewrite three-point monotone schemes in the form

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{(u_{j+1}^n)^2-(u_{j-1}^n)^2}{4\Delta x}=R(u_j^n,u_{j+1}^n)-R(u_{j-1}^n,u_j^n)$$

where the numerical viscosity R can be defined in a unique manner as

$$R(u,v)=\frac{Q(u,v)(v-u)}{2}=\frac{\lambda}{2}\Big(\frac{u^2}{2}+\frac{v^2}{2}-2g(u,v)\Big).$$

For instance:

$$R^{LF}(u,v)=\frac{v-u}{2},$$

$$R^{EO}(u,v)=\frac{\lambda}{4}(v|v|-u|u|),$$

$$R^{G}(u,v)=\begin{cases}\frac{\lambda}{4}\mathrm{sign}(|u|-|v|)(v^2-u^2), & v\le 0\le u,\\ \frac{\lambda}{4}(v|v|-u|u|), & \text{elsewhere.}\end{cases}$$

Properties

These three schemes are wel-known to satisfy the following properties:

- They converge to the entropy solution
- They are monotonic
- They preserve the total mass of solutions
- They are OSLC consistent:

$$\frac{u_{j-1}^n - u_{j+1}^n}{2\Delta x} \leq \frac{2}{n\Delta t}$$

- $L^1 \to L^\infty$ decay with a rate $O(t^{-1/2})$
- Similarly they verify uniform BV_{loc} estimates

Main result

Theorem (Lax-Friedrichs scheme)

Consider $u_0 \in L^1(\mathbf{R})$ and Δx and Δt such that $\lambda \Big| u^n \Big|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t/\Delta x$. Then, for any $p \in [1,\infty)$, the numerical solution u_Δ given by the Lax-Friedrichs scheme satisfies

$$\lim_{t\to\infty} t^{\frac{1}{2}(1-\frac{1}{p})} \Big| u_\Delta(t) - w(t) \Big|_{L^p(\mathbb{R})} = 0,$$

where the profile $w = w_{M_\Delta}$ is the unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = \frac{(\Delta x)^2}{2} w_{xx}, & x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$.

Main result

Theorem (Engquist-Osher and Godunov schemes)

Consider $u_0 \in L^1(\mathbf{R})$ and Δx and Δt such that $\lambda \left| u^n \right|_{\infty,\Delta} \leq 1$, $\lambda = \Delta t / \Delta x$. Then, for any $p \in [1, \infty)$, the numerical solutions u_Δ given by Engquist-Osher and Godunov schemes satisfy the same asymptotic behavior but for the hyperbolic $N - wave$ $w = w_{p_\Delta, q_\Delta}$ unique solution of

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, \quad x \in \mathbf{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \quad \lim_{t \to 0} \int_0^x w(t,z)dz = \begin{cases} 0, & x < 0, \\ -p_\Delta, & x = 0, \\ q_\Delta - p_\Delta, & x > 0, \end{cases} \end{cases}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$ and
$p_\Delta = -\min_{x \in \mathbb{R}} \int_{-\infty}^x u_\Delta^0(z)dz$ and $q_\Delta = \max_{x \in \mathbb{R}} \int_x^\infty u_\Delta^0(z)dz$.

Example

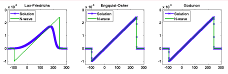
Let us consider the inviscid Burgers equation with initial data

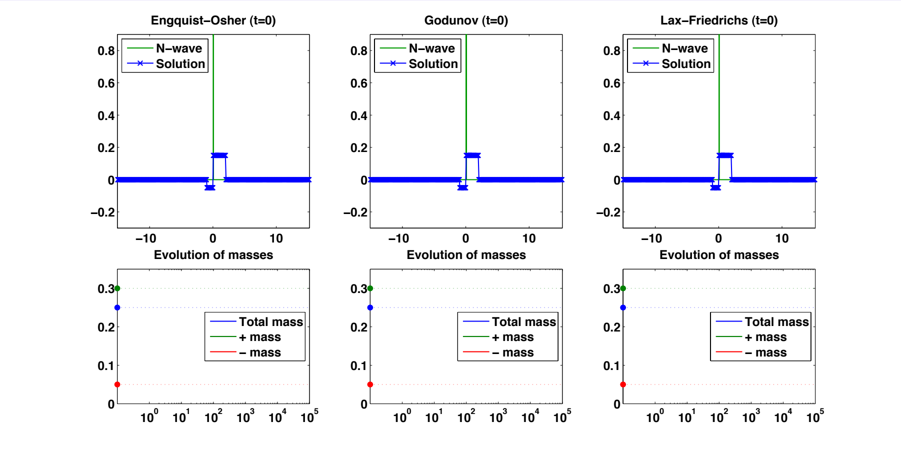
$$u_0(x) = \begin{cases} -0.05, & x \in [-1, 0], \\ 0.15, & x \in [0, 2], \\ 0, & \text{elsewhere.} \end{cases}$$

The parameters that describe the asymptotic N-wave profile are:

$$M = 0.25 \ , \quad p = 0.05 \quad \text{and} \quad q = 0.3.$$

We take $\Delta x = 0.1$ as the mesh size for the interval $[-350, 800]$ and $\Delta t = 0.5$. Solution to the Burgers equation at $t = 10^5$:




Engquist–Osher (t=0)
Godunov (t=0)
Lax–Friedrichs (t=0)
N–wave
Solution
Evolution of masses
Total mass
+ mass
– mass

Similarity variables

Let us consider the change of variables given by:

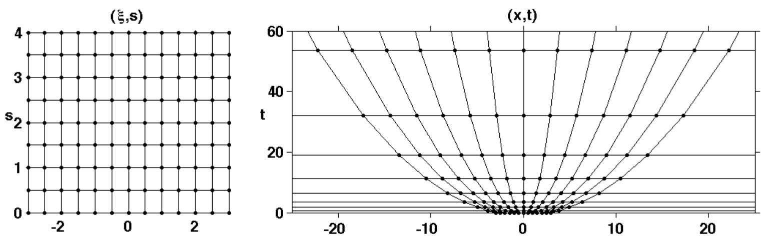
$$s = \ln(t+1), \qquad \xi = x/\sqrt{t+1}, \qquad w(\xi,s) = \sqrt{t+1}\; u(x,t),$$

which turns the continuous Burgers equation into

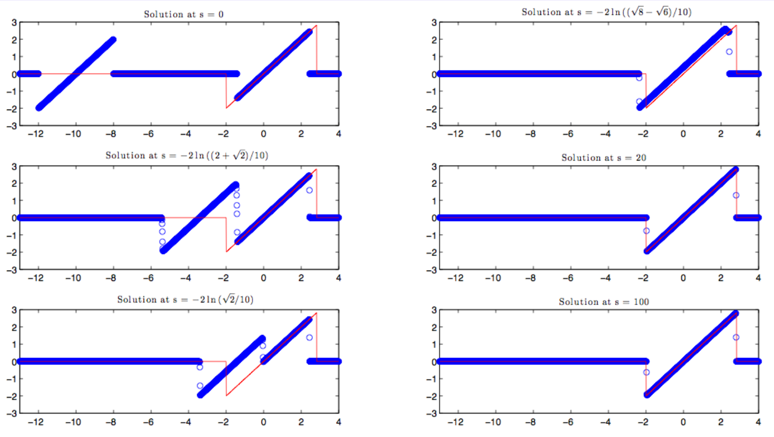
$$w_s + \left(\frac{1}{2}w^2 - \frac{1}{2}\xi w\right)_\xi = 0, \quad \xi \in \mathbf{R}, s > 0.$$

The asymptotic profile of the N-wave becomes a steady-state solution:

$$N_{p,q}(\xi) = \begin{cases} \xi, & -\sqrt{2p} < \xi < \sqrt{2q}, \\ 0, & \text{elsewhere}, \end{cases}$$



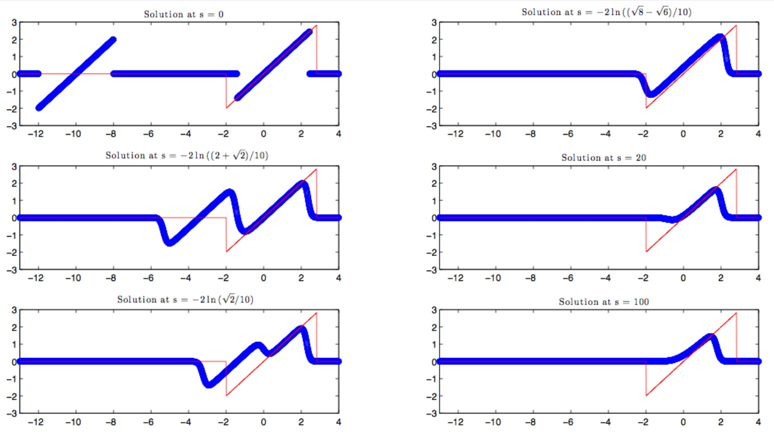
Examples



Convergence of the numerical solution using Engquist-Osher scheme (circle dots) to the asymptotic N-wave (solid line). We take $\Delta\xi = 0.01$ and $\Delta s = 0.0005$.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$.

Examples



Numerical solution using the Lax-Friedrichs scheme (circle dots), taking $\Delta\xi = 0.01$ and $\Delta s = 0.0005$. The N-wave (solid line) is not reached, as it converges to the diffusion wave.

Snapshots at $s = 0$, $s = 2.15$, $s = 3.91$, $s = 6.55$, $s = 20$ and $s = 100$.

Physical vs. Similarity variables

Comparison of numerical and exact solutions at $t = 1000$. We choose $\Delta\xi$ such that the $\left|\cdot\right|_{1,\Delta}$ error is similar. The time-steps are $\Delta t = \Delta x/2$ and $\Delta s = \Delta\xi/20$, respectively, enough to satisfy the CFL condition.

For $\Delta x = 0.1$:

	Nodes	Time-steps	$\left\|\cdot\right\|_{1,\Delta}$	$\left\|\cdot\right\|_{2,\Delta}$	$\left\|\cdot\right\|_{\infty,\Delta}$
Physical	1501	19987	0.0867	0.0482	0.0893
Similarity	215	4225	0.0897	0.0332	0.0367

For $\Delta x = 0.01$:

	Nodes	Time-steps	$\left\|\cdot\right\|_{1,\Delta}$	$\left\|\cdot\right\|_{2,\Delta}$	$\left\|\cdot\right\|_{\infty,\Delta}$
Physical	15001	199867	0.0093	0.0118	0.0816
Similarity	2000	39459	0.0094	0.0106	0.0233

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Joint work with M. Ersoy and E. Feireisl, JDE, 2013.

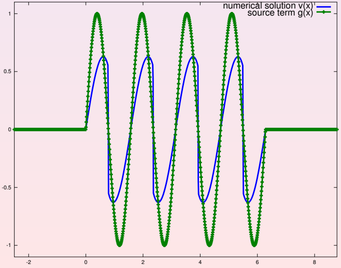
Consider the scalar steady driven conservation law

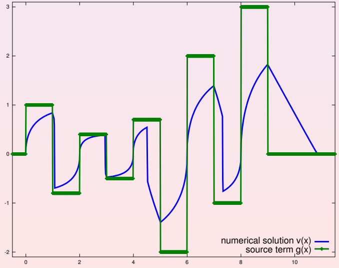
$$\partial_x[f(v(x))] + v(x) = g(x), \; x \in \mathbf{R}. \tag{1}$$

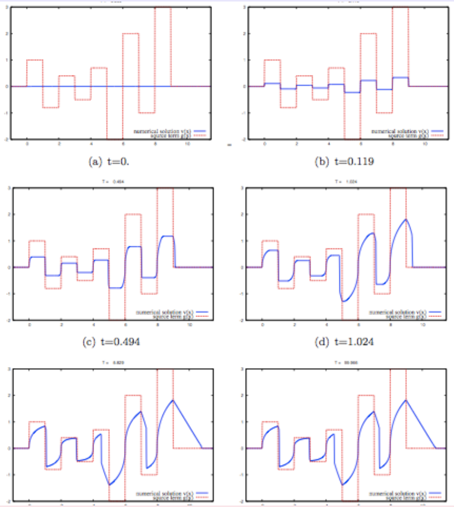
In the context of scalar conservation laws (nonlinear semigroups of L^1-contractions), these solutions can be viewed as limits as $t \to \infty$ of solutions of the evolution problem:

$$\partial_t u(t,x) + \partial_x f(u(t,x)) + u(t,x) = g(x), \; u(0,x) = u^0. \tag{2}$$

Entropy L^1-solutions exist and are unique in both cases.







Convergence towards the stateady state as $t \to \infty$

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Time evolution control problem. Joint work with A. Porretta

Consider the finite dimensional dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0 \end{cases} \tag{3}$$

where $A \in M(N, N)$, $B \in M(N, M)$, the control $u \in L^2(0, T; \mathbf{R}^M)$, and $x_0 \in \mathbf{R}^N$.
Given a matrix $C \in M(N, N)$, and some $x^* \in \mathbf{R}^N$, consider the optimal control problem

$$\min_u \; J^T(u) = \frac{1}{2}\int_0^T (|u(t)|^2 + |C(x(t) - x^*)|^2) dt \, .$$

There exists a unique optimal control $u(t)$ in $L^2(0, T; \mathbf{R}^M)$, characterized by the optimality condition

$$u = -B^* p \, , \qquad \begin{cases} -p_t + A^* p = C^* C(x - x^*) \\ p(T) = 0 \end{cases} \tag{4}$$

The steady state control problem

The same problem can be formulated for the steady-state model

$$Ax = Bu.$$

Then there exists a unique minimum $\bar{u}$, and a unique optimal state $\bar{x}$, of the stationary "control problem"

$$\min_{u} \ J_s(u) = \frac{1}{2}(|u|^2 + |C(x - x^*)|^2), \qquad Ax = Bu, \tag{5}$$

which is nothing but a constrained minimization in $\mathbf{R}^N$; and by elementary calculus, the optimal control $\bar{u}$ and state $\bar{x}$ satisfy

$$A\bar{x} = B\bar{u}, \quad \bar{u} = -B^*\bar{p}, \qquad \text{and} \quad A^*\bar{p} = C^*C(\bar{x} - x^*).$$

We assume that

$$\text{The pair } (A, B) \text{ is controllable,} \tag{6}$$

or, equivalently, that the matrices A, B satisfy the Kalman rank condition

$$Rank\left[B\ AB\ A^2B \ldots\ A^{N-1}B\right] = N\ . \tag{7}$$

Then there exists a linear stabilizing feedback law $L \in M(M, N)$ and c, $\mu > 0$ such that

$$\begin{cases} x_t + Ax = B(Lx) \\ x(0) = x_0 \end{cases} \Longrightarrow \quad |x(t)| \leq ce^{-\mu t}|x_0| \qquad \forall t > 0\,. \tag{8}$$

Concerning the cost functional, we assume that the matrix C is such that

$$\text{The pair } (A, C) \text{ is observable} \tag{9}$$

which means that the following algebraic condition holds:

$$Rank\left[C\ CA\ CA^2 \ldots\ CA^{N-1}\right] = N\ . \tag{10}$$

Under the above controllability and observability assumptions, we have the following result.

Theorem

Assume that (7) and (10) hold true. Then we have

$$\frac{1}{T} \min_{u \in L^2(0,T)} J^T \quad \xrightarrow{T \to \infty} \quad \min_{u \in \mathbf{R}^N} J_s$$

and

$$\frac{1}{T} \int_0^T \left(|u(t) - \bar{u}|^2 + |C(x(t) - \bar{x})|^2 \right) dt \to 0$$

where $\bar{u}$ is the optimal control of J_s and $\bar{x}$ the corresponding optimal state.

In particular, we have

$$\frac{1}{(b-a)T} \int_{aT}^{bT} x(t)\, dt \to \bar{x} \quad , \quad \frac{1}{(b-a)T} \int_{aT}^{bT} u(t)\, dt \to \bar{x}$$

for every $a, b \in [0, 1]$.

Time scaling = Singular perturbations

Note that the problem in the time interval $[0, T]$ as $T \to \infty$ can be rescaled into the fixed time interval $[0, 1]$ by the change of variables $t = Ts$.
In this case the evolution control problem takes the form

$$\varepsilon x_s + Ax = Bu, \; s \in [0, 1].$$

In the limit as $\varepsilon \to 0$ the steady-state equation emerges:

$$Ax = Bu.$$

This becomes a classical singular perturbation control problem.
Note however that, in this setting, the role that the controllability and observability properties of the system play is much less clear than when dealing with it as $T \to \infty$.

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Motivation

Often the data of the system under consideration or even the PDE (its parameters) describing the dynamics are not fully known.

In those cases it is relevant to address control problems so to ensure that the control mechanisms:

- Are robust with respect to parameter variations.
- Guarantee a good control theoretical response of the system at least in an averaged sense.

Parameter dependent control problem

Consider the finite dimensional linear control system

$$\begin{cases} x'(t) = A(\nu)x(t) + Bu(t),\ 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{11}$$

In (11) the (column) vector valued function $x(t,\nu) = \big(x_1(t,\nu), \ldots, x_N(t,\nu)\big) \in \mathbb{R}^N$ is the state of the system, $A(\nu)$ is a $N \times N-$matrix and $u = u(t)$ is a M-component control vector in $\mathbb{R}^M$, $M \leq N$.

- The matrix A is assumed to depend on a parameter ν in a continuous manner. To fix ideas we will assume that the parameter ν ranges within the interval $(0, 1)$.
- Note however that the control operator B is independent of ν, the same as the initial datum $x_0 \in \mathbb{R}^N$ to be controlled.

Averaged controllability

Given a control time $T > 0$ and a final target $x^1 \in \mathbb{R}^N$ we look for a control u such that the solution of (11) satisfies

$$\int_0^1 x(T,\nu)d\nu = x^1. \tag{12}$$

This concept of averaged controllability differs from that of simultaneous controllability in which one is interested on controlling all states simultaneously and not only its average.

When A is independent of the parameter ν, controllable systems can be fully characterized in algebraic terms by the rank condition

$$rank\left[B,\ AB,\dots,A^{N-1}B\right] = N. \tag{13}$$

The following holds:

Theorem

Averaged controllability holds if and only the following rank condition is satisfied:

$$rank\left[B, \int_0^1 [A(\nu)] d\nu B, \ldots, \int_0^1 [A(\nu)]^{N-1} d\nu B, ...\right] = N. \qquad (14)$$

Several remarks are in order:

- Note that, contrary to the case where A is fully determined, independent of ν, in (14) we are considering all the averages of all the powers of $A(\nu)$ to any order. This is so since, in the present setting, Cayley-Hamilton's Theorem cannot be applied to ensure that $\int_0^1 [A(\nu)]^N d\nu$ can be written as a linear combination $\int_0^1 [A(\nu)]^k d\nu$ for $k = 0, 1, ..., N-1$, as it happens in the case where A is fully determined, independent of ν.
- The averaged rank condition can be interpreted and simplified when all the matrices $A(\nu)$ are multiples of the same constant matrix A:

$$A(\nu) = \sigma(\nu) A. \tag{15}$$

In this case

$$\int_0^1 [A(\nu)]^k d\nu = \int_0^1 [\sigma(\nu)]^k d\nu A^k, \; k \geq 0$$

$$\left[B, \int_0^1 [A(\nu)] d\nu B, \ldots, \int_0^1 [A(\nu)]^{N-1} d\nu B, ...\right]$$
$$= \left[B, \int_0^1 [\sigma(\nu)] d\nu AB, \ldots, \int_0^1 [\sigma(\nu)]^{N-1} d\nu A^{N-1} B, ...\right] \tag{16}$$

Averaged observability

The adjoint system depends also on the parameter ν:

$$\begin{cases} -\varphi'(t) = A^*(\nu)\varphi(t), & t \in (0,\ T) \\ \varphi(T) = \varphi^0. \end{cases} \tag{18}$$

Note that, for all values of the parameter ν, we take the same datum for φ at $t = T$. This is so because our analysis is limited to the problem of averaged controllability.

We have the duality identity

$$< \int_0^1 x(T,\nu)d\nu, \varphi^0 >= \int_0^T < u(t), \int_0^1 B^*\varphi d\nu > dt + < x^0, \int_0^1 \varphi(0,\nu)d\nu \tag{19}$$

Accordingly, the controllability condition (12) can be recast as follows:

$$< x^1, \varphi^0 >= \int_0^T < u(t), B^* \int_0^1 \varphi d\nu > dt + < x^0, \int_0^1 \varphi(0,\nu)d\nu >, \forall \varphi^0 \in \mathbb{R}$$

This is the Euler-Lagrange equation associated to the minimization of the following quadratic functional over the class of solutions of the adjoint system:

$$J\left(\varphi^0\right) = \frac{1}{2}\int_0^T \left| B^* \int_0^1 \varphi(t,\nu) d\nu \right|^2 dt - < x^1, \varphi^0 > + < x^0, \int_0^1 \varphi(0,\nu) d\nu >$$

The functional $J : \mathbb{R}^N \to \mathbb{R}$ is trivially continuous and convex.
Let us assume for the moment that the functional J has a minimizer $\hat{\varphi}^0$.
This would automatically lead to the control

$$u(t) = B^* \int_0^1 \hat{\varphi}(t,\nu) d\nu, \tag{20}$$

$\hat{\varphi}$ being the solution of the parametrized adjoint system associated to the minimizer $\hat{\varphi}^0$. For the existence of the minimizer of J it is sufficient to prove the coercivity of the functional J:

$$|\varphi^0|^2 + |\int_0^1 \varphi(0,\nu) d\nu|^2 \leq C \int_0^T \left| B^* \int_0^1 \varphi(t,\nu) d\nu \right|^2 dt, \; \forall \varphi^0 \in \mathbb{R}^N. \tag{21}$$

Since we are working in the finite-dimensional context, inequality (21) is equivalent to the following uniqueness property:

$$B^* \int_0^1 \varphi(t,\nu) d\nu = 0 \quad \forall t \in [0, T] \Rightarrow \varphi^0 \equiv 0. \tag{22}$$

To analyze this inequality we use the following representation of the adjoint state:

$$\varphi(t,\nu) = \exp[A^*(\nu)(T-t)]\varphi^0.$$

Then, the fact that

$$B^* \int_0^1 \varphi(t,\nu) d\nu = 0 \quad \forall t \in [0, T]$$

is equivalent to

$$B^* \int_0^1 \exp[A^*(\nu)(t-T)] d\nu \, \varphi^0 = 0 \quad \forall t \in [0, T].$$

The result follows using the time analyticity of the matrix exponentials, and the classical argument consisting in taking consecutive derivatives at time $t = T$.

Comparison with simultaneous controllability

The notion of averaged observability differs and is weaker than the one of simultaneous controllability. Consider the simplest case:

$$\begin{cases} x_j'(t) = A_j x_j(t) + Bu(t), \ 0 < t < T, \\ x_j(0) = x_j^0, \end{cases} \tag{23}$$

with $j = 1, 2$. Contrarily to the problem of averaged controllability now the initial data of the system also depends on j.
The problem of simultaneous control requires

$$x_1(T) = x_2(T) = 0. \tag{24}$$

For, we need to consider the adjoint system with different possible data at $t = T$ for its different components:

$$-\varphi_j'(t) = A_j^* \varphi_j(t), t \in (0, \ T); \varphi_j(T) = \varphi_j^0, \ \text{ for } \ j = 1, 2. \tag{25}$$

The corresponding observability problem then reads

$$|\varphi_1^0|^2 + |\varphi_2^0|^2 \leq C \int_0^T \left|B^*[\varphi_1 + \varphi_2]\right|^2 dt, \; \forall \varphi_j^0 \in \mathbb{R}^N, \; j = 1, 2, \qquad (26)$$

For averaged controllability it is sufficient this to hold in the particular case where $\varphi_1^0 = \varphi_2^0$.

Table of Contents

Lots to be done on:

- Development of numerical algorithms preserving large time asymptotics for nonlinear PDEs (other works of our team on dispersive equations, dissipative wave equations,...)
- The analysis of how time-evolution control and steady-state one are related for nonlinear problems.
- Robust and averaged control of uncertain systems.

All this needs to be made in a multidisciplinary environment so to assure impact on Climate Sciences