Wave propagation on networks

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Outline

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Motivation

UNDERSTANDING WAVE PROPAGATION IS RELEVANT IN MANY WAYS AND IN MANY FIELDS

- Noise reduction in cavities and vehicles.
- Laser control in Quantum mechanical and molecular systems.
- Seismic waves, earthquakes.
- Flexible structures.
- Environment: the Thames barrier.
- Optimal shape design in aeronautics.
- Human cardiovascular system.
- Oil prospection and recovery.
- Irrigation systems.

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THE GOAL

Understand how waves propagate in complex media, in a way that we can use it in designing them or in determining its properties out of (partial) observations and measurements.

Of course there is a large variety of possible problems to be considered:

- Linear/ non linear;
- Deterministic / Stochastic
- Time-domain / Frequency domain
- Regular / Irregular media
- Direct / Inverse problems
- Discrete / Continuous models

And, accordingly, the variety of mathematics to be used and developed is huge too.

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THE 1-D STRING

A natural way of formulating these problems is as follows:

> *Can we recover full information about solutions, and the media in which they evolve out measurements done somewhere on it (its boundary, for instance)?*

Consider the $1-d$ wave equation with fixed-end conditions:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\ \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

Is it true that???

$$E(0) \le C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt,$$

where the energy of solutions, which is conserved in time, is

$$E(t) = \frac{1}{2} \int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2 \right] dx = E(0), \ \ \forall 0 \le t \le T.$$

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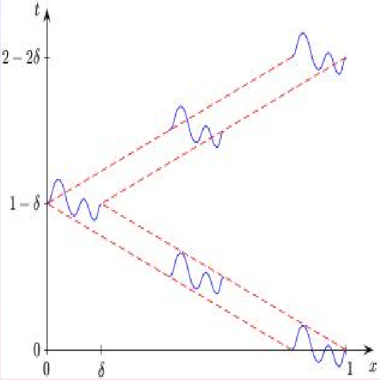
The answer

The inequality holds iff $T \geq 2$:



$$E(0) \leq C(T) \int_0^T |\varphi_x(1,t)|^2 dt$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

WHY?

- Explicit D'Alembert's formula:

$$\varphi(x,t) = f(x+t) + g(x-t);$$

- Fourier series:

Ingham's Theorem. (1936) *Let $\{\mu_k\}_{k\in\mathbf{Z}}$ be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \forall k \in \mathbf{Z}.$$

Then, for any $T > 2\pi/\gamma$ there exists $C(T,\gamma) > 0$ such that

$$\frac{1}{C(T,\gamma)} \sum_{k\in\mathbf{Z}} | a_k |^2 \leq \int_0^T \left| \sum_{k\in\mathbf{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in\mathbf{Z}} | a_k |^2$$

for all sequences of complex numbers $\{a_k\} \in \ell^2$

$$\varphi(x,t) = \sum_{k\in\mathbf{Z}} a_k e^{ik\pi t} \sin(k\pi x).$$

$$\varphi_x(1,t) = \sum_{k\in\mathbf{Z}} (-1)^k k a_k e^{ik\pi t}$$

Furthermore, if $T > 2$,

$$\int_0^T \Big| \sum_{k\in\mathbf{Z}} (-1)^k k a_k e^{ik\pi t} \Big|^2 dt \sim \sum_{k\in\mathbf{Z}} k^2 |a_k|^2.$$

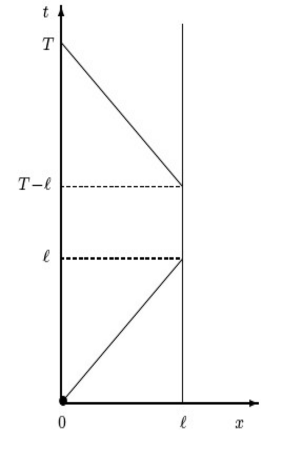
On the other hand,

$$E_0 \sim \sum_{k\in\mathbf{Z}} k^2 |a_k|^2.$$

In fact, in this case, using the orthogonality properties of trigonometric polynomials we can prove that the same holds if $T = 2$.

- Sidewise energy propagation:

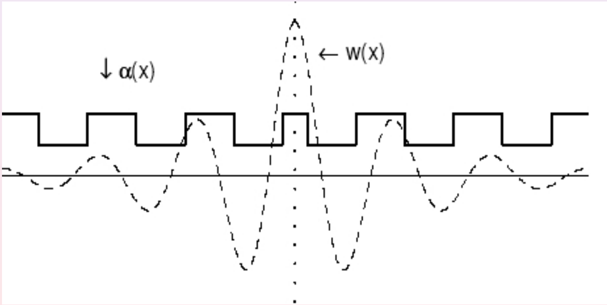
$$\varphi_{tt} - \varphi_{xx} = \varphi_{xx} - \varphi_{tt}.$$



But, except for this case

Proving this kind of inequalities is rarely an easy matter.

In fact, our intuition fails for rough coefficients. These results fail to hold when coefficients do not have one derivative (say BV-coefficients). In particular one can build $C^{0,\alpha}$ coefficients for which the above inequalities fail because of the existence of localized waves.



$$\rho(x)\varphi_{tt} - (a(x)\varphi_x)_x = 0.$$

- F. Colombini & S. Spagnolo, Ann. Sci. ENS, 1989
- M. Avellaneda, C. Bardos & J. Rauch, Asymptotic Analysis, 1992.
- C. Castro & E. Z. Archive Rational Mechanics and Analysis, 2002.

Similar difficulties appear when dealing with numerical schemes (discrete media$\sim$ irregular media) and, as we shall see, also for graphs and/or networks.

Pointwise measurements in the interior

Take $x_0 \in (0,1)$. How much energy we can recover from measurements done on x_0?

$$\varphi(x_0, t) = \sum_{k \in \mathbf{Z}} a_k e^{ik\pi t} \sin(k\pi x_0).$$

Furthermore, if $T > 2$,

$$\int_0^T \left| \sum a_k e^{ik\pi t} \sin(k\pi x_0) \right|^2 dt \sim \sum \sin^2(k\pi x_0) |a_k|^2.$$

Obviously, two cases:

- The case $x_0 \notin \mathbf{Q}$: $\sin^2(k\pi x_0) \neq 0$ for all k and the quantity under consideration is a norm, i.e. it provides information on all the Fourier components of the solutions.
- The case: $x_0 \in \mathbf{Q}$, some of the weights $\sin^2(k\pi x_0)$ vanish an the quadratic term is not a norm.

But, even if, $\sin^2(k\pi x_0) \neq 0$ for all k, the norm under consideration is not the L^2-one we expect!!!!

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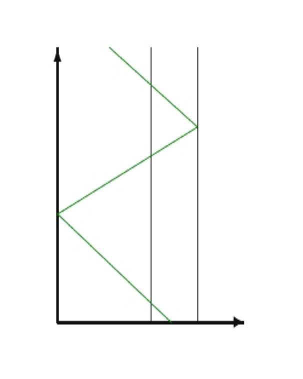
Can we explain this in terms of rays, and the propagation of waves (and antiwaves)?



If x_0 is rational we can build a finite number of rays and anti-rays that always intersect in x_0 for the time interval $(0, 2)$ of periodicity of solutions.

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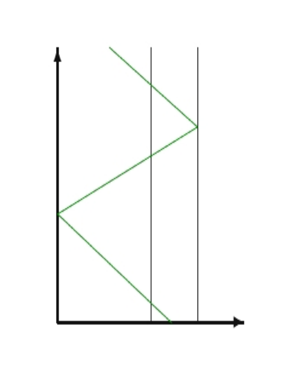

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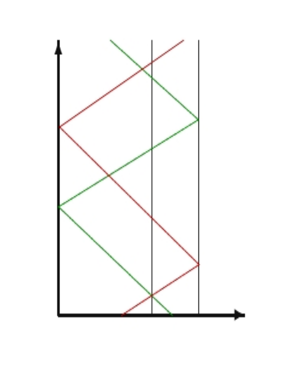

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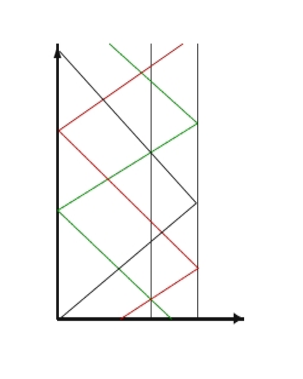

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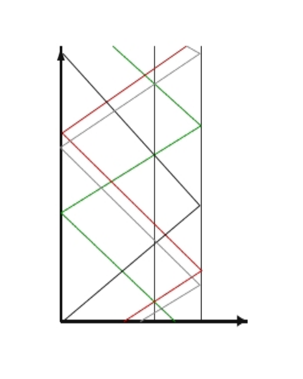

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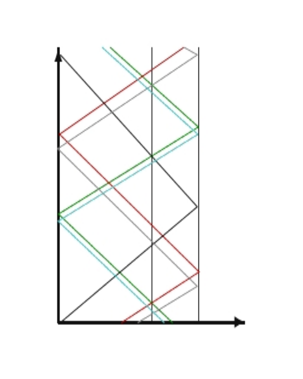

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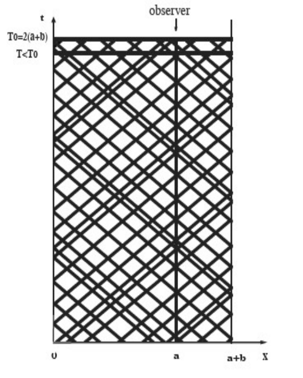

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The case x_0 irrational.

Can we expect that

$$\left|\sin(k\pi x_0)\right| \geq \alpha > 0, \quad \forall k?$$

This is impossible!!!!
Indeed, this would mean that

$$\left|k\pi x_0 - m\pi\right| \geq \beta$$

for all $k, m \in \mathbf{Z}$. And this is obviously false.
For suitable irrational numbers x_0 we can get

$$\left|k\pi x_0 - m\pi\right| \geq \beta/k.$$

And this is the best we can get.
In this case we get an observation inequality but with a loss of one derivative.
For some other irrational numbers (Liouville ones, for instance) the degeneracy may be arbitrary fast.

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Conclusion: Making measurements in the interior of the domain is a much less robust process than doing it on the boundary. In some cases we fail to capture all the Fourier components and, even if we are able to do it, this does not happen in the energy space, but there is a loss of at least one derivative.

$$E_{x_0} \leq C \int_0^T |\varphi_x(1,t)|^2 dt.$$

Note that, in this case, the answer does not come only by an analysis of the propagation of characteristics but that the diophantine approximation theory enters.

Note that the time needed for this to hold is $T = 2$ and not the characetristic time one could expect: $2(x_0, 1 - x_0)$.

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The same issue can be addressed using D'Alembert's formula. Then, the problem reads as follows:

$$\psi(t+\ell_1)-\psi(t-\ell_1) = f(t) \in L^2_t; \quad \psi(t+\ell_2)-\psi(t-\ell_2) = g(t) \in L^2_t.$$

Can we get an estimate of the form

$$||\psi||_* \leq C\left[||f||_{L^2_t} + ||g||_{L^2_t}\right]????$$

Again, the answer depends on whether ℓ_1/ℓ_2 is rational or not, and the class of irrationality to which it belongs.

Outline

How much energy can we recover from one or several external vertices?

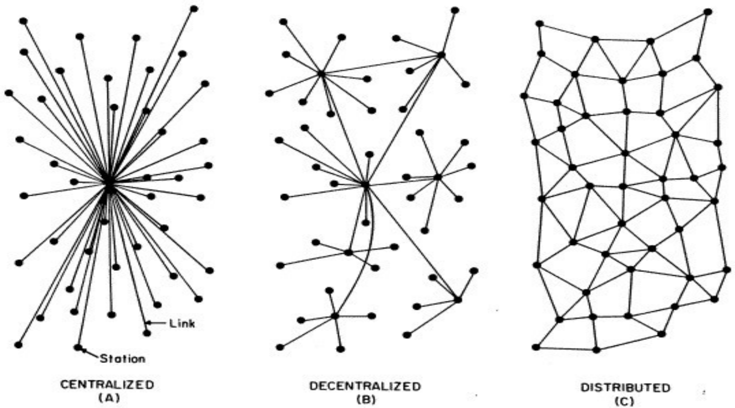


FIG. I — Centralized, Decentralized and Distributed Networks

The model for the vibration of a network:

$$
\begin{array}{ll}
\phi^i_{xx} - \phi^i_{tt} = 0 & \text{in } \mathbb{R} \times [0, \ell_i], \quad i = 1, ..., M, \\
\phi^{i(j)}(t, \mathbf{v}_j) = 0 & t \in \mathbb{R}, \quad j = 1, ..., N, \\
\phi^i(t, \mathbf{v}) = \phi^j(t, \mathbf{v}) & t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \ i, j \in I_{\mathbf{v}}, \\
\sum_{i \in I_{\mathbf{v}}} \partial_n \phi^i(t, \mathbf{v}) = 0 & t \in \mathbb{R}, \quad \mathbf{v} \in \mathcal{V}_{\mathcal{M}}, \\
\phi^i(0, x) = \phi^i_0(x), \quad \phi^i_t(0, x) = \phi^i_1(x) & x \in [0, \ell_i], \quad i = 1, ..., M;
\end{array}
$$

This is simply the wave equation on the network:

$$\Phi_{tt} - \Delta_{\mathcal{N}} \Phi = 0,$$

with null Dirichlet boundary conditions on the external vertices and initial conditions.

The energy of the system is conserved and it reads

$$E(t) = \sum_i \int_0^{\ell_i} \left[|\phi_t(x, t)|^2 + \phi_x(x, t)|^2 \right] dt,$$

where the sum runs over the set of edges in the nework.

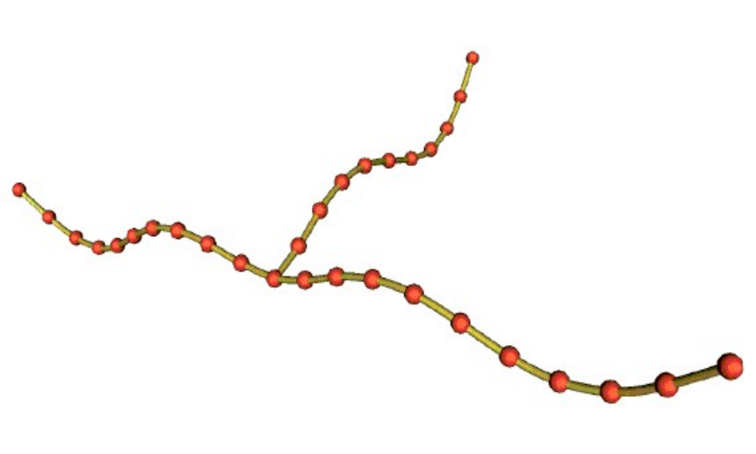
Three examples in increasing complexity:

- The star;
- The tree;
- General planar network.

Outline

The star (tripoid).

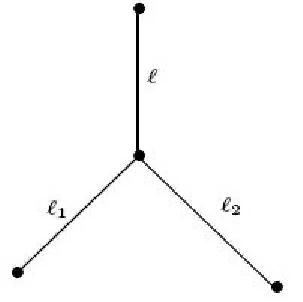
Generically vibrations excite all components. This means that measurements done in any of the vertices should give global information on solutions.

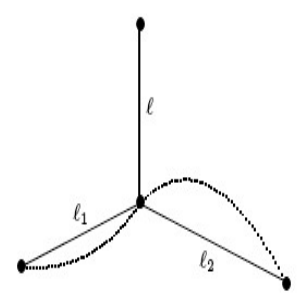


The star:

In some cases the possibility of making global measurements from only single external vertex fails!!!!

This happens, for instance, when ℓ_1/ℓ_2 is rational.





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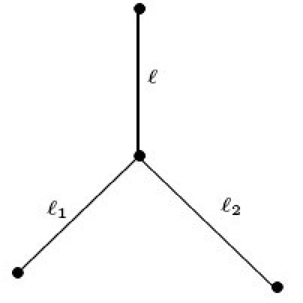

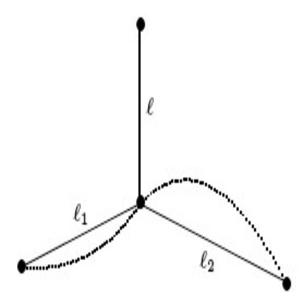

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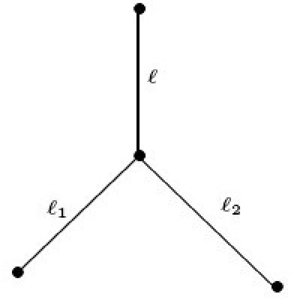
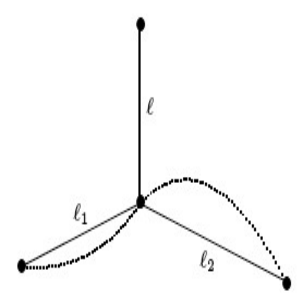
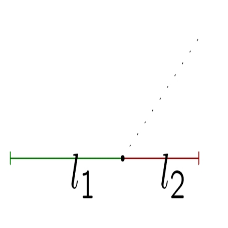




Assume ℓ_1/ℓ_2 irrational. How much of the total energy can we recover by means of measurements done on $x = 0$?

The answer is exactly the same as for the $1-d$ string with observation in some interior point: Even if some irrationality property is imposed one always looses a number of derivatives in the observation process. The minimal time is $T > 2(\ell + \ell_1 + \ell_2)$. Why this analogy?

- The energy propagates along the observed string;
- We end up getting a single string composed of ℓ_1 and ℓ_2 and with information on the joint.



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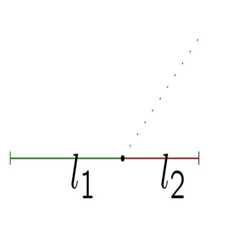

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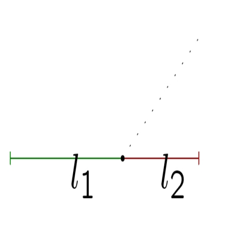

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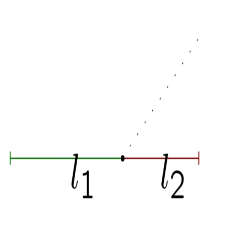
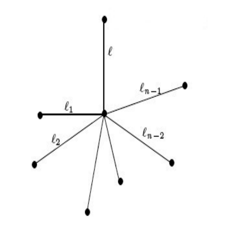
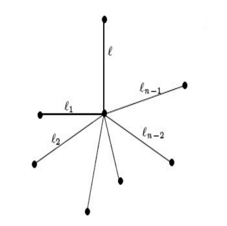




Similar results holds for general stars:

- As soon as two lenghts are mutually irrational one looses a number of Fourier components;
- One can recover all Fourier components under irrationality conditions on the ratio of ach pair of lengths.
- The precise energy we recover depends on diophantine approximation properties.
- The results are sharp, as one can show by a wave + anti-wave argument following characteristics.



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What about tress?

It is well known that (Lagnese-Leugering-Schmidt, Avdonin, ...) if one makes measurements on all but one external vertex, then one can recover the total energy of solutions.

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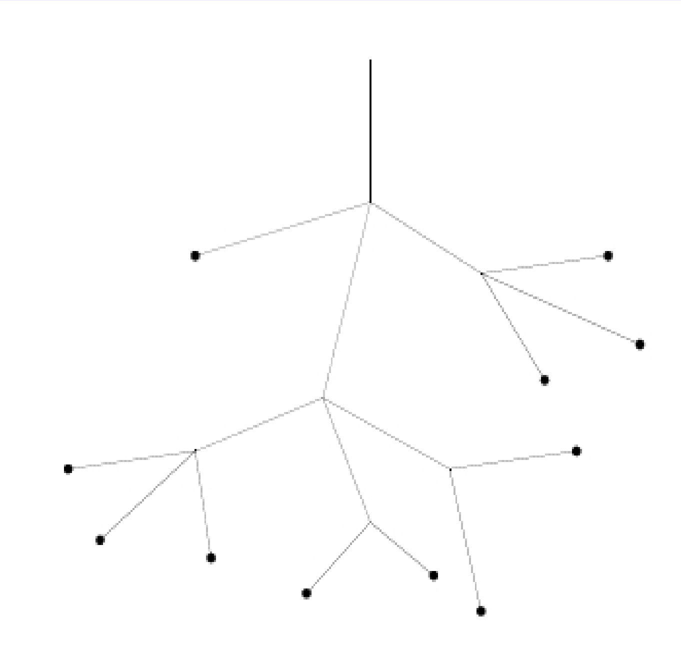

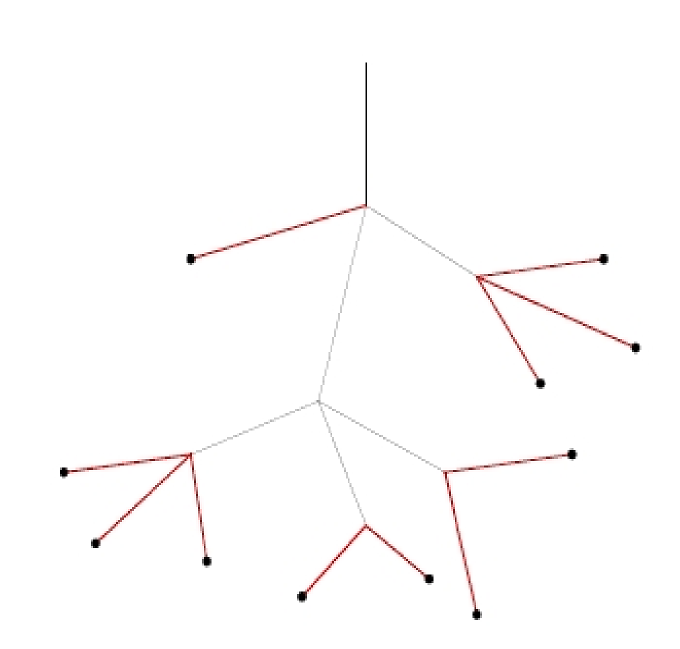

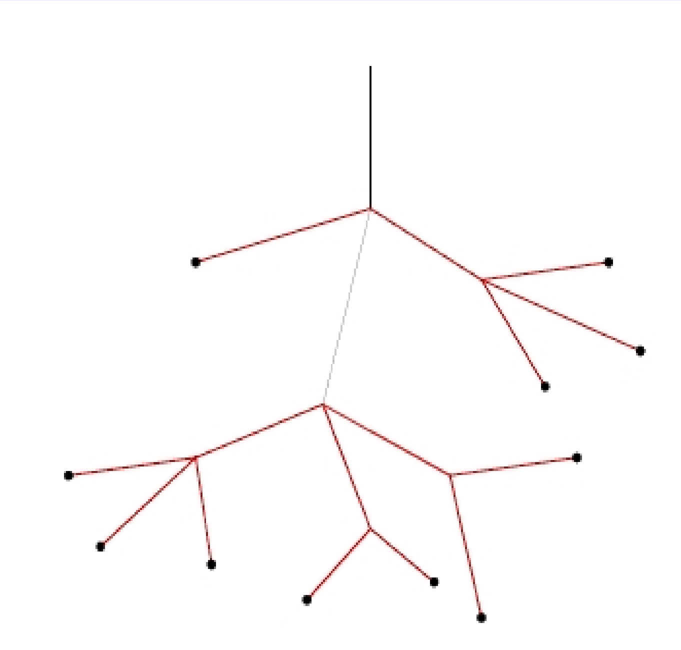

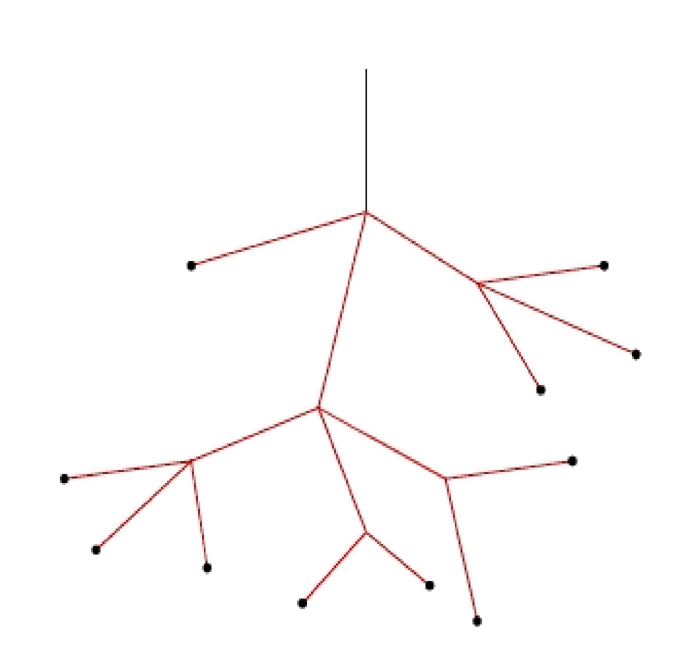

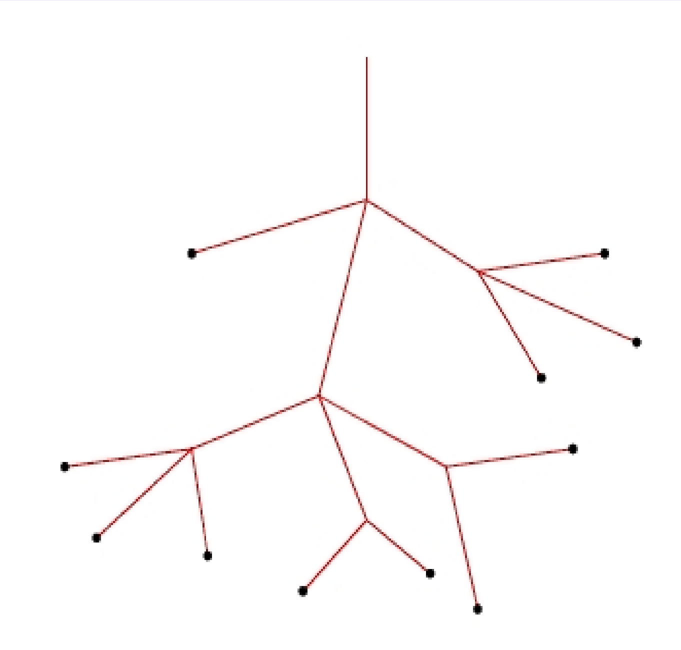

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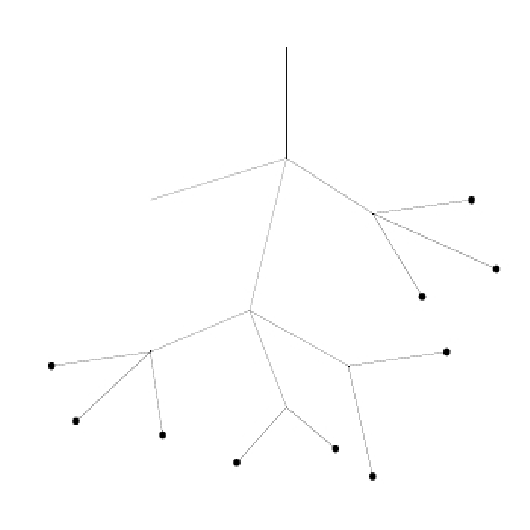

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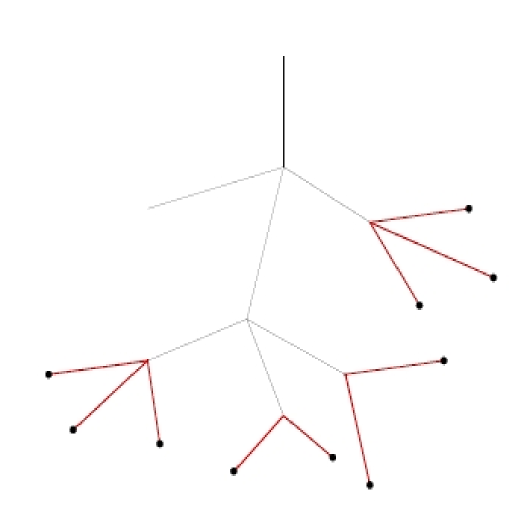

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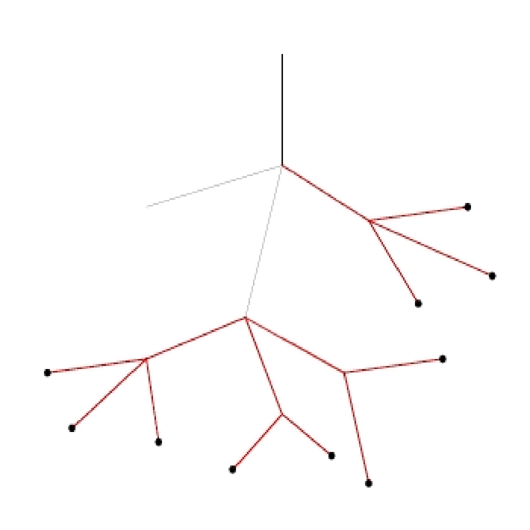

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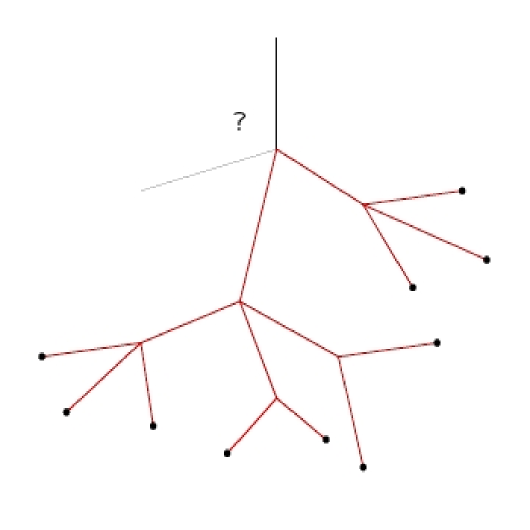
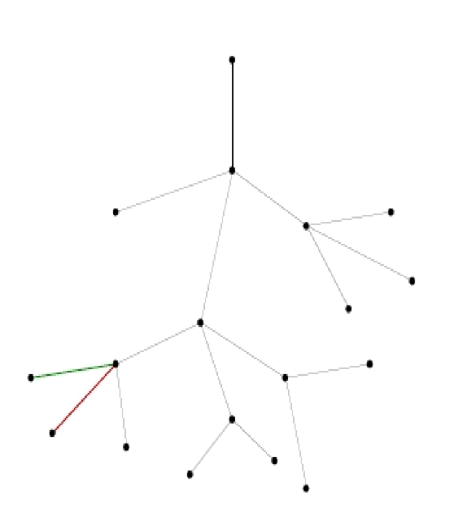


Observation on one single extrenal vertex

The sharp condition is: *The spectra of each pair of subtrees with a common vertex should have empty intersection.*



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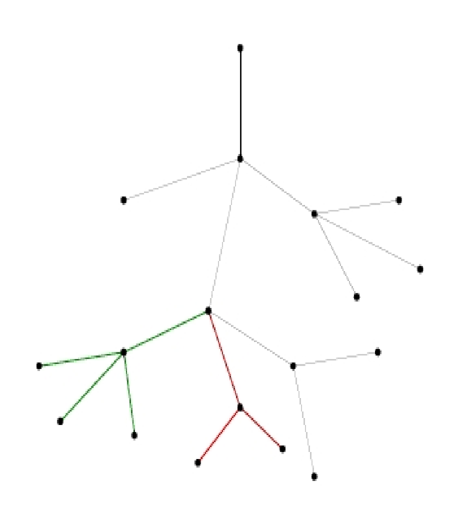

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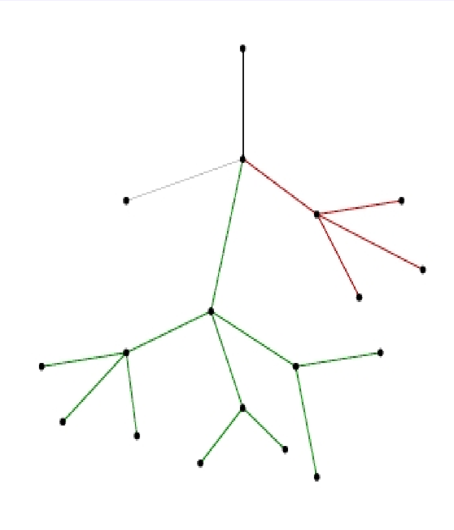

- This condition is sharp. Whenever two spectra have a common point one can build an isolated eigenfunction with support on those two subtrees and with the common vertex as a nodal point. This eigenfunction leaves the rest of the network at rest.
- This condition is the natural extension of the one on irrationality for stars: Note that ℓ_1/ℓ_2 being irrational is equivalent to $\sigma_1 \cap \sigma_2 = \emptyset$.
- This condition is generically true.

- The sharp energy one is able to measure depends on each tree. But, in all cases, it can be characterized in terms of the Fourier expansion of solutions:

$$\varphi(x,t)=\sum_{k\in Z} a_k e^{i\sqrt{\lambda_k}t} w_k;$$

$$\sum_{k\in \mathbf{Z}} \rho_k |a_k|^2 \le C\int_0^T \left|\varphi_x(O,t)\right|^2 dt,$$

and $\rho_k > 0$ for all $k \in \mathbf{Z}$ if and only if the condition of disjoint intersection for the spectra of subtrees holds.

Outline

Preliminaries on the Theory of Non Harmonic Fourier Series

Given a sequence (μ_k) of distinct real numbers

$$R(\mu_k) := \sup\left\{r : \quad \left\{\sum c_k e^{i\mu_k t}\right\} \text{ is dense } C([-r,r])\right\}$$

is called its *completeness radius*.
Haraux and Jaffard (1991) derived the following result, as a Corollary of the celebrated Beurling-Malliavin Theorem (1967):

1) For every $T > 2R(\lambda_k)$,

$$\int_0^T \left|\sum_{k\in\mathbb{Z}} a_k e^{i\mu_k t}\right|^2 dt \geq \sum_k \rho_k \left|a_k\right|^2, \qquad (1)$$

for any finite sequence (a_k), with $\rho_k > 0$ independent of (a_k).

2) If $T < 2R(\mu_k)$ the previous inequality may not hold whatever the weights (ρ_k) are.

In the present case, $\mu_k = \sqrt{\lambda_k}$, λ_k being the eigenvalues on the network. Since the asymptotic density of the eigenvalues of the network coincides with L[1], the total length of the network, then $R(\mu_k) = L$ as well.

As a consequence of this we deduce that:

Theorem

The necessary and sufficient condition such that for all $T > 2L$ a suitable energy (with suitable weights, coding non-trivial information on each eigencomponent) can be recovered by means of a measurement made on a single external vertex is that there are no eigenfunctions of the network vanishing on the corresponding edge.

[1] J. von Below, S. Nicaise,...

- Colored networks and multiple measurements: If measurements are made on the interior nodes how many do we need to measure on?
- Other models: heat and Schrödinger equations.
Kannai transform allows transfering the results we have obtained for the wave equation on the network to other models: (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta_{\mathcal{N}}}\varphi = \frac{1}{4\pi t}\int_{-\infty}^{+\infty} e^{-s^2/4t} W(s)ds$$

where $W(x,s)$ solves the wave equation on the same network with data $(\varphi, 0)$.

$$W_{ss} - \Delta_{\mathcal{N}} W = 0 \quad + \quad K_t - K_{ss} = 0 \quad \rightarrow \quad U_t - \Delta_{\mathcal{N}} U = 0,$$

$$W_{ss} - \Delta_{\mathcal{N}} W = 0 \quad + \quad iK_t - K_{ss} = 0 \quad \rightarrow \quad iU_t - \Delta_{\mathcal{N}} U = 0.$$

CONCLUSIONS:

- Wave propagation on networks needs to combine the classical methods on the theory of wave propagation, with tools coming from graph theory and, even from Number Theory. The later makes the topic extremely subtle and results unstable.

Wave propagation on networks

$$=$$

Wave propagation + Graph Theory + Number Theory.

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- Characterization of networks with localized modes?
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To learn more on this topic:

R. Dáger and E. Z. *Wave Propagation, Observation and Control in $1-d$ Flexible Multi-Structures.* Mathématiques et Applications, **50**, 2006.