Control and numerics: Continuous versus discrete approaches

Enrique Zuazua

Ikerbasque & Basque Center for Applied Mathematics (BCAM)
Bilbao, Basque Country, Spain
zuazua@bcamath.org
http://www.bcamath.org/zuazua/

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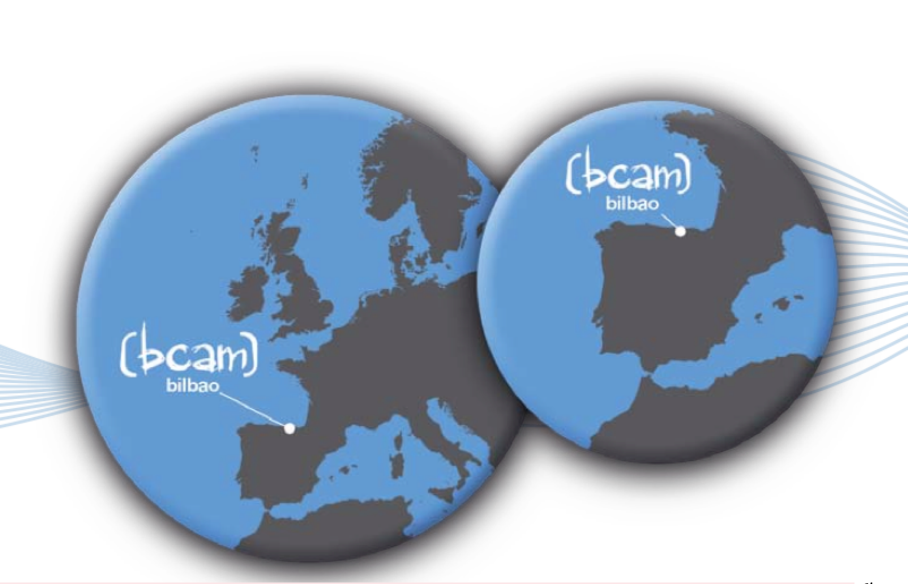
Joint work with S. Ervedoza and A. Marica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Motivation

Control problems for PDE are important for at least two reasons:

- They **emerge in most real applications**:

  PDE as the models of Continuum and Quantum Mechanics.

  Control and/or Optimization as essential step in all processes.

- They **demand a better master** of the standard PDE models and new analytical tools.

This need of new analytical tools is enhanced when facing numerical simulation problems!

Furthermore, these kind of techniques are of application in some other fields, such as **inverse problems**, **optimal shape design** and **parameter identification** problems.

Topics to be addressed:

1. The wave equation: propagation of discrete waves
2. Heterogenous grids
3. Perspectives

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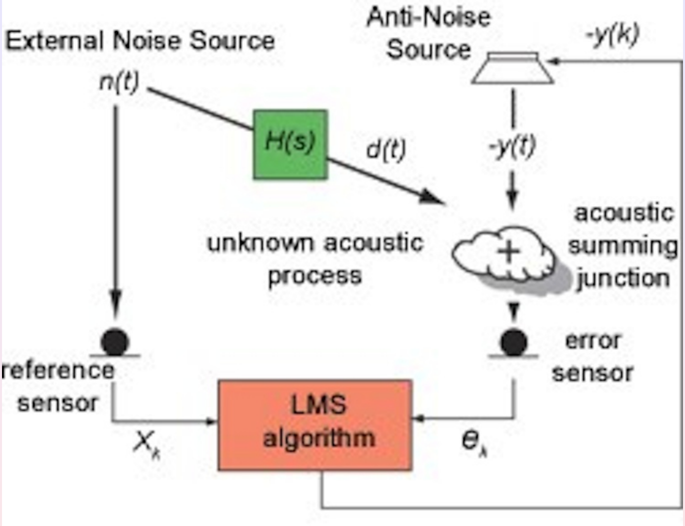


http://www.ind.rwth-aachen.de/research/noise_reduction.html

Outline

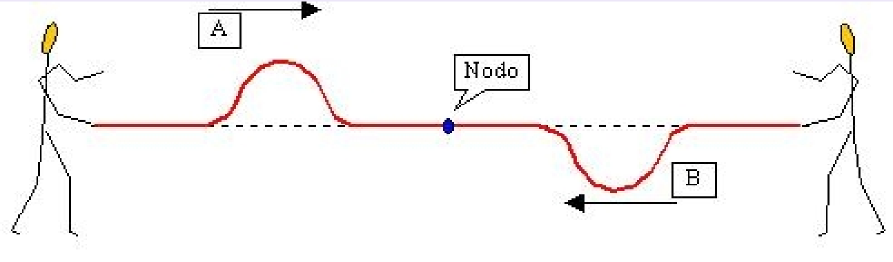
Control of $1-d$ vibrations of a string

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.
The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$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B

The dual observation problem

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\, 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\, \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

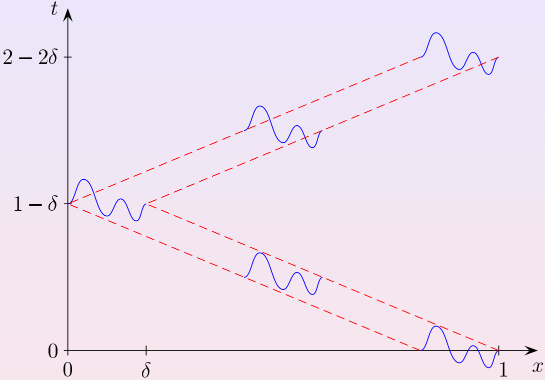
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2 \right] dx = E(0), \ \ \forall 0 \le t \le T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \le C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \geq 2$.



$$E(0) \leq \int_0^T |\varphi_x(1,t)|^2 dt$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

Construction of the Control

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x(1, t),$$

where u is the solution of the adjoint system corresponding to initial data $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $\varphi_x(1,t) \in L^2(0,T)$ (hidden regularity). Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY.

The continuous numerical approach: Gradient algorithms

The control was characterized as being the minimizer over $H_0^1(0,1) \times L^2(0,1)$ of

$$J(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1}\times H_0^1} .$$

We produce an algorithm in which:

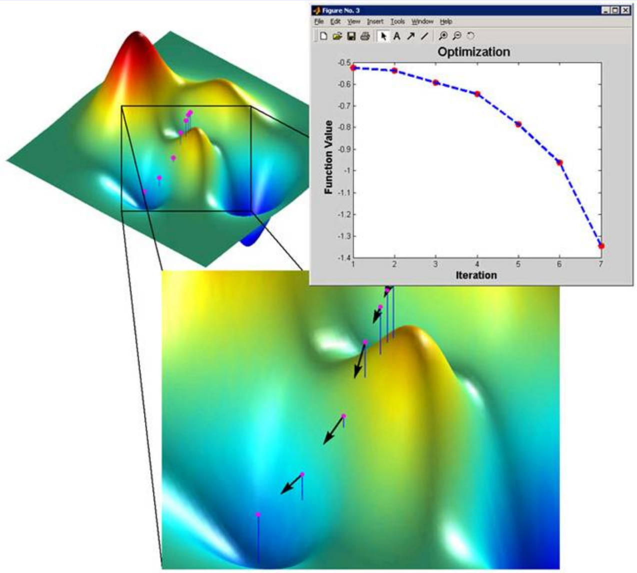
- We apply a gradient iteration algorithm to J.
  This leads to an iterative process

  $$(\varphi_k^0, \varphi_k^1), \quad k \geq 1$$

  so that

  $$\partial_x \varphi_k(1,t) = v_k(t) \to v(t), \quad \text{as } k \to \infty.$$

- We replace J by some numerical approximation J_h with an order h^θ, and apply a discrete version of the iterative process above to build approximations of v_k.



Note however that computing gradients, in practice, may be hard.

Classical steepest descent:

$J : H \to \mathbf{R}$. Two main assumptions:

$$< \nabla J(u) - \nabla J(v), u - v > \geq \alpha |u - v|^2, \quad |\nabla J(u) - \nabla J(v)|^2 \leq M |u - v|^2.$$

Then, for

$$u_{k+1} = u_k - \rho \nabla J(u_k),$$

we have

$$|u_k - u^*| \leq (1 - 2\rho\alpha + \rho^2 M)^{k/2} |u_1 - u^*|.$$

Convergence is guaranteed for $0 < \rho < 1$ small enough ($\rho < \alpha/M$).

Compare with the continuous marching gradient system

$$u'(\tau) = -\nabla J(u(\tau)).$$

The following holds:

Theorem

(S. Ervedoza & E. Z., 2011)
In

$$K \sim C|\log(h)|$$

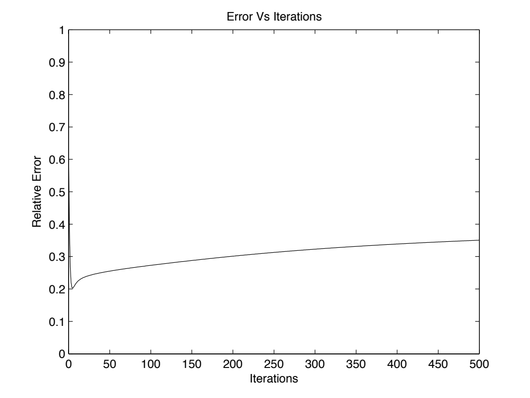
iterations, the controls v_h^K *obtained after applying* K *iterations of the gradient algorithm to* J_h *fulfill:*

$$||v - v_h^K|| \leq C|\log(h)|^{\max(\theta,1)} h^\theta.$$

Note that for the classical Finite Difference and Finite Element methods for the wave equation the convergence order is $\theta = 2/3$.

We have developed the continuous program successfully!

Note that the error estimate deteriorates if $K >> C|\log(h)|$!!!



... and, therefore, the method has to be used with much care since, after all, we are dealing with an **unstable, non-robust algorithm**....

Convergence of the descent algorithm for the continuous model and the convergence of the numerical scheme in the classical sense of numerical analysis leads to:

$$||v - v_h^k|| \leq ||v - v^k|| + ||v^k - v_h^k|| \leq C[\sigma(\rho)^k + kh^\theta].$$

Note that, in here, nothing has been used about the actual properties of control of the numerical scheme.

The estimate deteriorates when $kh^\theta >> \sigma(\rho)^k$.

The continuous method can not be implemented, in a reliable manner, as one would expect/wish: To apply a descent or iterative algorithm for a discrete functional J_h, without worrying about possible divergence of the process beyond a certain threshold of iterations.

The same occurs to other methods, based on different iterative algorithms for building continuous controls, as for instance, the one developed by N. Cindea et al.[1] based on D. Russell's[2] method of "stabilization implies control", also closely related to the works by Auroux and Blum on the nudging method for data assimilation for Burgers like equations.

[1] N. Cindea, S. Micu & M. Tucsnak, An approximation method for exact controls of vibrating systems. SICON, 49 (3), (2011), 1283–1305.

[2] D. Russell, Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions, SIAM Rev., 20 (1978), 639–739.

Outline

But one might want to take a shortcut controlling a finite-dimensional reduced dynamics.

Set $h = 1/(N+1) > 0$ and consider the mesh

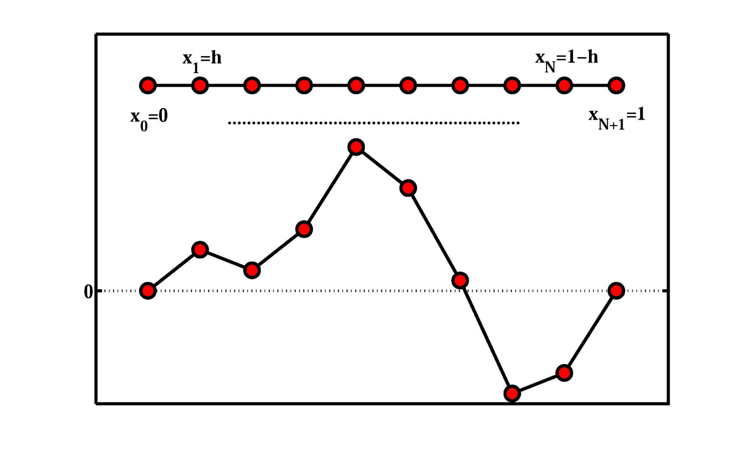
$$x_0 = 0 < x_1 < ... < x_j = jh < x_N = 1 - h < x_{N+1} = 1,$$

which divides $[0, 1]$ into $N + 1$ subintervals

$$I_j = [x_j, x_{j+1}], \, j = 0, ..., N.$$

Finite difference semi-discrete approximation of the wave equation:

$$\begin{cases} \varphi_j'' - \frac{1}{h^2}\left[\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j\right] = 0, & 0 < t < T, \, j = 1, \ldots, N \\ \varphi_j(t) = 0, & j = 0, \, N+1, \, 0 < t < T \\ \varphi_j(0) = \varphi_j^0, \, \varphi_j'(0) = \varphi_j^1, & j = 1, \ldots, N. \end{cases}$$



From finite-dimensional dynamical systems to infinite-dimensional ones in purely conservative dynamics.....

Then it should be sufficient to minimize the discrete functional

$$J_h(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T \frac{|\varphi_N(1,t)|^2}{h^2} dt + h\sum_{j=1}^{N} \varphi_j^1 y_j^0 - h\sum_{j=1}^{N} \varphi_j^0 y_j^1,$$

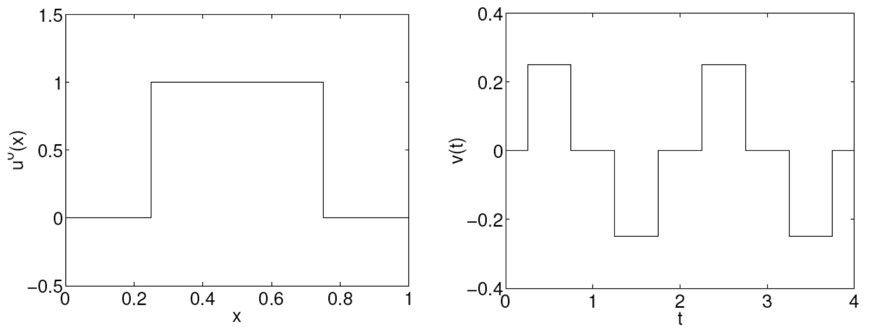
which is a discrete version of the functional J of the continuous wave equation since

$$-\frac{\varphi_N(t)}{h} = \frac{\varphi_{N+1} - \varphi_N(t)}{h} \sim \varphi_x(1,t).$$

Then

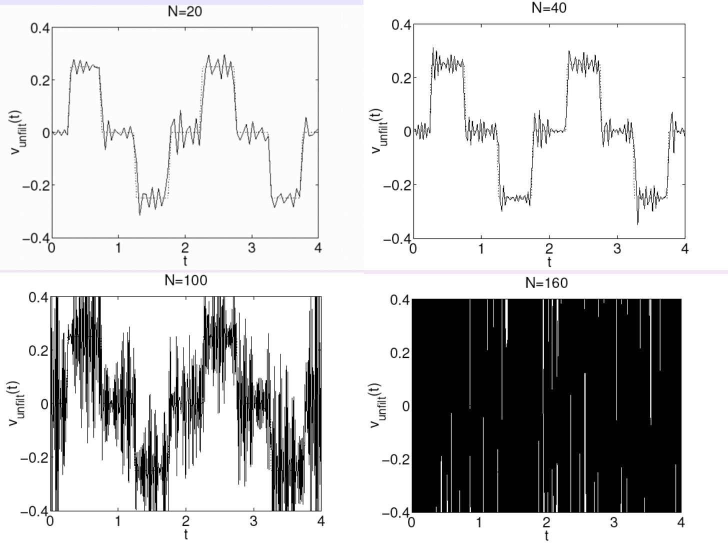
$$v_h(t) = -\frac{\varphi_N^\star(t)}{h}.$$

A NUMERICAL EXPERIMENT



Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.

Plot of the time evolution of the exact control for the wave equation in time $T = 4$.



The control diverges as $h \to 0$.

The discrete approach naively or directly applied diverges as well.

In this case our algorithm gets the minimizer of J_h. But the minimizer of J_h is very far form that of J: This is a clear case in which Γ-convergence with respect to the parameter $h \longrightarrow 0$ fails.

WHY?

The Fourier series expansion shows the analogy between continuous and discrete dynamics.

Discrete solution:

$$\vec{\varphi} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h t}\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h t}\right) \right) \vec{w}_k^h.$$

Continuous solution:

$$\varphi = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x)$$

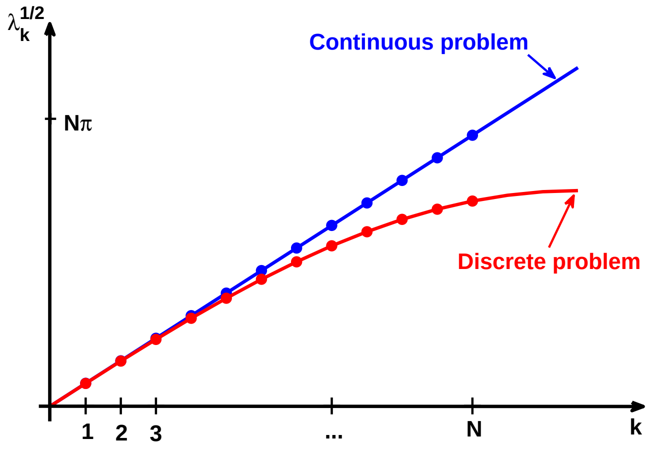
Recall that the discrete spectrum is as follows and converges to the continuous one:

$$\lambda_k^h = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right)$$

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0$$

$$w_k^h = (w_{k,1}, \dots, w_{k,N})^T : w_{k,j} = \sin(k\pi jh),\ k, j = 1, \dots, N.$$

The only relevant differences arise at the level of the dispersion properties and the group velocity. High frequency waves do not propagate, remain captured within the grid, without never reaching the boundary. This makes it impossible the uniform boundary control and observation of the discrete schemes as $h \to 0$.

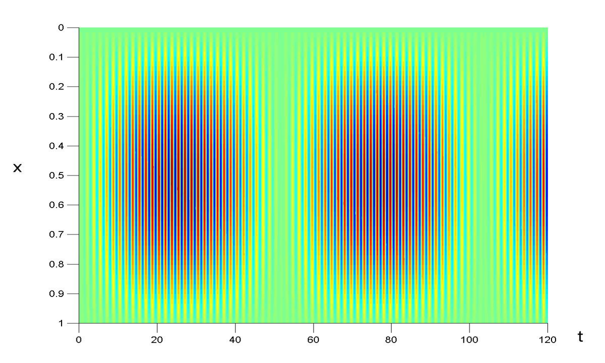


Graph of the square roots of the eigenvalues both in the continuous and in the discrete case. The gap is clearly independent of k in the continuous case while it is of the order of h for large k in the discrete one.

A numerical phamtom

$$\vec{\varphi} = \exp\left(i\sqrt{\lambda_N(h)}\,t\right)\vec{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}\,t\right)\vec{w}_{N-1}.$$

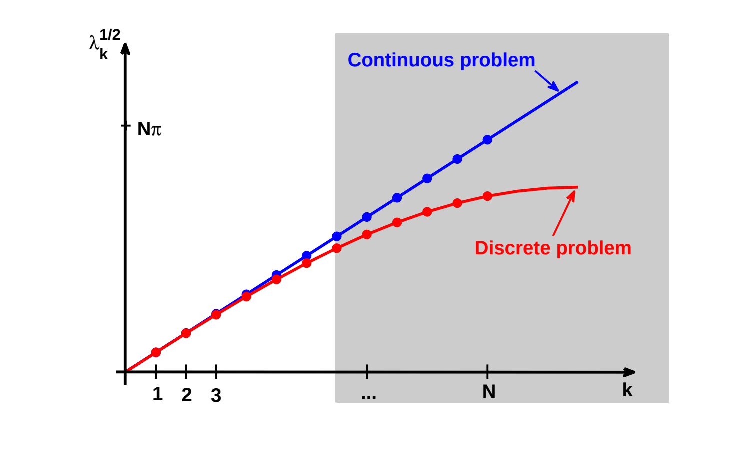
Spurious semi-discrete wave combining the last two eigenfrequencies with very little gap: $\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h.$



$h = 1/61$, $(N = 60)$, $0 \leq t \leq 120.$

Outline

Fourier filtering



To filter the high frequencies, keeping the components $k \leq \delta/h$ with $0 < \delta < 1$. Then the group velocity remains uniformly bounded below and uniform observation holds in time $T(\delta) > 2$ such that $T(\delta) \to 2$ as $\delta \to 0$.

Relaxed controls:

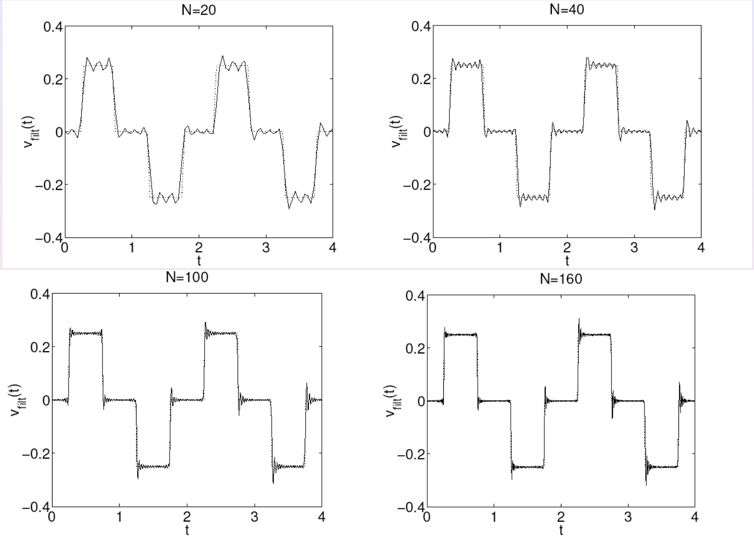
Then, the filtering algorithm can be implemented as follows:

- Minimize J_h over the class of filtered solutions with filtering parameter $0 < \delta < 1$ and $T > T(\delta)$;
- This yields controls v_h^δ such that
  - $v_h^\delta \to v$ as $h \to 0$;
  - The corresponding states $\vec{y}_h$ satisfiy:

$$\pi_\delta(\vec{y}_h) \equiv \pi_\delta(\vec{y}_h{}') \equiv 0.$$

This is a relaxed version of the controllability condition.

Numerical experiment, revisited, with filtering



With appropriate filtering the control converges as $h \to 0$.

The discrete approach when applied directly fails, but it can cured borrowing ideas from the continuous analysis. The bonus is that:

- We compute numerical approximations of the controls that perform well, in an identified manner, controlling a Fourier projection of solutions at the discrete level.
- The algorithm converges is stable and robust, an the error diminishes as the number of iterations $\to \infty$.

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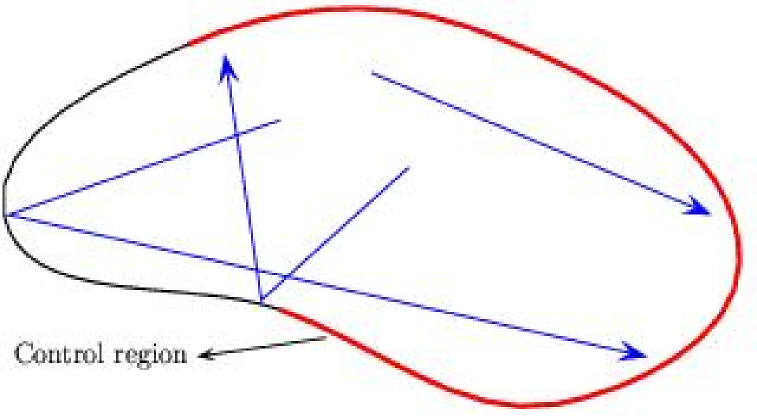
The multi-dimensional case.

Similar results are true in several space dimensions. The region in which the observation/control applies needs to be large enough to capture all rays of Geometric Optics. This is the so-called Geometric Control Condition introduced by Ralston (1982) and Bardos-Lebeau-Rauch (1992).

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2. Let Γ_0 be an open and non-empty subset of Γ and $T > 0$.

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ y = v(x,t) 1_{\Gamma_0} & \text{on} \quad \Sigma = \Gamma \times (0, T) \\ (x, 0) = y^0(x), y_t(x, 0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$



Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools form Microlocal Analysis.

In all cases the control is equivalent to an observation problem for the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ \varphi = 0 & \text{on} \quad \Sigma = \Gamma \times \times (0, T) \\ \varphi(x,0) = \varphi^0(x), \varphi_t(x,0) = \varphi^1(x) & \text{in} \quad \Omega. \end{cases}$$

Is it true that:

$$E_0 \leq C(\Gamma_0, T) \int_{\Gamma_0} \int_0^T \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt \quad ?$$

And a sharp discussion of this inequality requires of Microlocal analysis.
Partial results may be obtained by means of multipliers: $x \cdot \nabla\varphi$, φ_t, φ,...

THE 5-POINT FINITE-DIFFERENCE SCHEME

$$\varphi''_{j,k} - \frac{1}{h^2}\left[\varphi_{j+1,k} + \varphi_{j-1,k} - 4\varphi_{j,k} + \varphi_{j,k+1} + \varphi_{j,k-1}\right] = 0.$$

The energy of solutions is constant in time:

$$\begin{aligned} E_h(t) &= \frac{h^2}{2}\sum_{j=0}^{N}\sum_{k=0}^{N}\Big[\mid \varphi'_{jk}(t) \mid^2 \\ &\quad + \left|\frac{\varphi_{j+1,k}(t) - \varphi_{j,k}(t)}{h}\right|^2 + \left|\frac{\varphi_{j,k+1}(t) - \varphi_{j,k}(t)}{h}\right|^2\Big]. \end{aligned}$$

Without filtering observability inequalities fail in this case too.

Understanding how filtering should be used requires of a microlocal analysis of the propagation of numerical waves combining von Neumann analysis and Wigner measures developments (N. Trefethen, P. Gérard, P. L. Lions & Th. Paul, G. Lebeau, F. Macià, ...).

The von Neumann analysis.

Symbol of the semi-discrete system for solutions of wavelength h

$$p_h(\xi,\tau) = \tau^2 - 4\left(\sin^2(\xi_1/2) + \sin^2(\xi_2/2)\right),$$

versus $p(\xi,\tau) = \tau^2 - [|\xi_1|^2 + |\xi_2|^2]$.
Both symbols coincide for $(\xi_1,\xi_2) \sim (0,0)$.
Solving the bicharacteristic flow we get the discrete rays:

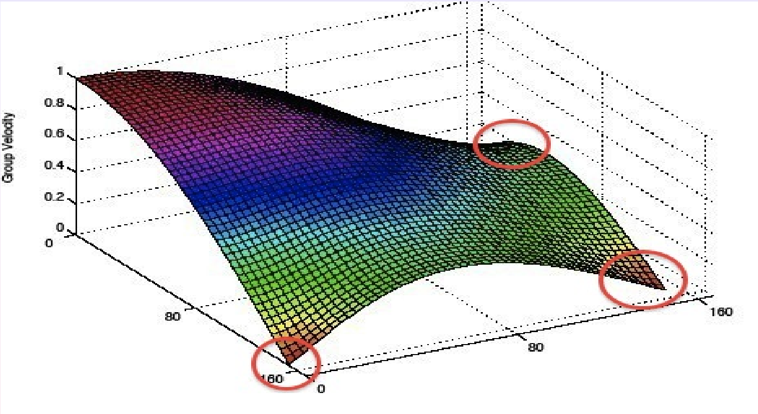
$$x_j(t) = -\frac{\sin(\xi_j)}{\tau}t + x_{j,0}, \quad (\text{versus } x_j(t) = -\frac{\xi_j}{\tau}t + x_{j,0}.)$$

RAYS ARE STILL STRAIGHT LINES. BUT! The velocity is

$$|x'(t)| \equiv \left[\left|\frac{\sin(\xi_1)}{\tau}\right|^2 + \left|\frac{\sin(\xi_2)}{\tau}\right|^2\right]^{1/2}$$

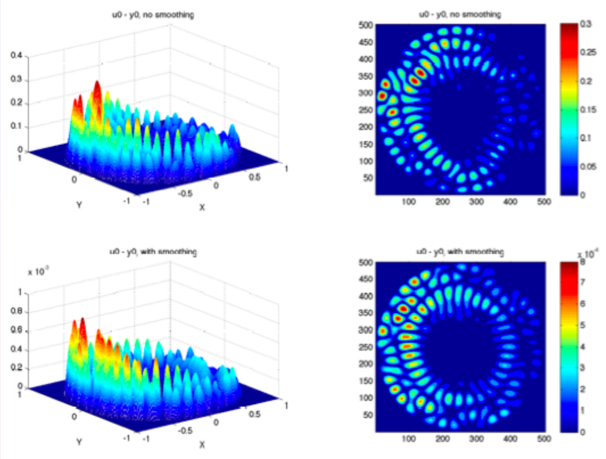
THE VELOCITY OF PROPAGATION VANISHES !!!!!!! in the following eight points

$$\xi_1 = 0, \pm\pi, \xi_2 = 0, \pm\pi, \quad (\xi_1,\xi_2) \neq (0,0).$$



Group velocity in dimension two, $h = 1/50$

Controls in multi-d may develop complex and unexpected patterns, in view of the laws of Geometric Optics.



G. Lebeau and M. Nodet, Experimental Study of the HUM Control Operator for Linear Waves, Experimental Mathematics, 2010.

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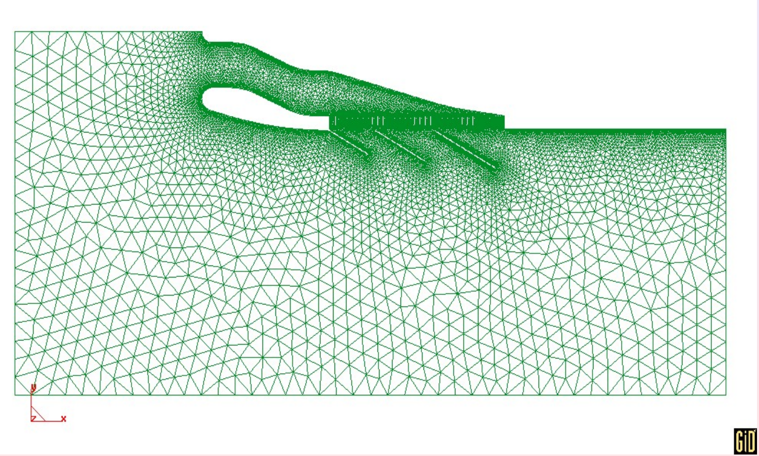
In practice, grids are not uniform and, accordingly, the Fourier and von Neumann analysis above is not sufficient to describe the propagation of high frequency numerical solutions. A new symbolic calculus is needed to define the discrete rays.



Problem formulation

The wave equation with variable coefficients on $\mathbb{R}$:

$$\rho(y)u_{tt} - (\sigma(y)u_y)_y = 0, t > 0, \; y \in \mathbb{R}. \tag{1}$$

Energy conserved in time:

$$\mathcal{E}_{\rho,\sigma}(u^0, u^1) := \frac{1}{2} \int_{\mathbb{R}} (\rho(y)|u_t(y,t)|^2 + \sigma(y)|u_y(y,t)|^2) \, dy.$$

Finite difference approximations

Let $h > 0$ be the mesh size parameter, $g : \mathbb{R} \to \mathbb{R}$ be an increasing function on $\mathbb{R}$, $\mathcal{G}^h := \{x_j := jh, j \in \mathbb{Z}\}$ and $\mathcal{G}^h_g := \{g_j := g(x_j), j \in \mathbb{Z}\}$ the *uniform grid* of size h of $\mathbb{R}$ and the *non-uniform* one obtained by transforming the uniform one through the map g.

Finite difference semi-discretization of (1) on the non-uniform grid $\mathcal{G}^h_g$:

$$\rho(g_j)u_{j,tt}(t) - \frac{\sigma(g_{j+1/2})\frac{u_{j+1}(t)-u_j(t)}{g_{j+1}-g_j} - \sigma(g_{j-1/2})\frac{u_j(t)-u_{j-1}(t)}{g_j-g_{j-1}}}{\frac{g_{j+1}-g_{j-1}}{2}} = 0. \qquad (2)$$

Energy is conserved in time

$$\mathcal{E}^h_{\rho,\sigma,g}(\mathbf{u}^{h,0}, \mathbf{u}^{h,1}) := \frac{h}{2} \sum_{j\in\mathbb{Z}} \left[\partial^h g_j \rho(g_j)|u_{j,t}(t)|^2 + \frac{\sigma(g_{j+1/2})}{\partial^{h,+} g_j} |\partial^{h,+} u_j(t)|^2\right].$$

This schemes provides a convergent numerical approximation.

The *principal symbol* :

$$\wp(x,t,\xi,\tau) := -g'(x)\rho(g(x))\tau^2 + 4\sin^2\left(\frac{\xi}{2}\right)\frac{\sigma(g(x))}{g'(x)}. \tag{3}$$

The *null bi-characteristic lines* associated to this *principal symbol* are the solutions of the Hamiltonian system:

$$\begin{cases} \dot{X}(s) = \partial_\xi \wp = 2\sin(\Xi(s))\frac{\sigma(g(X(s)))}{g'(X(s))}, \\ \dot{t}(s) = \partial_\tau \wp = -g'(X(s))\rho(g(X(s)))\tau, \\ \dot{\Xi}(s) = -\partial_x \wp = (g'(\cdot)\rho(g(\cdot)))'(X(s)) - 4\sin^2\left(\frac{\Xi(s)}{2}\right)\left(\frac{\sigma(g(\cdot))}{g'(\cdot)}\right)'(X(s)), \\ \dot{\tau}(s) = -\partial_t \wp = 0, \end{cases} \tag{4}$$

$(X(t),\Xi(t))$ solves the *Hamiltonian system*:

$$(X)'(t) = \mp c_g(X(t))\cos\left(\frac{\Xi(t)}{2}\right),\ (\Xi)'(t) = \pm c_g'(X(t))2\sin\left(\frac{\Xi(t)}{2}\right) \tag{5}$$

with

$$c_g(x) := \sqrt{\sigma(g(x))/\rho(g(x))}/g'(x).$$

Using Wigner transforms, high frequency solutions can be shown to propagate and concentrate along those rays provided $c_g \in C^{1,1}(\mathbb{R})$. This means that the coefficients σ and ρ need to be in $C^{1,1}(\mathbb{R})$ and the grid transformation $g \in C^{2,1}(\mathbb{R})$.

- Gérard, P., Markowich, P., Mauser, N., Poupaud, Ph. *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math., 1997.
- Lions P.-L., Paul, *Sur les mesures de Wigner*, Rev. Matemática Iberoamericana, 1993.
- Macia, F. *Wigner measures in the discrete setting: high frequency analysis of sampling and reconstruction operators*, SIAM J. Math. Anal., 2004.

Numerical simulations

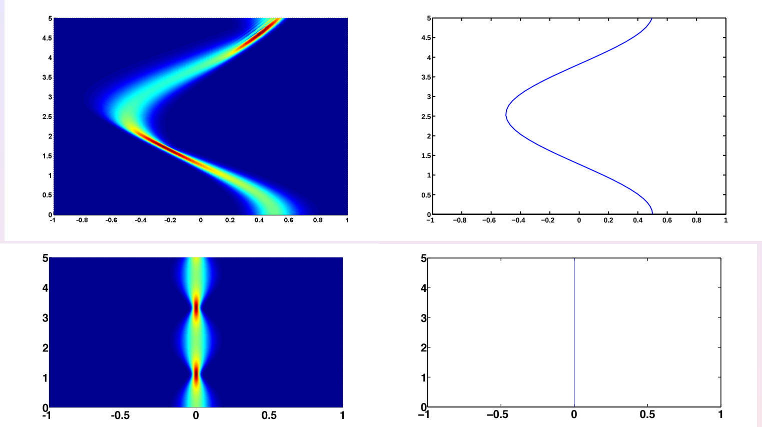


Figure: The numerical solution and the corresponding bicharacteristic ray with $g(x) = \tan(\pi x/4)$.

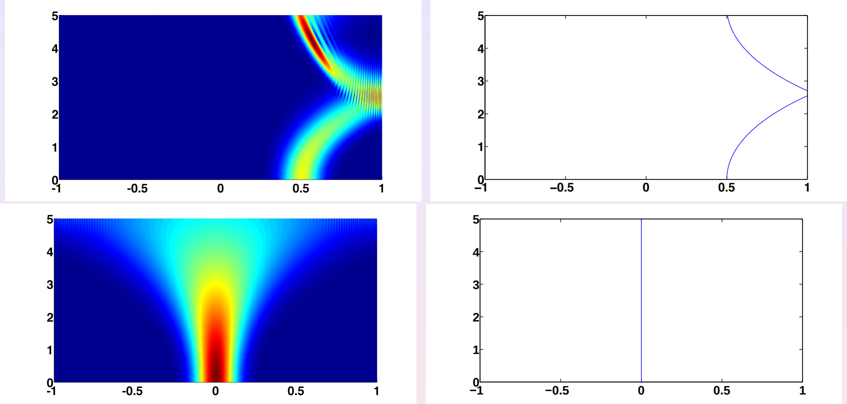


Figure: The numerical solution and the corresponding bicharacteristic ray with $g(x) = 2\sin(\pi x/6)$.

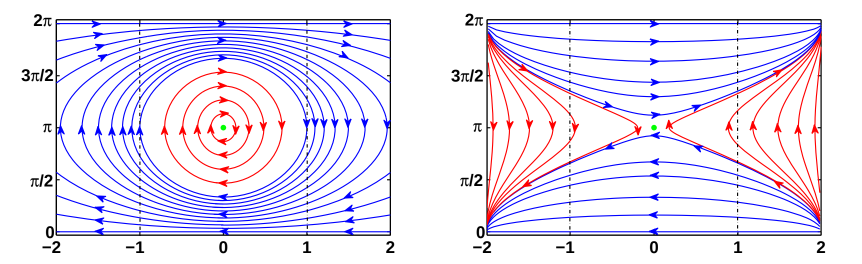


Figure: The phase portrait for the grid transformations $g(x) = \tan(\pi x/4)$ and g(x)=2sin($\pi x/6$).

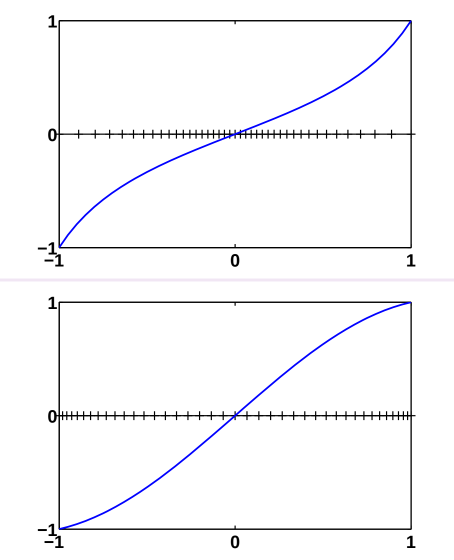
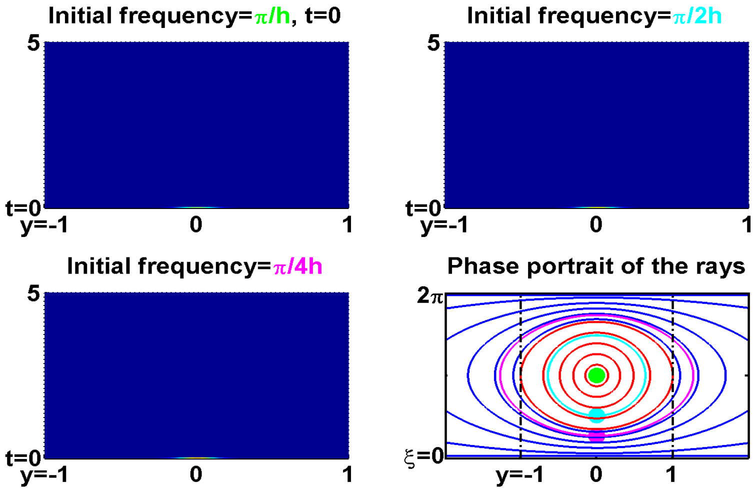
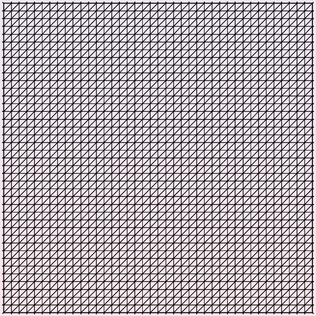
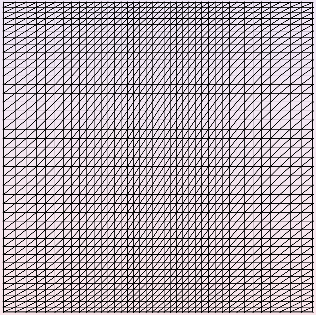


Figure: Grid corresponding to the transformations $g(x) = \tan(\pi x/4)$ and $g(x) = \sin(\pi x/6)$ respectively.


Initial frequency=π/h, t=0
5
t=0
y=-1
0
1
Initial frequency=π/2h
5
t=0
y=-1
0
1
Initial frequency=π/4h
5
t=0
y=-1
0
1
Phase portrait of the rays
2π
ξ=0
y=-1
0
1

The analysis above can be extended to the multi-dimensional case, provided grids can be mapped smoothly into uniform grids as in the example below:





But there is istill plenty to be done to understand the behavior of discrete waves over very irregular meshes:

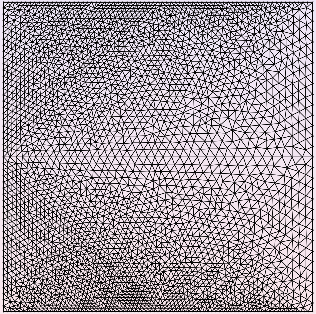
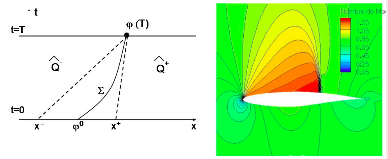


Table of Contents

Conclusions

- Efficient and rigorous numerical computation of controllers can be built but often combining tools from the continuous and the discrete approaches.
- Heterogeneous grids may lead to novel unexpected phenomena of discrete wave propagation.
- Plenty is still to be done in the interfaces between PDE, Control, Numerics, Harmonic Analysis,...
- Similar issues are relevant in many other contexts as well, for instance, **control of conservation laws in the presence of shocks** (S. Ulbrich, M. Giles, C. Bardos & O. Pironneau, A. Bressan & A. Marson, E. Godlewski & P. A. Raviart, C. Castro, F. Palacios & E. Z., ...)



Perspectives

- Multi-resolution filtering techniques.
- Adaptivity.
- More singular and heterogeneous grids.
- Numerical control of waves in random media and in the presence of noise.
- Robust controllers.
- Multiphysics systems: thermoelasticity, fluid-structure interaction,...
- Networks.

Some references:

- E. Z. Propagation, observation, and control of waves approximated by finite difference methods. SIAM Review, 47 (2) (2005), 197-243.
- S. Ervedoza and E. Z. The Wave Equation: Control and Numerics, in "Control and stabilization of PDE's", P. M. Cannarsa y J. M. Coron, eds., "Lecture Notes in Mathematics", CIME Subseries, Springer Verlag, to appear.
- L. Ignat and E. Z. Convergence of a multi-grid method for the control of waves, J. European Math. Soc., 11 (2009), 351-391.
- A. Marica and E. Z., Propagation of $1-d$ waves in regular discrete heterogeneous media: A Wigner measure approach, to appear.