# Dynamic versus steady state control

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- Engineering and physical processes do not obey a single modeling paradigm. Often times, both time-dependent and steady models are available and appropriate.
- This yields a number of different possibilities when facing optimal design, control and inverse problems.
- What model do we adopt? The time-dependent or the steady state one.
- Steady-state models are often understood as a simplification of the time evolution one, assuming (some times rigorously but most often without proof) that the time-dependent solution stabilizes around the steady-state as $t \to \infty$.

**Main question**

Does the solution of the time-evolution control (or design or inverse problem) converge as $t \to \infty$ to the control of the steady state problem as well?

- This issue is particularly important in aeronautical optimal design,[1] a mature but still rapidly evolving field where huge challenges arise and, in which, in particular, many problems related to design and control are still widely open.
- Often times people employ steady state models and the corresponding control ones while, from a mathematical point of view, the evolution problem is better understood.
- The reason for this is very simple: In the context of Nonlinear PDE steady state problems are hard to solve. In particular uniqueness is hard to prove. Accordingly sensitivity analysis is difficult as well. By the contrary, for evolution problems, under suitable assumptions on the nonlinearity, initial-boundary value problems are uniquely solvable, solutions depending smoothly on the data.

[1] A. Jameson. "Optimization Methods in Computational Fluid Dynamics (with Ou, K.), Encyclopedia of Aerospace Engineering, Edited by Richard Blockley and Wei Shyy, John Wiley  Sons, Ltd., 2010."

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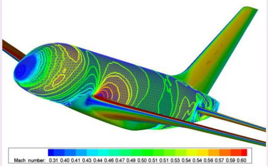
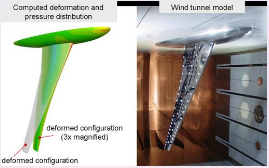
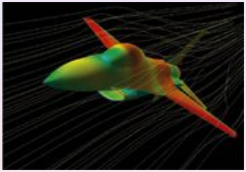
# Shape design in aeronautics

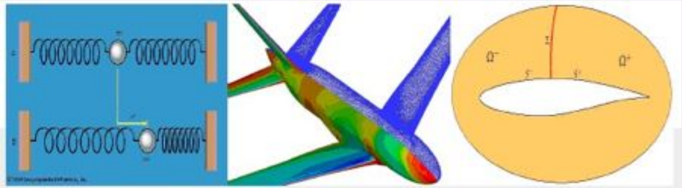


# Optimal shape design in aeronautics. Two aspects:

- Shocks.
- Oscillations.

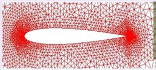


Optimal shape ∼ Active control. The shape of the cavity or airfoil controls the surrounding flow of air.

Two approaches:

Discrete: Discretization + gradient

- Advantages: Discrete clouds of values. No shocks. Automatic differentiation, ...
- Drawbacks:
  - "Invisible" geometry.




  - Scheme dependent.

Continuous: Continuous gradient + discretization.

- Advantages: Simpler computations. Solver independent. Shock detection.
- Drawbacks:
  - Yields approximate gradients.
  - Subtle if shocks.

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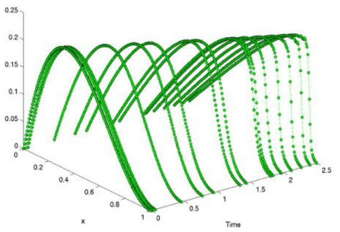
The relevant models in aeronautics (Fluid Mechanics):

- Navier-Stokes equations;
- Euler equations;
- Turbulent models: Reynolds-Averaged Navier-Stokes (RANS), Spalart-Allmaras Turbulence Model, $k - \varepsilon$ model;

  ....
- Burgers equation (as a $1 - d$ theoretical laboratory).

Solutions may develop shocks or quasi-shock configurations.

- For shock solutions, classical calculus fails;
- For quasi-shock solutions the sensitivity is so large that classical sensitivity clalculus is meaningless.



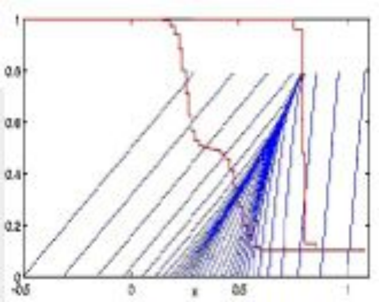
## Burgers equation

- Viscous version:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0.$$

- Inviscid one:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$



In the inviscid case, the simple and "natural" rule

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \to \frac{\partial \delta u}{\partial t} + \delta u\frac{\partial u}{\partial x} + u\frac{\partial \delta u}{\partial x} = 0$$

breaks down in the presence of shocks

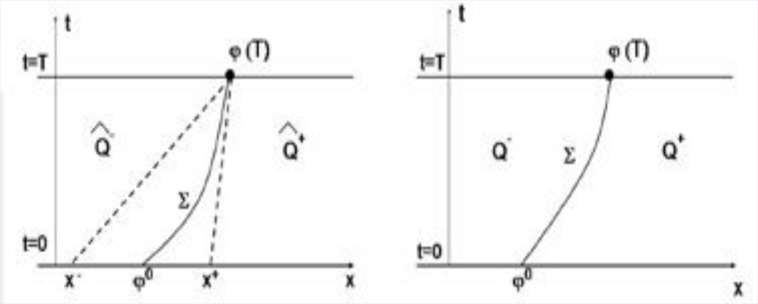
$\delta u =$ discontinuous, $\frac{\partial u}{\partial x} =$ Dirac delta $\Rightarrow \delta u\frac{\partial u}{\partial x}$????

The difficulty may be overcame with a suitable notion of measure valued weak solution using Volpert's definition of conservative products and duality theory (Bouchut-James, Godlewski-Raviart,...)

A new viewpoint: Solution = Flow solution + shock location.

The pair $(u, \varphi)$ solves:

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0, & \text{in } Q^- \cup Q^+, \\ \varphi'(t)[u]_{\varphi(t)} = \left[u^2/2\right]_{\varphi(t)}, & t \in (0,T), \\ \varphi(0) = \varphi^0, & \\ u(x,0) = u^0(x), & \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}. \end{cases}$$



The corresponding linearized system is:

$$\begin{cases} \partial_t \delta u + \partial_x (u\delta u) = 0, \quad \text{in } Q^- \cup Q^+, \\ \\ \delta\varphi'(t)[u]_{\varphi(t)} + \delta\varphi(t)\left(\varphi'(t)[u_x]_{\varphi(t)} - [u_x u]_{\varphi(t)}\right) \\ \quad +\varphi'(t)[\delta u]_{\varphi(t)} - [u\delta u]_{\varphi(t)} = 0, \quad \text{in } (0,T), \\ \\ \delta u(x,0) = \delta u^0, \quad \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}, \\ \delta\varphi(0) = \delta\varphi^0, \end{cases}$$

Majda (1983), Bressan-Marson (1995), Godlewski-Raviart (1999), Bouchut-James (1998), Giles-Pierce (2001), Bardos-Pironneau (2002), Ulbrich (2003), ...

None seems to provide a clear-cut recipe about how to proceed within an optimization loop.

## A new method

A new method: Splitting + alternating descent algorithm.

*C. Castro, F. Palacios, E. Z., M3AS, 2008.*

Ingredients:

- The shock location is part of the state.

  State = Solution as a function + Geometric location of shocks.

- Alternate within the descent algorithm:
  - Shock location and smooth pieces of solutions should be treated differently;
  - When dealing with smooth pieces most methods provide similar results;
  - Shocks should be handeled by geometric tools, not only those based on the analytical solving of equations.

Lots to be done: Pattern detection, image processing, computational geometry,... to locate, deform shock locations,....

# Alternating descent / steepest descent.

- **Steepest descent:**

$$u_{k+1} = u_k - \rho \nabla J(u_k).$$

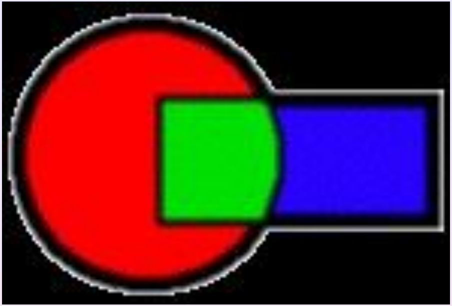
Discrete version of continuous gradient systems

$$u'(\tau) = -\nabla J(u(\tau)).$$

- **Alternating descent:** $J = J(x, y)$, $u = (x, y)$:

$$u_{k+1/2} = u_k - \rho J_x(u_k); \quad u_{k+1} = u_{k+1/2} - \rho J_y(u_k).$$

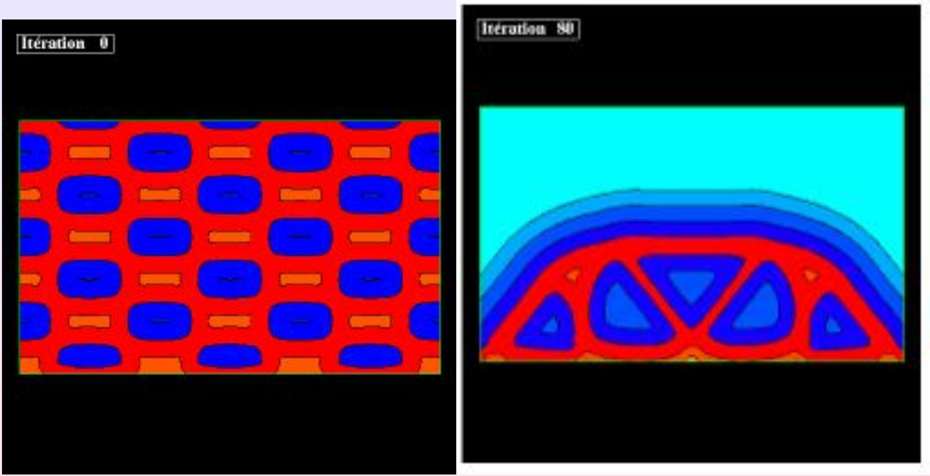
What's the continuous analog? Does it correspond to a class of dynamical systems for which the stability is understood?



The logo of the web page “Domain decomposition”, one of the most widely used computational techniques for solving PDE in domains (“divide and conquer”). Inspired on the works by Karl Hermann Amandus Schwarz (1843 – 1921) and Marius Sophus Lie ( 1842 – 1899 ):

$$\exp(A+B) = \lim_{n\to\infty} \left[ \exp(A/n) \exp(B/n) \right]^n .$$

Compare with the use of shape and topological derivatives in elasticity:




G. Allaire's web page at Ecole Polytechnique, Paris.

## An example: Inverse design of initial data

Consider

$$J(u^0) = \frac{1}{2}\int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

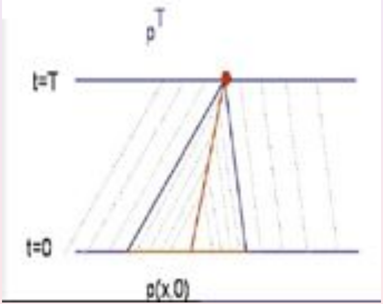
$u^d$ = step function.
Gateaux derivative:

$$\delta J = \int_{\{x<\varphi^0\}\cup\{x>\varphi^0\}} p(x,0)\delta u^0(x)\, dx + q(0)[u]_{\varphi^0}\delta\varphi^0,$$

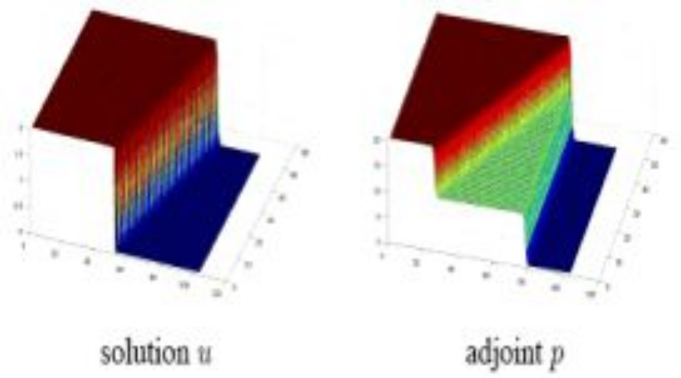
$(p,q)$ = adjoint state

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } Q^- \cup Q^+, \\ [p]_\Sigma = 0, \\ q(t) = p(\varphi(t),t), \ \text{in } t \in (0,T) \\ q'(t) = 0, \ \text{in } t \in (0,T) \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\} \\ q(T) = \frac{\frac{1}{2}\left[(u(x,T)-u^d)^2\right]_{\varphi(T)}}{[u]_{\varphi(T)}}. \end{cases}$$

- The gradient is twofold= variation of the profile + shock location.
- The adjoint system is the superposition of two systems = Linearized adjoint transport equation on both sides of the shock + Dirichlet boundary condition along the shock that propagates along characteristics and fills all the region not covered by the adjoint equations.



State $u$ and adjoint state $p$ when $u$ develops a shock:



## The discrete aproach

Recall the continuous functional

$$J(u^0) = \frac{1}{2}\int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

The discrete version:

$$J^\Delta(u^0_\Delta) = \frac{\Delta x}{2} \sum_{j=-\infty}^{\infty} (u_j^{N+1} - u_j^d)^2,$$

where $u_\Delta = \{u_j^k\}$ solves the 3-point conservative numerical approximation scheme:

$$u_j^{n+1} = u_j^n - \lambda \left( g_{j+1/2}^n - g_{j-1/2}^n \right) = 0, \quad \lambda = \frac{\Delta t}{\Delta x},$$

where, $g$ is the numerical flux

$$g_{j+1/2}^n = g(u_j^n, u_{j+1}^n), \; g(u,u) = u^2/2.$$

# Examples of numerical fluxes

$$
\begin{aligned}
g^{LF}(u,v) &= \frac{u^2+v^2}{4} - \frac{v-u}{2\lambda}, \\
g^{EO}(u,v) &= \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4}, \\
g^{G}(u,v) &= \begin{cases} \min_{w\in[u,v]} w^2/2, & \text{if } u \le v, \\ \max_{w\in[u,v]} w^2/2, & \text{if } u \ge v. \end{cases}
\end{aligned}
$$

# A new method: splitting+alternating descent

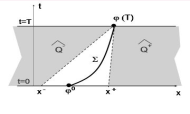
- Generalized tangent vectors $(\delta u^0, \delta\varphi^0) \in T_{u^0}$ s. t.

$$\delta\varphi^0 = \left( \int_{x^-}^{\varphi^0} \delta u^0 + \int_{\varphi^0}^{x^+} \delta u^0 \right) \Big/ [u]_{\varphi^0}.$$

do not move the shock $\delta\varphi(T) = 0$ and

$$\delta J = \int_{\{x<x^-\}\cup\{x>x^+\}} p(x,0)\delta u^0(x)\, dx,$$

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } \hat{Q}^- \cup \hat{Q}^+, \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\}. \end{cases}$$




For those descent directions the adjoint state can be computed by "any numerical scheme"!

- Analogously, if $\delta u^0 = 0$, the profile of the solution does not change, $\delta u(x, T) = 0$ and

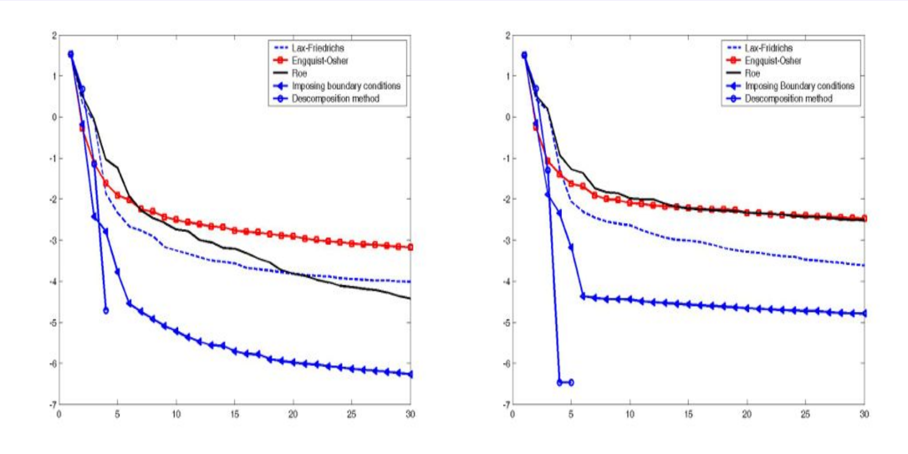
$$\delta J = -\left[\frac{(u(x,T) - u^d(x))^2}{2}\right]_{\varphi(T)} \frac{[u^0]_{\varphi^0}}{[u(\cdot, T)]_{\varphi(T)}} \delta\varphi^0.$$

This formula indicates whether the descent shock variation is left or right!

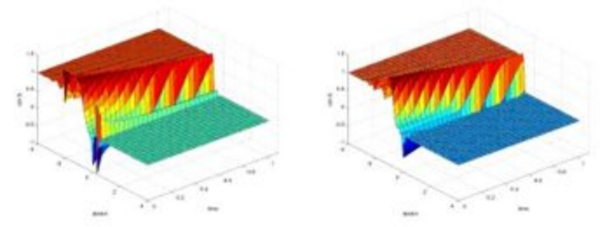
## WE PROPOSE AN ALTERNATING STRATEGY FOR DESCENT

In each iteration of the descent algorithm do two steps:

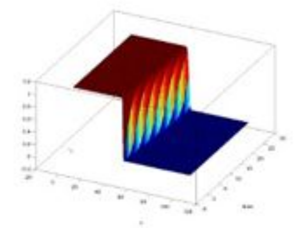
- Step 1: Use variations that only care about the shock location
- Step 2: Use variations that do not move the shock and only affect the shape away from it.




Splitting+Alternating wins!

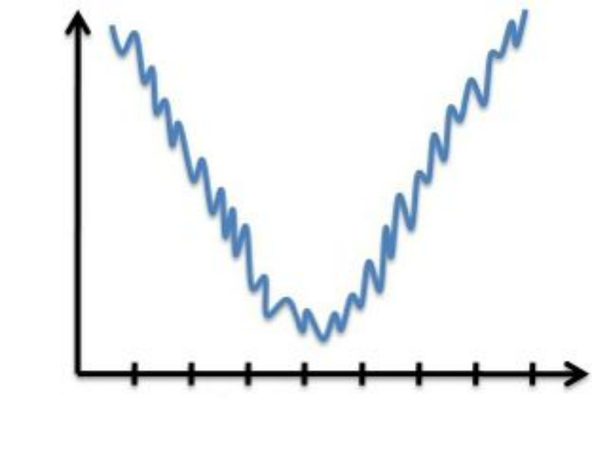


Results obtained applying Engquist-Osher's scheme and the one based on the complete adjoint system

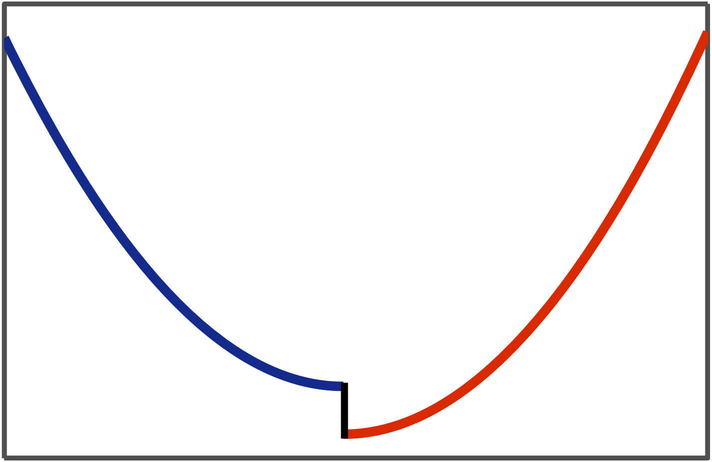


Splitting+Alternating method.

- Numerical schemes replace shocks by oscillations.
- The oscillations of the numerical solution introduce oscillations on the approximation of the functional $J$:



We suggest to stay as close as possible to the true landscape if the functional to be minimized accepting, and even taking advantage of, its possible discontinuities.



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## Joint work with M. Ersoy and E. Feireisl, in progress

Consider the scalar steady driven conservation law

$$\partial_x[f(v(x))] + v(x) = g(x), \ x \in \mathbf{R}. \tag{1}$$

In the context of scalar conservation laws (nonlinear semigroups of $L^1$-contractions), these solutions can be viewed as limits as $t \to \infty$ of solutions of the evolution problem:

$$\partial_t u(t,x) + \partial_x f(u(t,x)) + u(t,x) = g(x), \ u(0,x) = u^0. \tag{2}$$

Entropy $L^1$-solutions exist and are unique in both cases.
The steady state problem can be linearized with respect to variations of the right hand side term

$$\partial_x \left(f'(v)h\right) + h = \delta g. \tag{3}$$

The measure valued solutions $h$ of this problem can be characterized in an unique manner in the context of duality solutions of Bouchut and James [2]

[2] Bouchut F. and James F., One-dimensional transport equations with discontinuous coefficients, Nonlinear Analysis, Theory and Applications, **32** (7) (1998), 891-933.

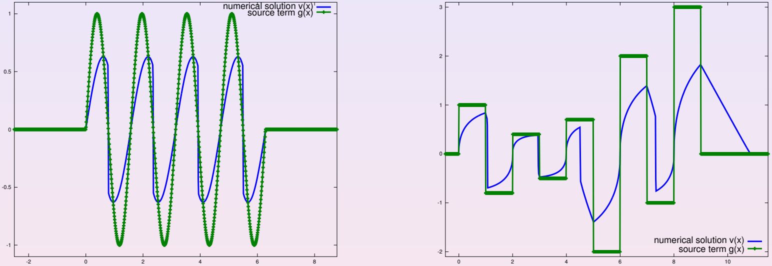
The singular part of the measure contains the sensitivity of its shock location. In this steady-state setting it holds

$$\delta s = [f'(v)\delta v]/[v].$$

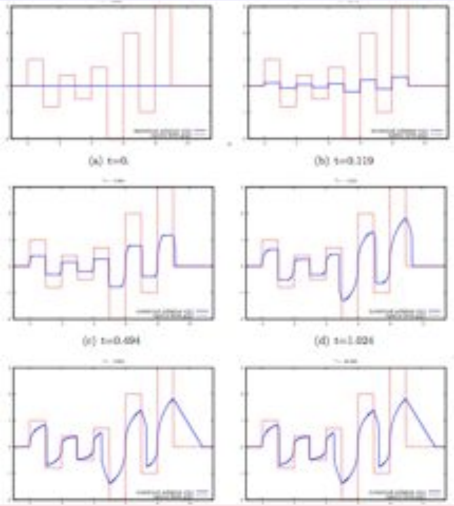
And this is precisely the asymptotic limit as $t \to \infty$ of the time evolution sensitivity of shocks:

$$\delta\varphi'(t)[u]_\varphi + \delta\varphi\left(\varphi'(t)[u_x]_\varphi - [(f(u))_x]_\varphi\right) + \varphi'(t)[\delta u]_\varphi - [f'(u)\delta u]_\varphi = 0,$$

since for the steady state solutions $[(f(u))_x]_s = -[u]_s$.




Two examples of steady state solutions




Convergence towards the stateady state as $t \to \infty$

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## Time evolution control problem. Joint work with A. Porretta, in progress

Consider the finite dimensional dynamics

$$\begin{cases} x_t + Ax = Bu \\ x(0) = x_0 \end{cases} \tag{4}$$

where $A \in \mathcal{M}_{N,N}$, $B \in \mathcal{M}_{N,M}$, the control $u \in L^2(0,T;\mathbf{R}^M)$, and $x_0 \in \mathbf{R}^N$. Given a matrix $C \in \mathcal{M}_{N,N}$, and some $x^* \in \mathbf{R}^N$, consider the optimal control problem

$$\min_u \ J^T(u) = \frac{1}{2}\int_0^T (|u(t)|^2 + |C(x(t)-x^*)|^2)dt\,.$$

There exists a unique optimal control $u(t)$ in $L^2(0,T;\mathbf{R}^M)$, characterized by the optimality condition

$$u = -B^*p\,, \qquad \begin{cases} -p_t + A^*p = C^*C(x-x^*) \\ p(T) = 0 \end{cases} \tag{5}$$

## The steady state control problem

The same problem can be formulated for the steady-state model

$$Ax = Bu.$$

Then there exists a unique minimum $\bar{u}$, and a unique optimal state $\bar{x}$, of the stationary "control problem"

$$\min_u \; J_s(u) = \frac{1}{2}(|u|^2 + |C(x - x^*)|^2), \qquad Ax = Bu, \tag{6}$$

which is nothing but a constrained minimization in $\mathbf{R}^N$; and by elementary calculus, the optimal control $\bar{u}$ and state $\bar{x}$ satisfy

$$A\bar{x} = B\bar{u}, \quad \bar{u} = -B^*\bar{p}, \qquad \text{and} \quad A^*\bar{p} = C^*C(\bar{x} - x^*).$$

We assume that

$$\text{The pair } (A, B) \text{ is controllable,} \tag{7}$$

or, equivalently, that the matrices $A$, $B$ satisfy the Kalman rank condition

$$\mathrm{Rank}\left[B\ AB\ A^2B \dots A^{N-1}B\right] = N\,. \tag{8}$$

Then there exists a linear stabilizing feedback law $L \in \mathcal{M}_{M,N}$ and $c$, $\mu > 0$ such that

$$\begin{cases} x_t + Ax = B(Lx) \\ x(0) = x_0 \end{cases} \implies \quad |x(t)| \leq ce^{-\mu t}|x_0| \qquad \forall t > 0\,. \tag{9}$$

Concerning the cost functional, we assume that the matrix $C$ is such that

$$\text{The pair } (A, C) \text{ is observable} \tag{10}$$

which means that the following algebraic condition holds:

$$\mathrm{Rank}\left[C\ CA\ CA^2 \dots CA^{N-1}\right] = N\,. \tag{11}$$

If $(A, B)$ is controllable, then there exists $c$, independent of $T$, such that, for every $f \in L^2(0, T; \mathbf{R}^N)$, $q_T \in \mathbf{R}^N$, the solution of

$$\begin{cases} -q_t + A^* q = f \\ q(T) = q_T \end{cases} \tag{12}$$

satisfies

$$|q(0)|^2 \leq c \left[ \int_0^T |B^* q|^2 dt + \int_0^T |f|^2 dt + e^{-2\mu T} |q_T|^2 \right], \tag{13}$$

where $\mu$ is as above.

Indeed, multiplying the adjoint equation by $x$ of (9), we get

$$\begin{aligned} q(0) \cdot x_0 &= q_T \cdot x(T) - \int_0^T q(x_t + Ax)\, dt + \int_0^T f \cdot x\, dt \\ &= q_T \cdot x(T) - \int_0^T B^* q \cdot Lx\, dt + \int_0^T f \cdot x\, dt, \end{aligned}$$

and using the exponential decay of $x(t)$ we obtain

$$|q(0) \cdot x_0| \leq C|x_0| \left[ \int_0^T |B^* q|^2 dt + \int_0^T |f|^2 dt \right]^{\frac{1}{2}} + C|x_0|\, e^{-\mu T} |q_T|$$

which suffices with $x_0 = q(0)$.

Under the above controllability and observability assumptions, we have the following result.

## Theorem

*Assume that (8) and (11) hold true. Then we have*

$$\frac{1}{T}\min_{u\in L^2(0,T)} J^T \quad \stackrel{T\to\infty}{\longrightarrow} \quad \min_{u\in \mathbf{R}^N} J_s$$

*and*

$$\frac{1}{T}\int_0^T \left(|u(t)-\bar{u}|^2 + |C(x(t)-\bar{x})|^2\right) dt \to 0$$

*where $\bar{u}$ is the optimal control of $J_s$ and $\bar{x}$ the corresponding optimal state.*

*In particular, we have*

$$\frac{1}{(b-a)T}\int_{aT}^{bT} x(t)\,dt \to \bar{x} \quad , \quad \frac{1}{(b-a)T}\int_{aT}^{bT} u(t)\,dt \to \bar{x}$$

*for every $a, b \in [0,1]$.*

## Proof

We use the optimality conditions defining the adjoint states $p$ and $\bar{p}$, which give

$$\begin{cases} (x-\bar{x})_t + A(x-\bar{x}) = B(u-\bar{u}) \\ u-\bar{u} = -B^*(p-\bar{p}) \\ -(p-\bar{p})_t + A^*(p-\bar{p}) = C^*C(x-\bar{x}) \end{cases}$$

Using the observability inequality (13) we have

$$|(p(0)-\bar{p})| \leq c\left[\left(\int_0^T |C(x-\bar{x})|^2\,dt\right)^{\frac{1}{2}} + \left(\int_0^T |B^*(p-\bar{p})|^2\,dt\right)^{\frac{1}{2}} + |\bar{p}|\right]. \quad (14)$$

Similarly, in the equation of $x-\bar{x}$ we use the observability inequality for $(A, C)$ which is ensured by (11):

$$|x(T)-\bar{x}| \leq c\left(\int_0^T |u-\bar{u}|^2 dt + \int_0^T |C(x(t)-\bar{x})|^2 dt + |x_0-\bar{x}|^2\right)^{\frac{1}{2}}. \quad (15)$$

From the optimality system we get

$$[(x-\bar{x})(p-\bar{p})]_t = B(u-\bar{u})(p-\bar{p}) - |C(x-\bar{x})|^2$$

which implies

$$\int_0^T \left(|u-\bar{u}|^2 + |C(x-\bar{x})|^2\right) dt = [(x_0-\bar{x})(p(0)-\bar{p})] + [(x(T)-\bar{x})\bar{p}]$$

hence

$$\int_0^T \left(|u-\bar{u}|^2 + |C(x-\bar{x})|^2\right) dt \leq c \tag{16}$$

by the previous estimates (14) and (15). We conclude

$$\frac{1}{T}\int_0^T \left(|u-\bar{u}|^2 + |C(x-\bar{x})|^2\right) dt \leq \frac{C}{T} \to 0\,.$$

This of course also implies the convergence of the averaged minimum level to the stationary minimum.

## Rate of convergence

Following [CLLP][3], if $B^*$ and $C$ are coercive we also have

$$|u-\bar{u}|^2+|C(x-\bar{x})|^2=|B^*(p-\bar{p})|^2+|C(x-\bar{x})|^2\geq\gamma\left(|p-\bar{p}|^2+|x-\bar{x}|^2\right)$$

hence we deduce from the optimality system

$$[(x-\bar{x})(p-\bar{p})]_t=-|B^*(p-\bar{p})|^2-|C(x-\bar{x})|^2\leq-\gamma|(x-\bar{x})(p-\bar{p})|\,,$$

for some $\gamma>0$. Since $(x-\bar{x})(p-\bar{p})$ is bounded at $t=0$ and $t=T$ due to (14), (15) and (16), we obtain

$$-e^{-\gamma(T-t)}K\leq[(x-\bar{x})(p-\bar{p})](t)\leq Ke^{-\gamma t}$$

for some $K>0$. Integrating we get

$$\int_{aT}^{bT}\left(\left|u-\bar{u}\right|^2+\left|x-\bar{x}\right|^2\right)ds\leq K\left(e^{-\gamma aT}+e^{-\gamma(1-b)T}\right)$$

which implies an exponential rate of convergence.

[3] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, A. Porretta, *Long time average of Mean Field Games*, Network Heterogeneous Media, 7 (2), 2012

## Time scaling = Singular perturbations

Note that the problem in the time interval $[0, T]$ as $T \to \infty$ can be rescaled into the fixed time interval $[0, 1]$ by the change of variables $t = Ts$.
In this case the evolution control problem takes the form

$$\varepsilon x_s + Ax = Bu,\ s \in [0, 1].$$

In the limit as $\varepsilon \to 0$ the steady-state equation emerges:

$$Ax = Bu.$$

This becomes a classical singular perturbation control problem.
Note however that, in this setting, the role that the controllability and observability properties of the system play is much less clear than when dealing with it as $T \to \infty$.

# Table of Contents

- The linear finite dimensional theory linking time evolution and steady state control problems can be extended to the PDE setting (heat and wave equations) under suitable controllability and observability assumptions that are by now well understood.
- The extension of this theory to the nonlinear setting is a widely open subject.
- This is particularly the case for models in aerodynamic optimal design.
- Often times the controllability and observability assumptions needed to link the time evolution and the steady state optimal control problems are ignored when "reducing" one problem to the other.