Control and numerics

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Outline

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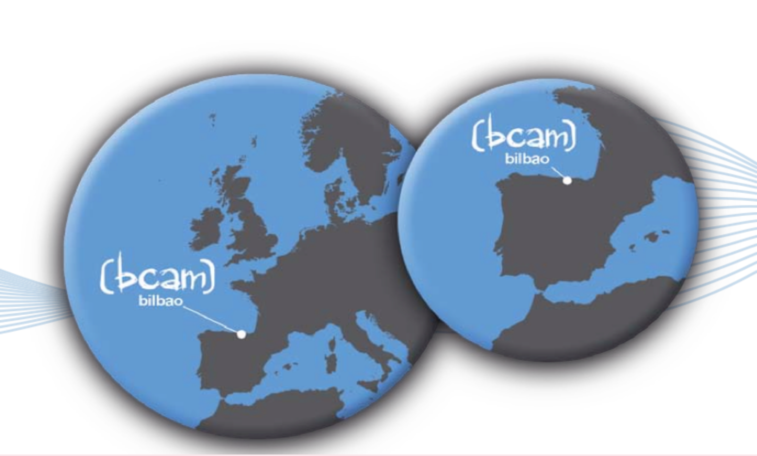
1. Flux Identification for hyperbolic conservation laws
2. The numerical control of waves
3. Optimal location of observers and controllers (joint with Y. Privat and E. Trélat)
4. Several open problems for Jacques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Flux identification.

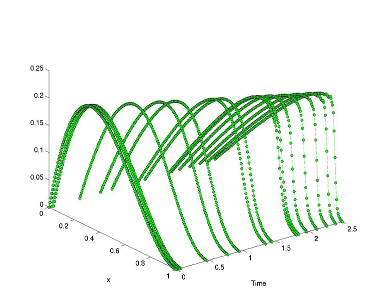
$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0, & \text{in } \mathbf{R} \times (0, T), \\ u(x, 0) = u^0(x), & x \in \mathbf{R}. \end{cases}$$

The control is the nonlinearity f. It is actually an inverse problem.

- F. James and M. Sepúlveda, Convergence results for the flux identification in a scalar conservation law. *SIAM J. Control Optim.* **37**(3) (1999) 869-891.
- C. Castro and E. Zuazua, Flux identification for 1-d scalar conservation laws in the presence of shocks, Mathematics of Computation, March 2011.

Solutions may develop shocks or quasi-shock configurations.

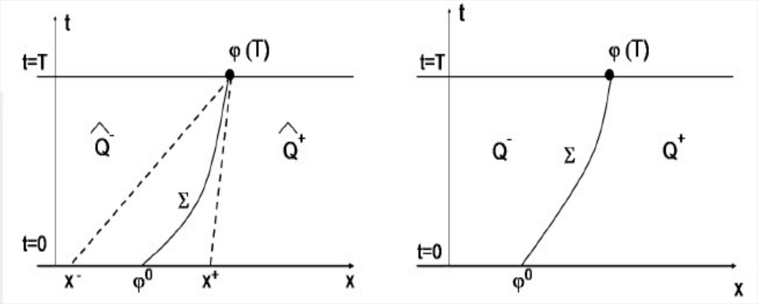
- For shock solutions, classical calculus fails: The derivative of a discontinuous function is a Dirac delta;
- For quasi-shock solutions the sensitivity (gradient) is so large that classical sensitivity calculus is meaningless.



Solution as a pair: flow+shock variables

Then the pair (u, φ)= (flow solution, shock location) solves:

$$\begin{cases} \partial_t u + \partial_x(\frac{u^2}{2}) = 0, & \text{in } Q^- \cup Q^+, \\ \varphi'(t)[u]_{\varphi(t)} = \left[u^2/2\right]_{\varphi(t)}, & t \in (0, T), \\ \varphi(0) = \varphi^0, & \\ u(x,0) = u^0(x), & \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}. \end{cases}$$



In the inviscid case, the simple and "natural" rule

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \to \frac{\partial \delta u}{\partial t} + \delta u\frac{\partial u}{\partial x} + u\frac{\partial \delta u}{\partial x} = 0$$

breaks down in the presence of shocks

$\delta u =$ discontinuous, $\frac{\partial u}{\partial x} =$ Dirac delta $\Rightarrow \delta u\frac{\partial u}{\partial x}$????

The difficulty may be overcame with a suitable notion of measure valued weak solution using Volpert's definition of conservative products and duality theory (Bouchut-James, Godlewski-Raviart,...)

The corresponding linearized system is:

$$\begin{cases} \partial_t \delta u + \partial_x (u \delta u) = 0, \quad \text{in } Q^- \cup Q^+, \\ \\ \delta\varphi'(t)[u]_{\varphi(t)} + \delta\varphi(t)\left(\varphi'(t)[u_x]_{\varphi(t)} - [u_x u]_{\varphi(t)}\right) \\ \quad +\varphi'(t)[\delta u]_{\varphi(t)} - [u\delta u]_{\varphi(t)} = 0, \quad \text{in } (0,T), \\ \\ \delta u(x,0) = \delta u^0, \quad \text{in } \{x < \varphi^0\} \cup \{x > \varphi^0\}, \\ \delta\varphi(0) = \delta\varphi^0, \end{cases}$$

Majda (1983), Bressan-Marson (1995), Godlewski-Raviart (1999), Bouchut-James (1998), Giles-Pierce (2001), Bardos-Pironneau (2002), Ulbrich (2003), ...
None seems to provide a clear-cut recipe about how to proceed within an optimization loop.

Continuous versus discrete

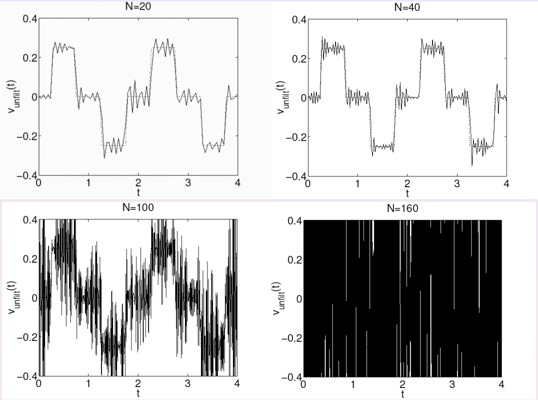
Two approaches:

- **Continuous:** PDE+ Optimal shape design $\rightarrow$ implement that numerically.
- **Discrete:** Replace PDE and optimal design problem by discrete version $\rightarrow$ Apply discrete tools

Do these processes lead to the same result?

$$\text{OPTIMAL DESIGN} + \text{NUMERICS}$$
$$=$$
$$\text{NUMERICS} + \text{OPTIMAL DESIGN?}$$

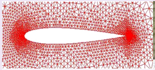
NO!!!!!!



E. Z., SIAM Review, 47 (2) (2005), 197-243.

Discrete: Discretization + gradient

- Advantages: Discrete clouds of values. No shocks. Automatic differentiation, ...
- Drawbacks:
  - "Invisible" geometry.



  - Scheme dependent.

Continuous: Continuous gradient + discretization.

- Advantages: Simpler computations. Solver independent. Shock detection.
- Drawbacks:
  - Yields approximate gradients.
  - Subtle if shocks.

SHOCKS: A MUST

- Discrete approach: You do not see them
- Continuous approach: They make life difficult

A new method

A new method: Splitting + alternating descent algorithm.
C. Castro, F. Palacios, E. Z., M3AS, 2008.
Ingredients:

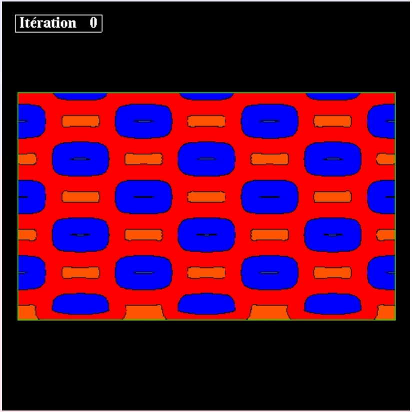
- The shock location is part of the state.

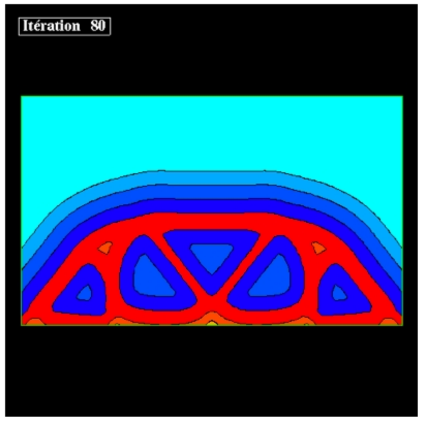
  State = Solution as a function + Geometric location of shocks.

- Alternate within the descent algorithm:
  - Shock location and smooth pieces of solutions should be treated differently;
  - When dealing with smooth pieces most methods provide similar results;
  - Shocks should be handeled by geometric tools, not only those based on the analytical solving of equations.

Lots to be done: Pattern detection, image processing, computational geometry,... to locate, deform shock locations,....

Compare with the use of shape and topological derivatives in elasticity:





An example: Inverse design of initial data

Consider

$$J(u^0) = \frac{1}{2} \int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

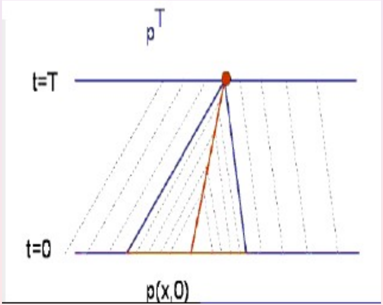
u^d = step function.
Gateaux derivative:

$$\delta J = \int_{\{x<\varphi^0\} \cup \{x>\varphi^0\}} p(x,0)\delta u^0(x)\, dx + q(0)[u]_{\varphi^0} \delta\varphi^0,$$

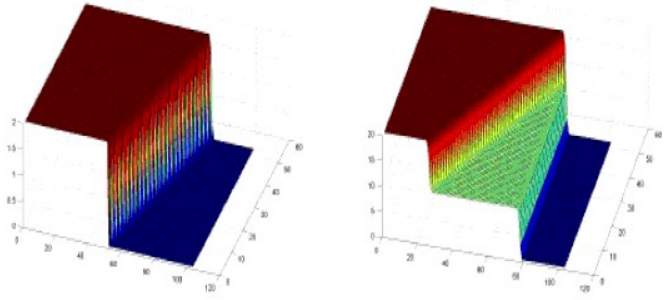
(p,q) = adjoint state

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } Q^- \cup Q^+, \\ [p]_\Sigma = 0, \\ q(t) = p(\varphi(t),t), \text{ in } t \in (0,T) \\ q'(t) = 0, \text{ in } t \in (0,T) \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\} \\ q(T) = \dfrac{\frac{1}{2}\left[(u(x,T)-u^d)^2\right]_{\varphi(T)}}{[u]_{\varphi(T)}}. \end{cases}$$

- The gradient is twofold= variation of the profile + shock location.
- The adjoint system is the superposition of two systems = Linearized adjoint transport equation on both sides of the shock + Dirichlet boundary condition along the shock that propagates along characteristics and fills all the region not covered by the adjoint equations.



State u and adjoint state p when u develops a shock:



solution u adjoint p

The discrete aproach

Recall the continuous functional

$$J(u^0) = \frac{1}{2} \int_{-\infty}^{\infty} |u(x,T) - u^d(x)|^2 dx.$$

The discrete version:

$$J^\Delta(u_\Delta^0) = \frac{\Delta x}{2} \sum_{j=-\infty}^{\infty} (u_j^{N+1} - u_j^d)^2,$$

where $u_\Delta = \{u_j^k\}$ solves the 3-point conservative numerical approximation scheme:

$$u_j^{n+1} = u_j^n - \lambda \left(g_{j+1/2}^n - g_{j-1/2}^n \right) = 0, \quad \lambda = \frac{\Delta t}{\Delta x},$$

where, g is the numerical flux

$$g_{j+1/2}^n = g(u_j^n, u_{j+1}^n), \, g(u,u) = u^2/2.$$

Examples of numerical fluxes

$$
\begin{aligned}
g^{LF}(u,v) &= \frac{u^2+v^2}{4} - \frac{v-u}{2\lambda},\\
g^{EO}(u,v) &= \frac{u(u+|u|)}{4} + \frac{v(v-|v|)}{4},\\
g^{G}(u,v) &= \begin{cases} \min_{w\in[u,v]} w^2/2, & \text{if } u \le v,\\ \max_{w\in[u,v]} w^2/2, & \text{if } u \ge v, \end{cases}
\end{aligned}
$$

The Γ-convergence of discrete minimizers towards continuous ones is guaranteed for the schemes satisfying the so called one-sided Lipschitz condition (OSLC):

$$\frac{u_{j+1}^n - u_j^n}{\Delta x} \leq \frac{1}{n\Delta t},$$

which is the discrete version of the Oleinick condition for the solutions of the continuous Burgers equations

$$u_x \leq \frac{1}{t},$$

which excludes non-admissible shocks and provides the needed compactness of families of bounded solutions.

As proved by Brenier-Osher, [1] Godunov's, Lax-Friedfrichs and Engquits-Osher schemes fulfil the OSLC condition.

[1] Brenier, Y. and Osher, S. The Discrete One-Sided Lipschitz Condition for Convex Scalar Conservation Laws, SIAM Journal on Numerical Analysis, **25** (1) (1988), 8-23.

A new method: splitting+alternating descent

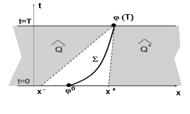
- Generalized tangent vectors $(\delta u^0, \delta\varphi^0) \in T_{u^0}$ s. t.

$$\delta\varphi^0 = \left(\int_{x^-}^{\varphi^0} \delta u^0 + \int_{\varphi^0}^{x^+} \delta u^0 \right) \Big/ [u]_{\varphi^0}.$$

do not move the shock $\delta\varphi(T) = 0$ and

$$\delta J = \int_{\{x<x^-\}\cup\{x>x^+\}} p(x,0)\delta u^0(x)\, dx,$$

$$\begin{cases} -\partial_t p - u\partial_x p = 0, & \text{in } \hat{Q}^- \cup \hat{Q}^+, \\ p(x,T) = u(x,T) - u^d, & \text{in } \{x < \varphi(T)\} \cup \{x > \varphi(T)\}. \end{cases}$$



For those descent directions the adjoint state can be computed by "any numerical scheme"!

- Analogously, if $\delta u^0 = 0$, the profile of the solution does not change, $\delta u(x,T) = 0$ and

$$\delta J = -\left[\frac{(u(x,T) - u^d(x))^2}{2}\right]_{\varphi(T)} \frac{[u^0]_{\varphi^0}}{[u(\cdot,T)]_{\varphi(T)}}\delta\varphi^0.$$

This formula indicates whether the descent shock variation is left or right!

WE PROPOSE AN ALTERNATING STRATEGY FOR DESCENT

In each iteration of the descent algorithm do two steps:

- Step 1: Use variations that only care about the shock location
- Step 2: Use variations that do not move the shock and only affect the shape away from it.

An open problem: Alternating descent / steepest descent.

- **Steepest descent:**

$$u_{k+1} = u_k - \rho \nabla J(u_k).$$

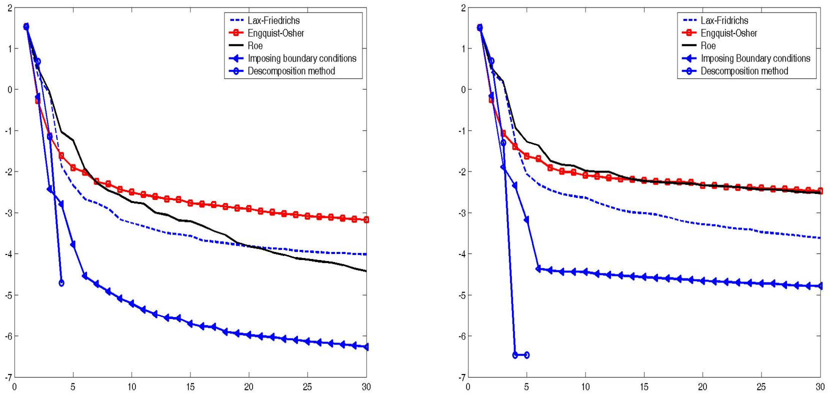
Discrete version of continuous gradient systems

$$u'(\tau) = -\nabla J(u(\tau)).$$

- **Alternating descent:** $J = J(x, y)$

$$u_{k+1/2} = u_k - \rho J_x(u_k); \quad u_{k+1} = u_{k+1/2} - \rho J_y(u_k).$$

What's the continuous analog? Does it correspond to a class of dynamical systems for which the stability is understood?



Splitting+Alternating wins!

Sol y sombra!

Splitting+alternating is more efficient:

- It is faster.
- It does not increase the complexity.
- Rather independent of the numerical scheme.

Two key ideas: Iterate + Alternate.

D. Auroux & J. Blum, Back and forth nudging algorithm for data assimilation problems : CRAS, Ser.1 340, 2005, 873-878.
Identification of data as limit of an iterative algorithm for dissipative semigroups.

Other related works:

- N. Cindea, S. Micu, M.Tucsnak, An approximation method for exact controls of vibrating systems. SIAM J. Control Optim. 49, pp. 1283-1305, June 2011.
- D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: Recent progress and open questions. SIAM Review 20 (1978), 639-739.

The numerical control of waves

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control on one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \quad 0 < t < T \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.
The goal is to stop the vibrations, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1.$$

THE 1-D OBSERVATION PROBLEM

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T \\ \varphi(x,0) = \varphi^0(x),\ \varphi_t(x,0) = \varphi^1(x), & 0 < x < 1. \end{cases}$$

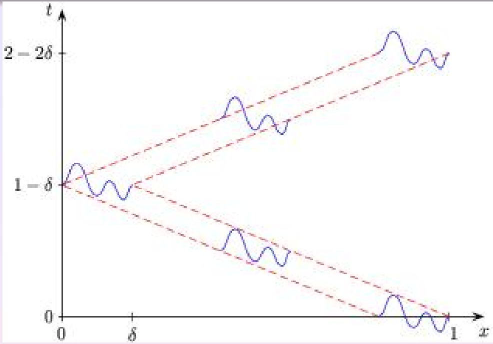
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|\varphi_x(x,t)|^2 + |\varphi_t(x,t)|^2\right] dx = E(0), \ \ \forall 0 \le t \le T.$$

The question is then reduced to analyze whether the folllowing inequality is true. This is the so called observability inequality:

$$E(0) \le C(T) \int_0^T |\varphi_x(1,t)|^2 \, dt.$$

The answer to this question is easy to gues: The observability inequality holds if and only if $T \ge 2$.



$$E(0) \leq C(T) \int_0^T |u_x(1,t)|^2 \, dt.$$

Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order of 2.

CONSTRUCTION OF THE CONTROL:

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions' HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \varphi_x(1, t),$$

where u is the solution of the adjoint system corresponding to initial data $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1},$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that $\varphi_x(1,t) \in L^2(0,T)$ (hidden regularity). Moreover,

COERCIVITY OF J = OBSERVABILITY INEQUALITY.

The continuous numerical approach: Gradient algorithms

The control was characterized as being the minimizer over $H_0^1(0,1) \times L^2(0,1)$ of

$$J(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T |\varphi_x(1,t)|^2 dt + \int_0^1 y^0 \varphi^1 dx - < y^1, \varphi^0 >_{H^{-1} \times H_0^1} .$$

We produce an algorithm in which:

- We replace J by some numerical approximation J_h with an order h^θ.
- We apply a gradient iteration algorithm to J_h.

The following holds:

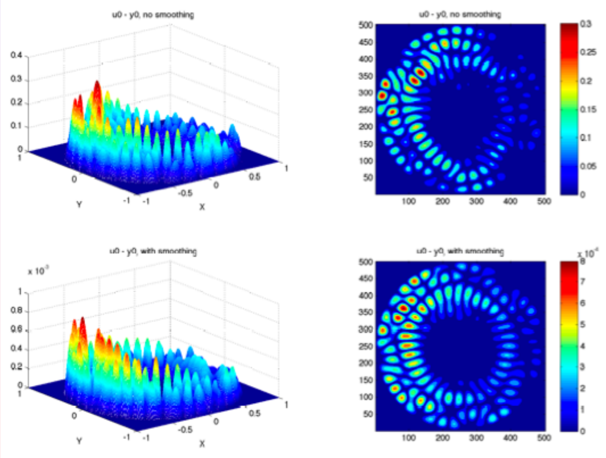
Theorem

(S. Ervedoza & E. Z., 2011)
In $K \sim C|\log(h)|$ *iterations, the controls* v_h^K *obtained after applying* K *iterations of the gradient algorithm to* J_h *fulfill:*

$$||v - v_h^K|| \leq C|\log(h)|^{\max(\theta,1)} h^\theta.$$

Note that for the classical Finite Difference and Finite Element methods for the wave equation the convergence order is $\theta = 2/3$.

We refer to the following article for a number of interesting simulations showing the complexity of the behavior of the controls: G. Lebeau and M. Nodet, Experimental Study of the HUM Control Operator for Linear Waves, Experimental Mathematics, 19 (1) (2010), 93-120.



- This continuous approach is easy to be implemented.
- It requires a significant amount of previous work at the PDE level. In particular showing that the minimizers of the functional J above keep the regularity of the data to be controlled: “Smooth data $\Longrightarrow$ Smooth Controls.”
- Note however that it does not yield any result on the control of the approximated finite-dimensional dynamics but only the convergence towards the continuous control.
- When implementing a gradient descent algorithm for solving this minimization problem one is led to an iterative procedure similar to Nudging by Auroux and Blum. The difference is that this time we combine forward resolution of the state equation and backward resolution of the adjoint and that both models are conservative.

Optimal location of observers and controllers (joint with Y. Privat and E. Trélat)

$$
\begin{gathered}
y_{tt} - y_{xx} = h_\omega, \quad (t,x) \in (0,T) \times (0,\pi), \\
y(t,0) = y(t,\pi) = 0, \quad t \in [0,T], \\
y(0,x) = y^0(x),\ y_t(0,x) = y^1(x), \quad x \in [0,\pi],
\end{gathered}
$$

- $T > 0$ fixed
- h_ω: internal control with support in $(0,T) \times \omega$
- $\omega \subset [0,\pi]$ of positive measure

Objective

Steer the system to $y(T,\cdot) = y_t(T,\cdot) = 0$.

Question

What is the "best possible" control domain ω of given measure?

The case $T = 2\pi$

Two problems naturally arise. That of finding the best location for a given initial datum to be controlled and that of doing it uniformly for all data.

First problem

Given $(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$, minimize

$$F(\chi_\omega) = \sum_{j=1}^{+\infty} \frac{\rho_j^2}{\int_\omega \sin^2(jx)\, dx}$$

over all $\omega \subset [0,\pi]$ of Lebesgue measure $|\omega| = L\pi$.

Second problem

Minimize

$$\sup\left\{ F(\chi_\omega) \mid \sum_{j=1}^{+\infty} \rho_j^2 = 1 \right\}$$

Solving of the first problem

1. Relaxation procedure

For every $a(\cdot) \in L^\infty(0,\pi)$, set

$$F(a) = \sum_{j=1}^{+\infty} \frac{\rho_j^2}{\int_0^\pi a(x)\sin^2(jx)\,dx}.$$

$$\overline{\mathcal{U}}_L = \{a \in L^\infty(0,\pi;(0,1)) \mid \int_0^\pi a(x)\,dx = L\pi\}.$$

Theorem

The relaxed problem

$$\min_{a\in\overline{\mathcal{U}}_L} F(a)$$

has a unique solution.

The application of the Pontryagin Maximum Principle leads to

$$a(x) = \begin{cases} 1 & \text{if } \varphi(x) > 0, \\ 0 & \text{if } \varphi(x) < 0, \end{cases}$$

with

$$\varphi(x) = p_y + \frac{1}{2} \sum_{j=1}^{+\infty} p_j \cos(2jx)$$

where $p_y \in \mathbb{R}$ is a parameter (implicitly determined), and

$$p_j = -\frac{\rho_j^2}{\int_0^\pi a(x) \sin^2(jx)\, dx}$$

The application of the Pontryagin Maximum Principle leads to

$$a(x) = \begin{cases} 1 & \text{if } \varphi(x) > 0, \\ 0 & \text{if } \varphi(x) < 0, \end{cases}$$

3. Back to the first problem

Theorem

If $\exists M, \delta > 0$ such that

$$\forall j \in \mathbb{N}^* \quad |\rho_j| \leq M e^{-\delta j},$$

then the first problem has a unique solution χ_ω, where ω is a measurable subset of $[0, \pi]$ of Lebesgue measure $L\pi$. Moreover,

- ω is symmetric with respect to $\pi/2$,
- there exists $\eta > 0$ such that $\omega \subset [\eta, \pi - \eta]$,
- ω has a finite number of connected components.

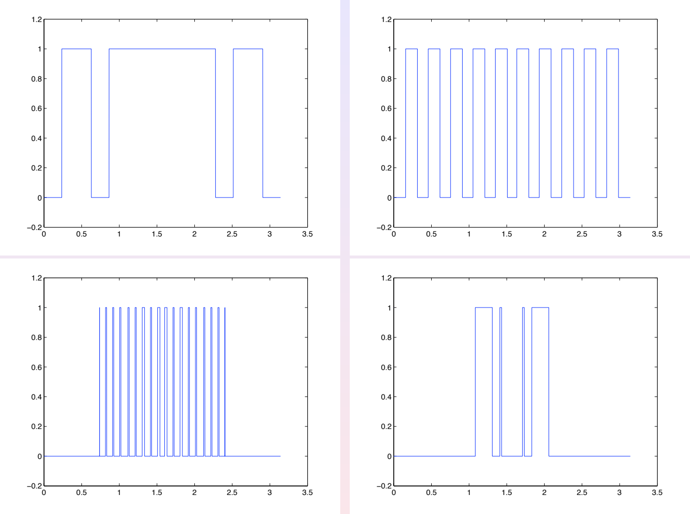
(see S. Mandelbrojt, *Quasi-analycité des séries de Fourier*, Ann. Scuola Normale Sup. Pisa, tome 4, no. 3 (1935), 225–229)

4. Characterization of the relaxation phenomenon

Characterization of all initial data (y^0, y^1) for which the relaxation phenomenon occurs: initial data whose associated coefficients ρ_j are such that the corresponding switching function φ vanishes identically on a subset I of positive measure (symmetric with respect to $\frac{\pi}{2}$).

$\longrightarrow$ explicit characterization in terms of Fourier series

$\longrightarrow$ explicit examples where relaxation occurs (with smooth initial data)



$$\big[N = 4,\ L = 0.7,\ \rho = (5, 0, 0.1, 5)\big]\ [N = 10,\ L = 0.5,\ \rho = (1, \ldots, 1)]$$
$$[N = 30,\ L = 0.1,\ \rho_j = 1/j,\ j = 1, \ldots, 30]\ [N = 50,\ L = 0.2,\ \rho_j = 1/j^2,\ j = 1, \ldots, 50]$$

Solving of the second problem

Second problem

$$\sup_{\substack{\omega\subset[0,\pi]\\|\omega|=L\pi}} \inf_{j\in\mathbb{N}^*} \int_\omega \sin^2(jx)\,dx$$

1. Relaxation procedure

$$\overline{\mathcal{U}}_L = \{a \in L^\infty(0,\pi;(0,1)) \mid \int_0^\pi a(x)\,dx = L\pi\}.$$

$$\longrightarrow \quad \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_0^\pi a(x)\sin^2(jx)\,dx$$

has an $\infty\#$ of solutions $a(x) = L + \sum_j b_j \sin(2jx)$, and

$$\sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_0^\pi a(x)\sin^2(jx)\,dx = \frac{L\pi}{2}.$$

2. Gap or no-gap?

A priori:

$$\sup_{\substack{\omega\subset[0,\pi]\\ |\omega|=L\pi}} \inf_{j\in\mathbb{N}^*} \int_\omega \sin^2(jx)\,dx \leq \sup_{a\in\mathcal{U}_L} \inf_{j\in\mathbb{N}^*} \int_0^\pi a(x)\sin^2(jx)\,dx = \frac{L\pi}{2}$$

Theorem

Assume that $L \in (0,1)\setminus\{1/2\}$. Then there is no gap: $\sup_\omega \inf_j \int_\omega \sin^2(jx)\,dx = \frac{L\pi}{2}$. However the second problem does not have any solution.

Technical proof, based on the following harmonic analysis lemma:

Let $\mathcal{F}$ the set of functions

$$f(x) = L + \sum_{j=1}^{+\infty} (a_j \cos(2jx) + b_j \sin(2jx)), \qquad \text{with } a_j \leq 0 \quad \forall j \in \mathbb{N}^*.$$

Then:

$$d(\mathcal{F}, \mathcal{U}_L) = 0$$

but there is no $\chi_\omega \in \mathcal{F}$.

(where $\mathcal{U}_L = \{\chi_\omega \mid \omega \subset [0, \pi], \ |\omega| = L\pi\}$)

Truncated version of the second problem

Since the second problem has no solution, we consider as in

P. Hébrard, A. Henrot, *A spillover phenomenon in the optimal location of actuators*, SIAM J. Control Optim. **44** (2005), 349–366.

a truncated version:

$$\sup_{\substack{\omega\subset[0,\pi]\\ |\omega|=L\pi}} \quad \min_{1\leq j\leq N} \int_\omega \sin^2(jx)\,dx$$

Truncated version of the second problem

$$\sup_{\substack{\omega\subset[0,\pi]\\ |\omega|=L\pi}} \min_{1\le j\le N} \int_\omega \sin^2(jx)\,dx$$

Theorem

The problem has a unique solution ω^N, satisfying:

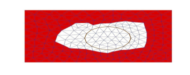
- ω^N is the union of at most N intervals;
- ω^N is symmetric with respect to $\pi/2$;
- there exists η_N such that $\omega^N \subset [\eta_N, \pi-\eta_N]$;
- there exists $L_N \in (0,1]$ such that, for every $L \in (0, L_N]$,

$$\int_{\omega^N} \sin^2 x\,dx = \int_{\omega^N} \sin^2(2x)\,dx = \cdots = \int_{\omega^N} \sin^2(Nx)\,dx.$$

- Equality of the criteria $\Rightarrow$ the optimal domain ω^N concentrates around the points $\frac{k\pi}{N+1}$, $k = 1, \dots, N$.
- Spillover phenomenon: the best domain ω^N for the N rst modes is the worst possible for the $N+1$ first modes.

Conclusion

- The optimal location of actuators and sensors is an interesting topic in engineering but mostly open form the analysis viewpoint.
- Different problems have been considered, and they may lead to very different phenomena(classical solutions, relaxation...)
- Complete analysis requires of the development of quite complex tools of harmonic analysis.
- The rigorous numerical analysis of these issues is still to be done (see Münch, Pedregal, Periago,... for numerical simulations using level sets methods).
- Generally speaking, lots to be done in the interface between optimal design and numerics:
  D. Chenais and E. Z. Finite Element Approximation of 2D Elliptic Optimal Design, JMPA, 85 (2006), 225-249.

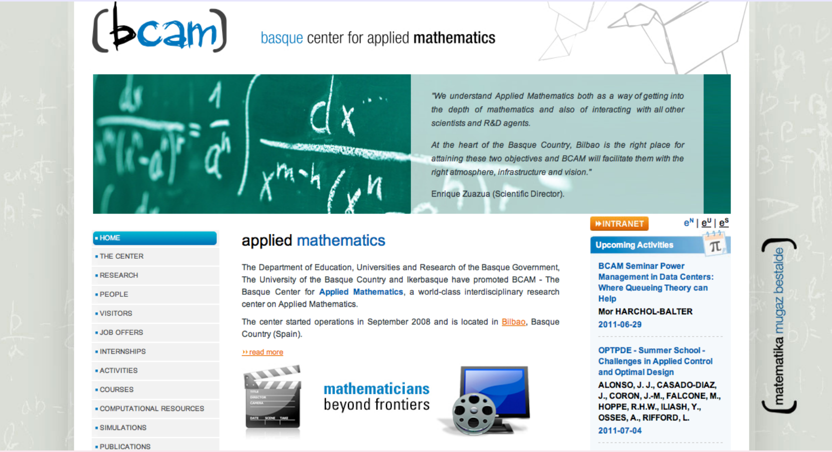


Several open problems for Jacques

So far we have:

- Described several applications of methods closely related to Auroux & Blum's nudging.
- In particular in what concerns the control of wave processes.
- Analyzed the problem of optimal location of actuators.

JB : MATEMATIKA MUGAZ BESTALDE



Bernardo Atxaga: 37 Galdera Mugaz Bestalde Dudan Kontaktu Bakarrari
37 Questions à mon seul contact de l'autre côté de la frontière

Esaidan, zoriontsuak al zarete mugaz bestaldeko biztanleak?
Stp, dis moi si les gens de l'autre côté de la frontière sont heureux ?

Mugaz bestaldean, hostoek ematen al diete babesa fruituei? Ba al dago marrubirik? Arrain abisalek ba al dute aurresentipenik eguzkiaz?
De l'autre côté de la frontière, la feuille protège-t-elle le fruit?
Y-a-t' il des fraises?
Les poissons abyssaux ont ils l'appréhension du soleil?

Asko al dira, asko al zarete mugaz bestaldeko erresuma hartan?
Egunero kaletik ikusten dudan jende hau, han bizi al da?
Les habitants de l'autre côté de la frontière êtes-vous, sont-ils nombreux?
Les gens que je vois tous les jours dans la rue, habitent-ils là bas?

Félicitations, Jacques.
Collègue et ami de "mugaz bestalde"