Mathematics of Planet Earth

Discovering the Earth

Putting mathematical glasses to discover:

- its shape and size: Eratosthenes (3rd century BC)
- its mass: Newton (18th century)
- its age: Kelvin, Perry, etc. (19th century)
- its interior: Dixon Oldham, Lehmann (20th century)
- its movements as a planet: Laskar (21st centuy)

Evaluating the mass of the Earth

We use Newton's gravitational law and deduce the mass of the Earth from the gravitational attraction of the Earth at the surface of the Earth. We get

$$M=5.98 \times 10^{24} \text{ kg}$$

The Earth is much too heavy to be homogeneous since the density of the crust is around 2.2-2.9 kg/dm^3 and the mean density of 5.52 kg/dm^3.

This means that the interior of the Earth is very heavy!

Discovering the Earth interior

Richard Dixon Oldham identified the different types of seismic waves recorded on seismographs:

. P-waves: the pressure waves travel through the viscous interior

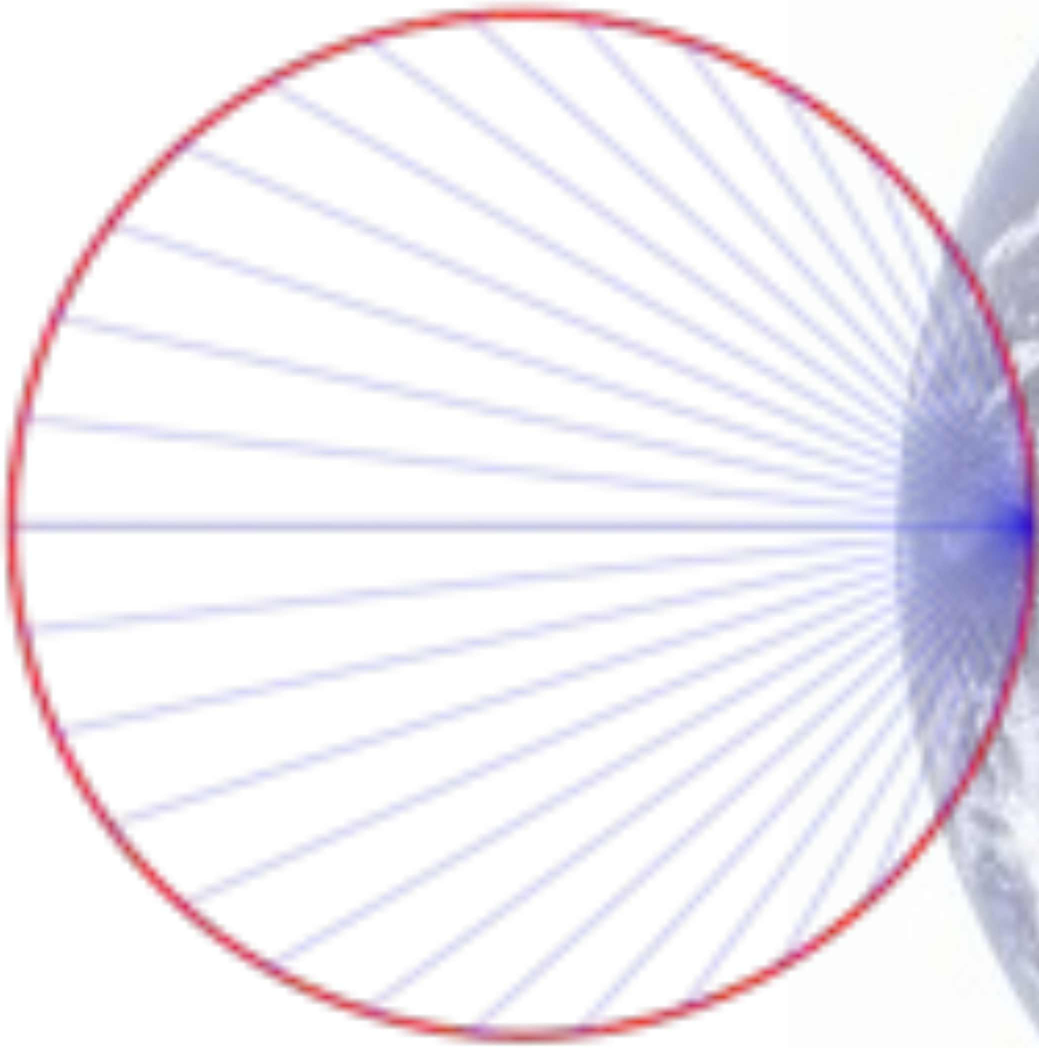
. S-waves: the shear waves are damped in the mantle, and hence not recorded far from the epicenter of an earthquake.

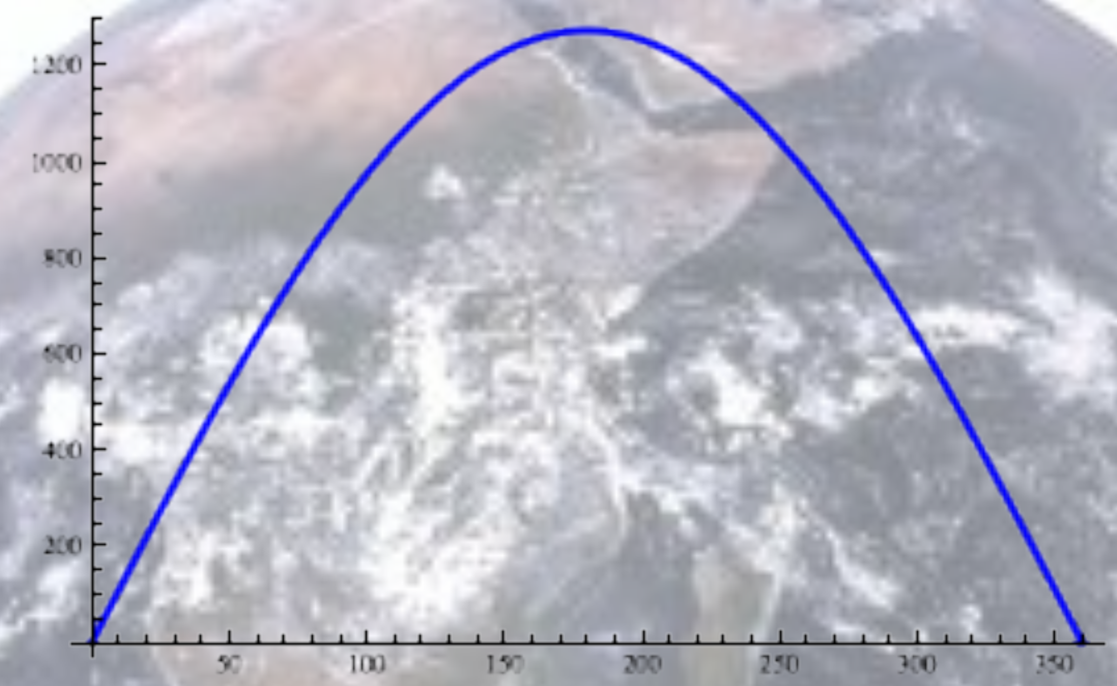
Inge Lehmann discovered the inner core of the Earth in 1936

Inge Lehmann was a mathematician. She worked at the Danish Geodetic Institute.

She used the measures of the different travel times of seismic waves generated by earthquakes to different stations over the Earth.

If the Earth were uniform then the signal would travel like that:

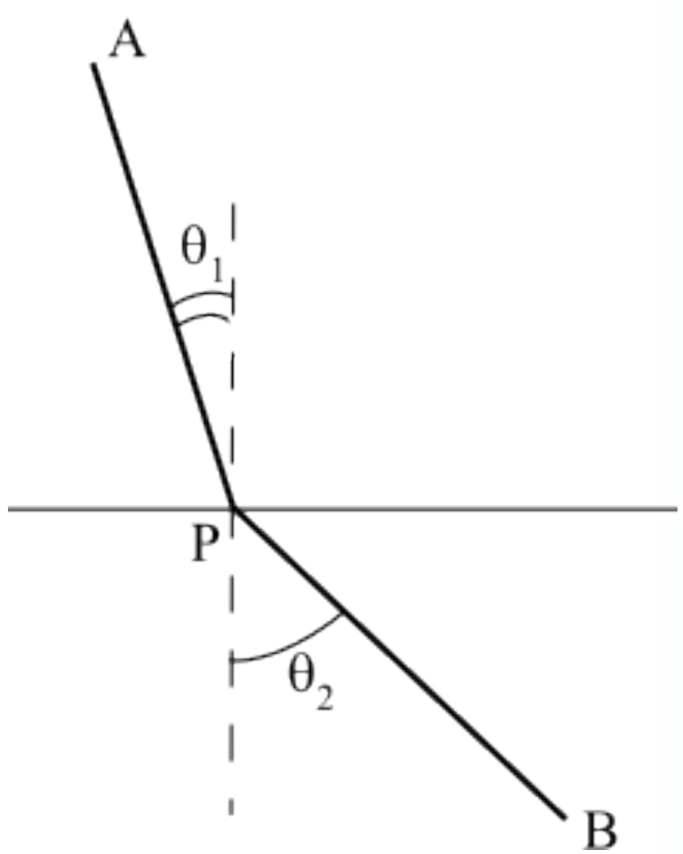




The travel time (in s) depending on the angle would be like that:

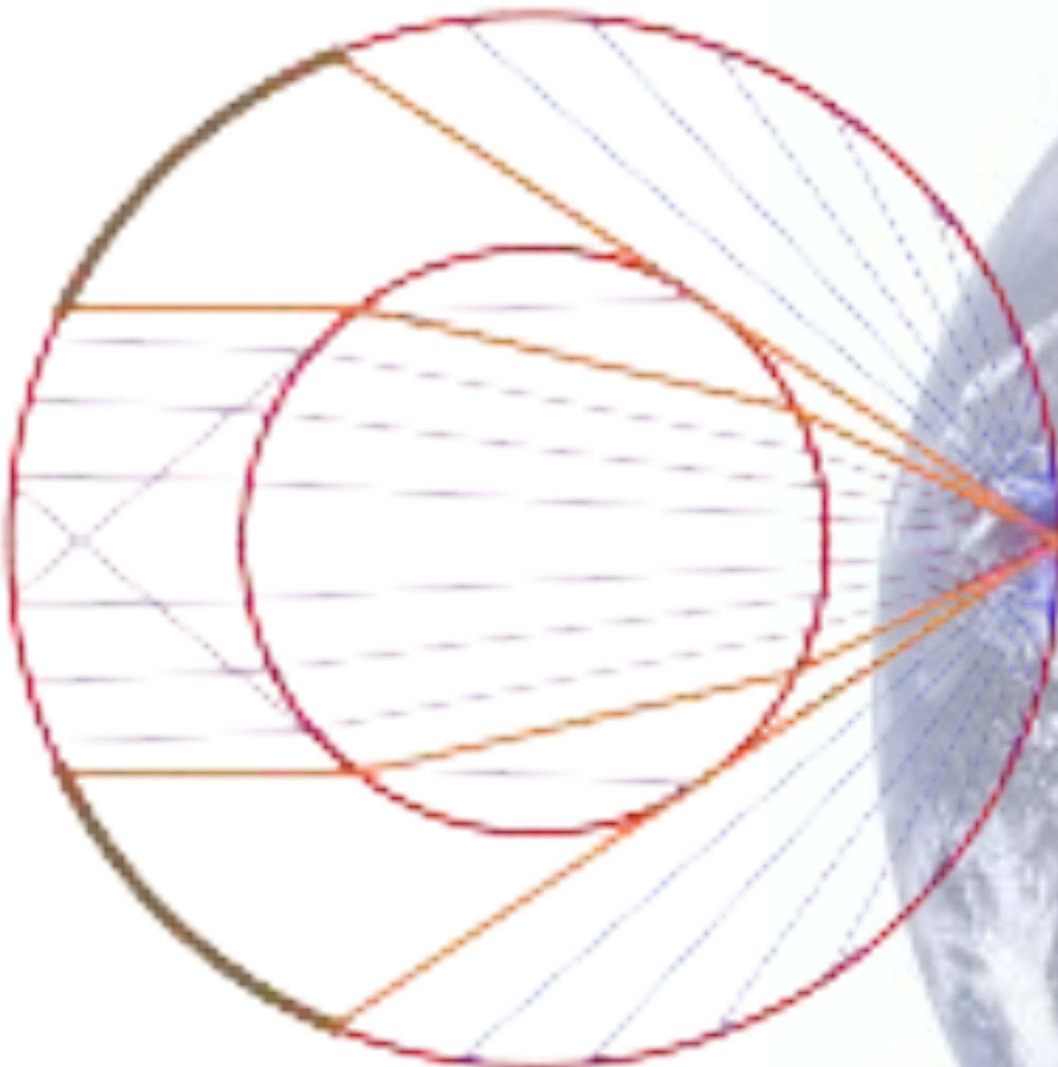
But the Earth has several layers in which the signal travels at different speeds.

When we change layer, the signal makes an angle according to the refraction law:



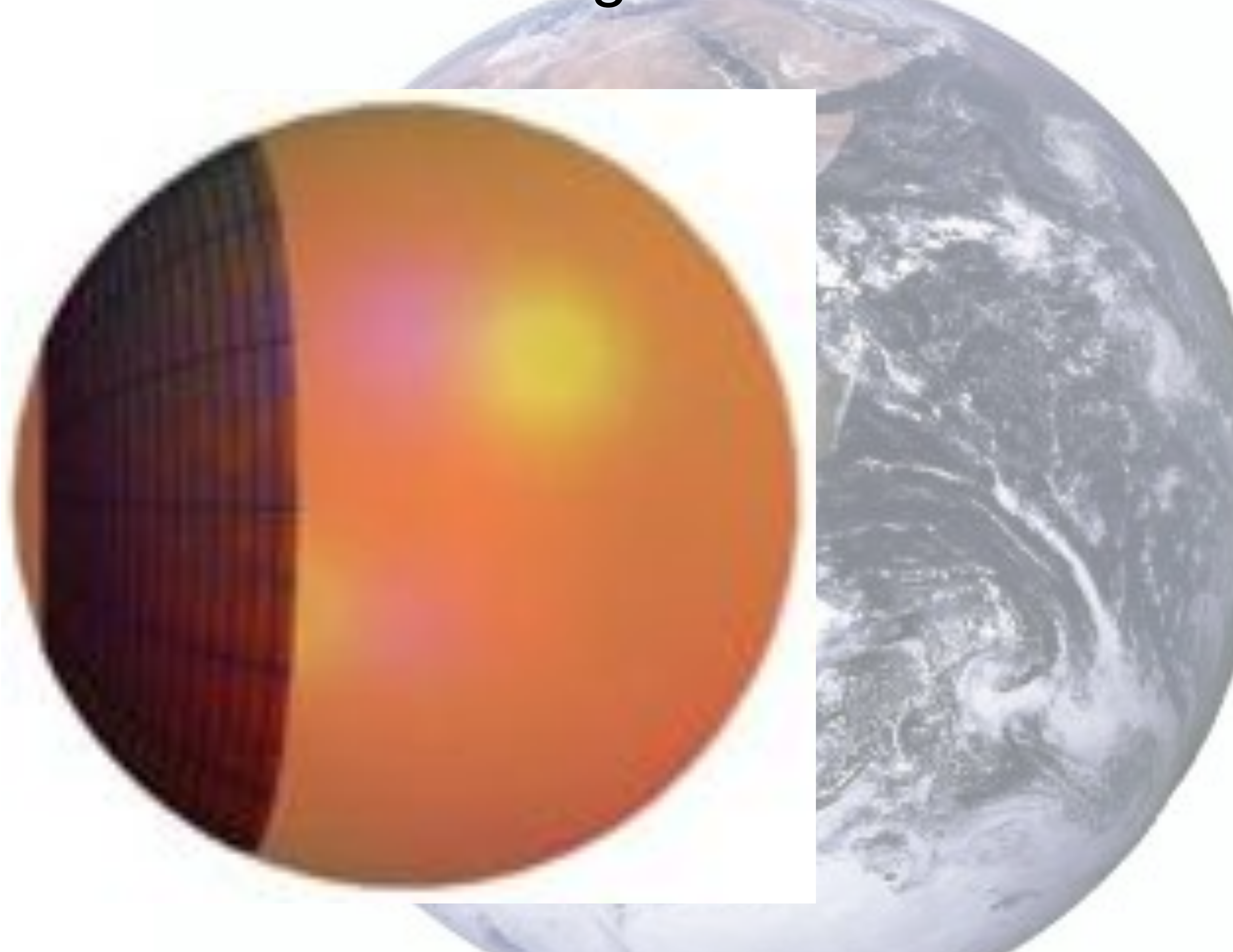
$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

This is what occurs when leaving the mantle and entering the core in which the signal slows down.

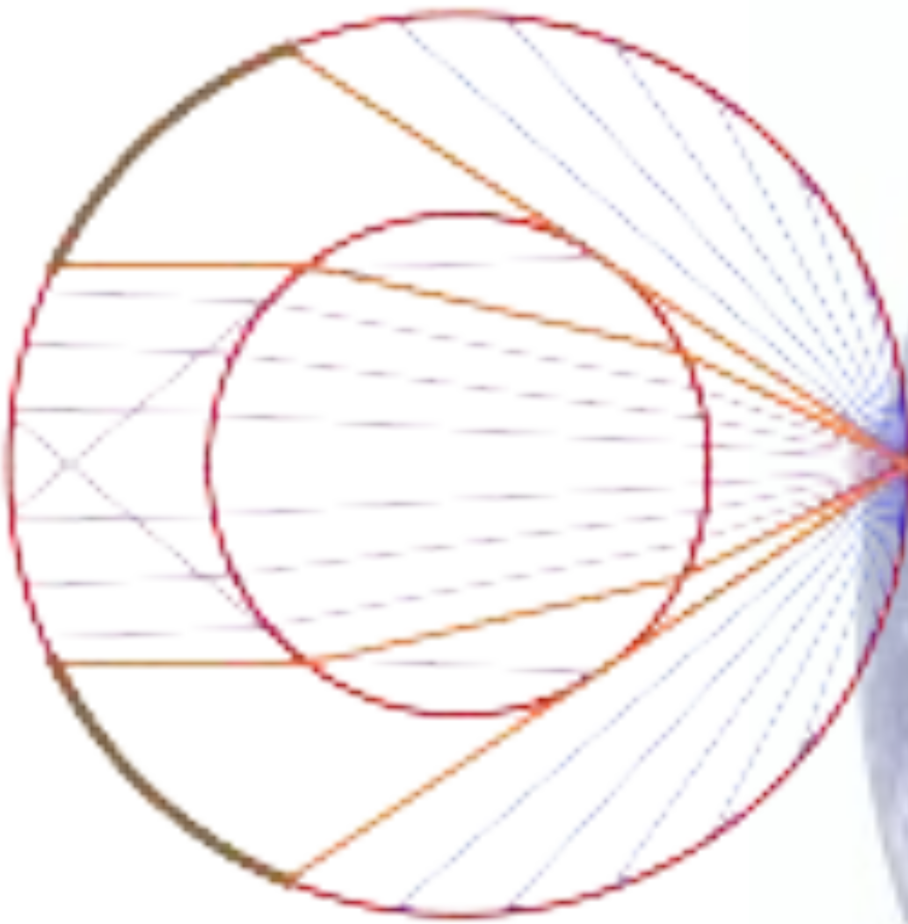


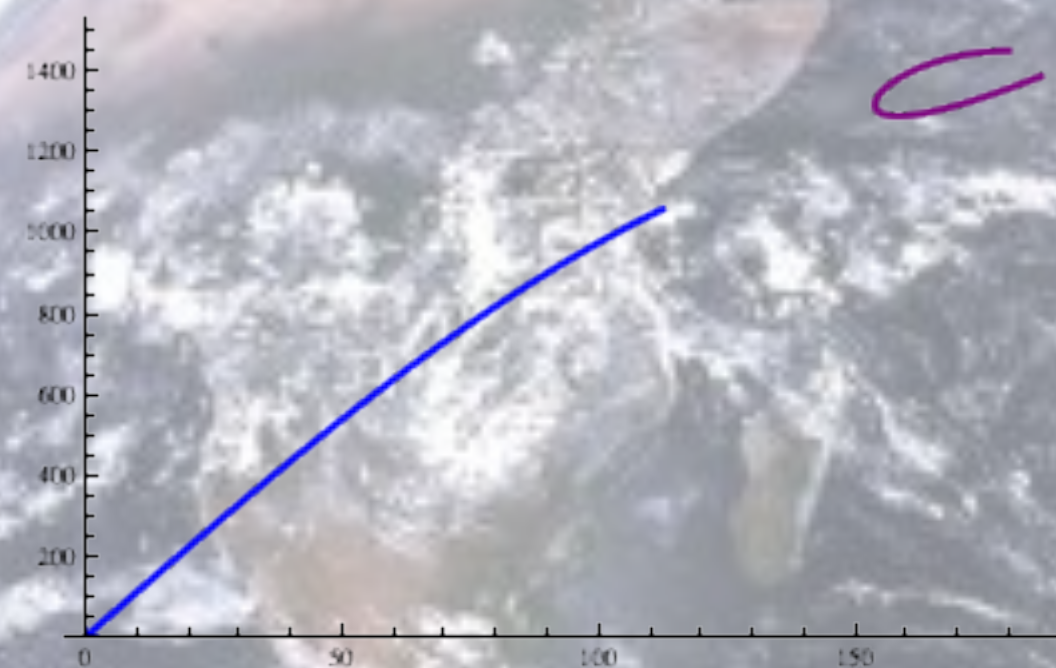
We see that no signals can be detected along the two brown arcs located between 112 degrees and 154 degrees.

This means that there should be no signal detected in an annular region like this



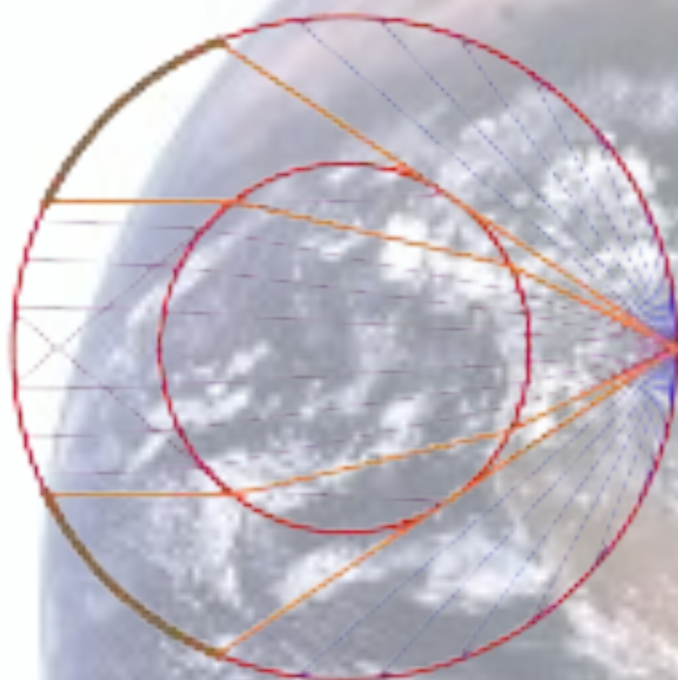
Also, the travel time is discontinuous





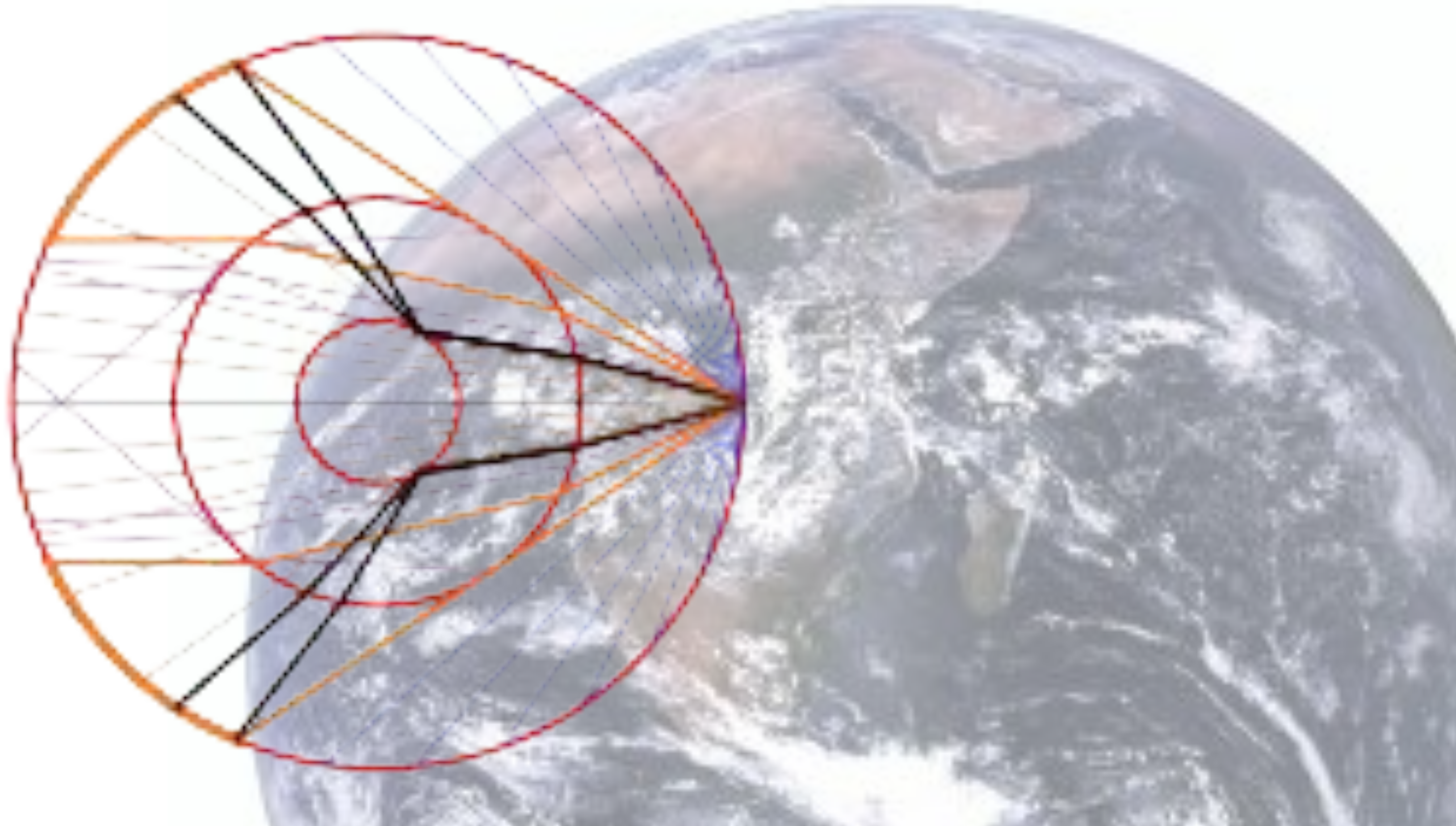
The travel time for rays starting in the upper half-sphere

But Inge Lehmann discovered that signals were registered in the forbidden region of the two brown arcs!



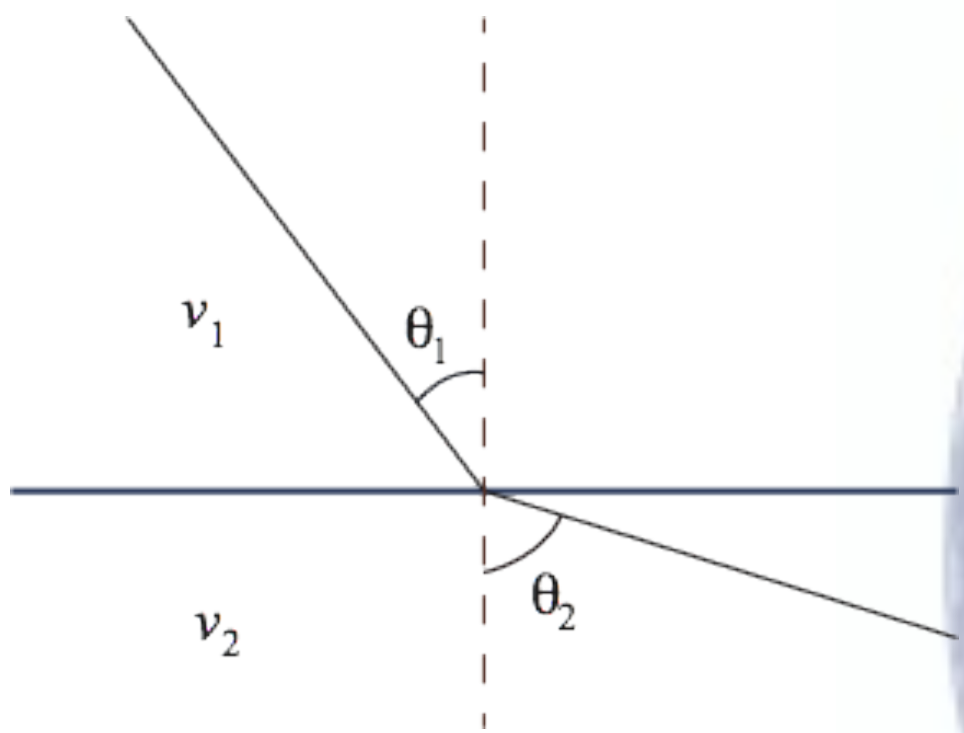
A model explaining the anomalies and the registered travel times for these signals is that the core is divided in two parts: the inner core and the outer core.

The outer core and the inner core



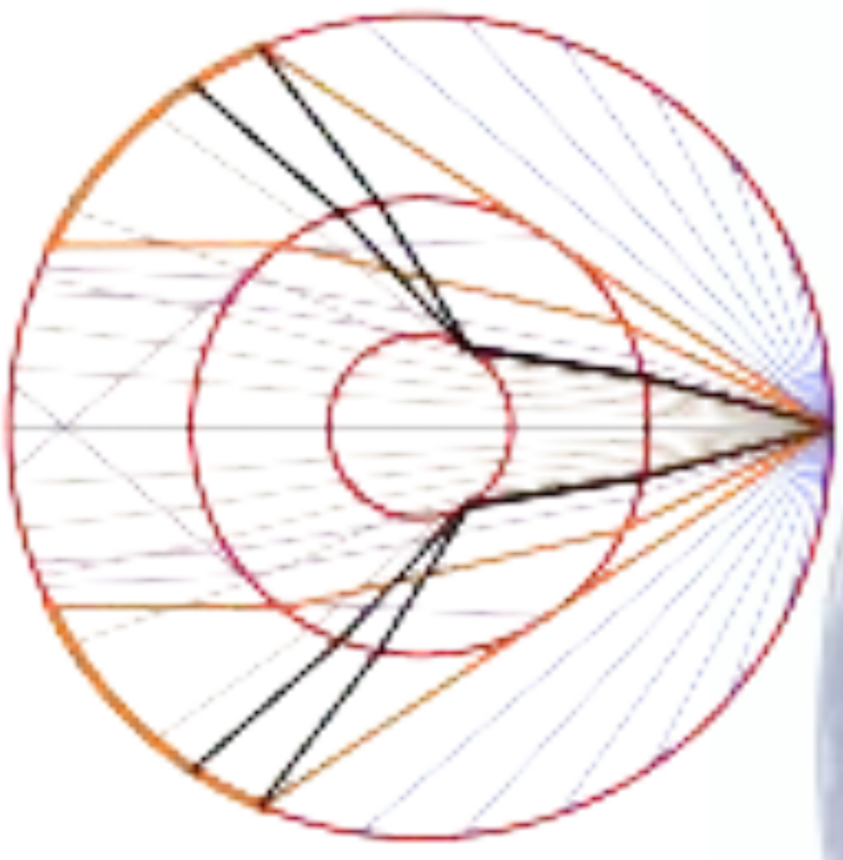
The signal travels faster in the inner core. So some rays cannot enter and are reflected. They are detected in the forbidden region of the orange arcs.

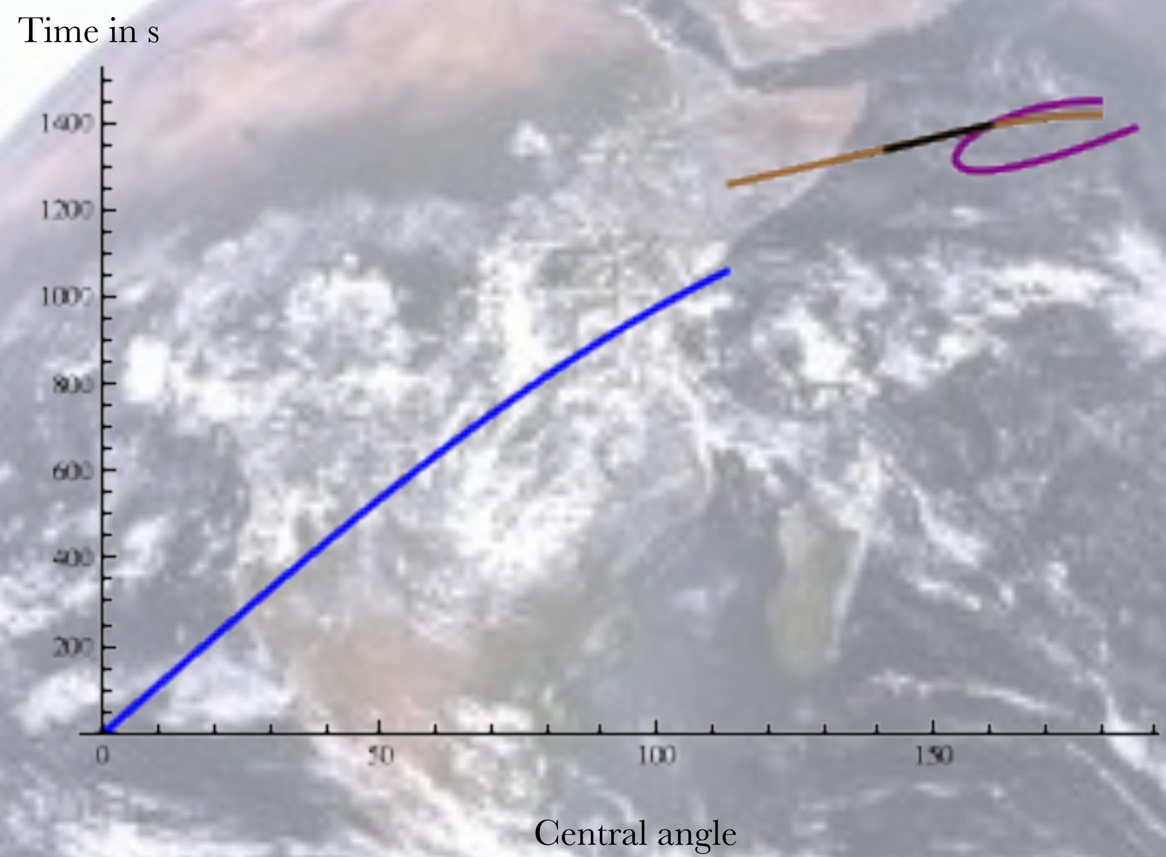
Indeed, in the refraction law, when θ_1 is too large and $v_1 < v_2$, the signal cannot enter the second layer and is reflected.



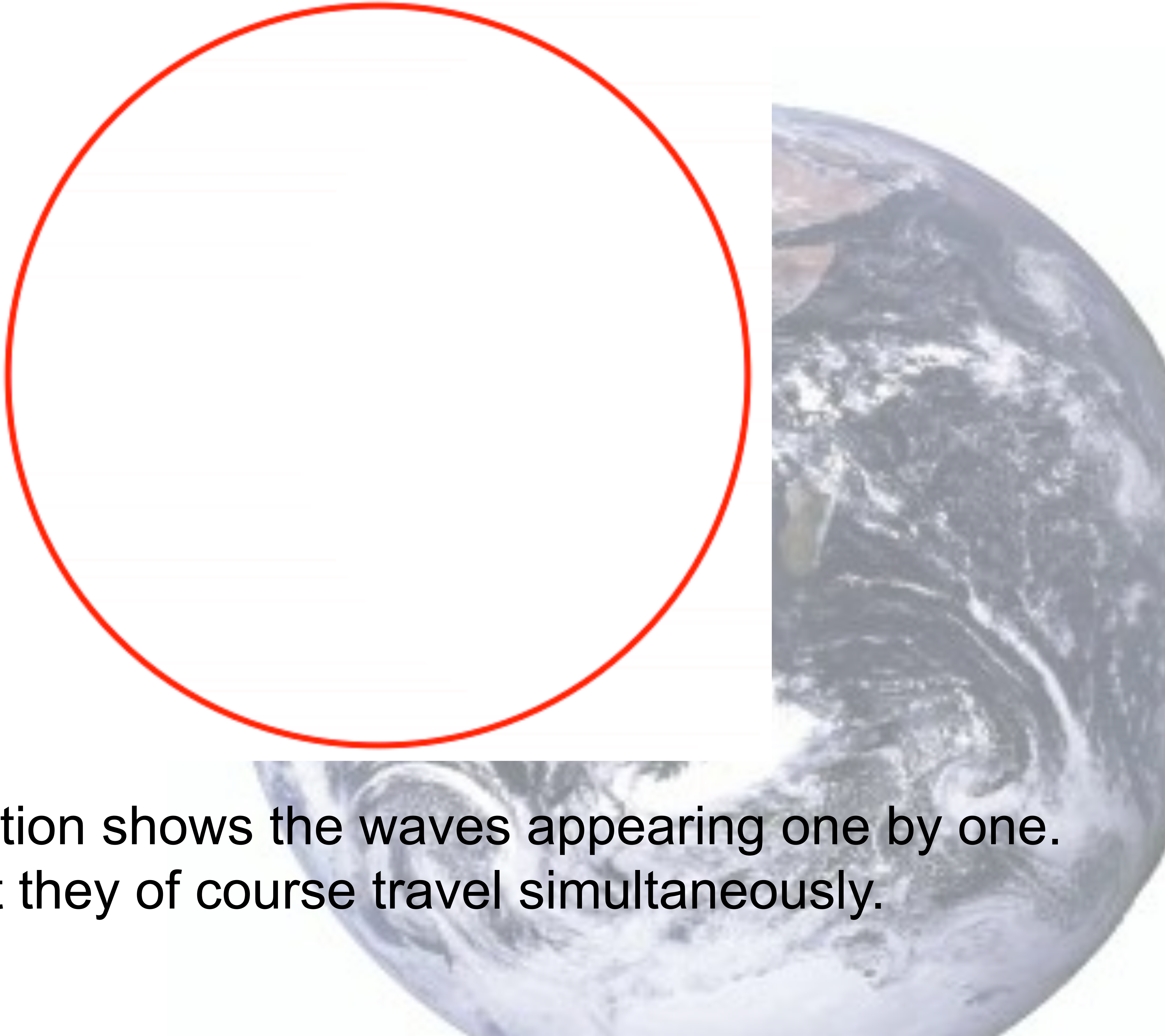
If $\sin\theta_2 \frac{v_1}{v_2}$ is greater than 1, then it cannot be equal to $\sin\theta_1$

The corresponding travel times of the signals



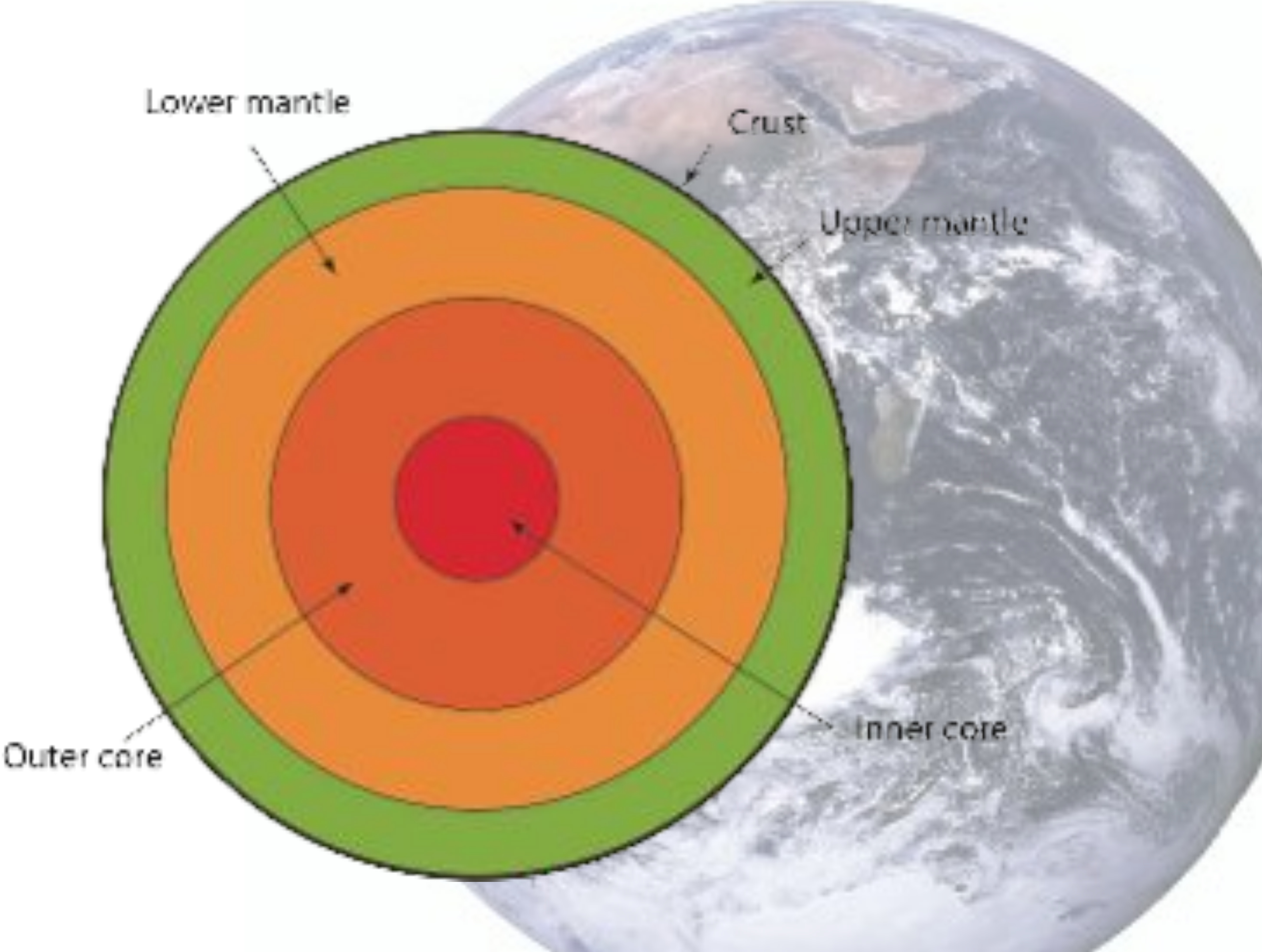


The travel times do not allow to distinguish between the black rays reflected on the inner core and the brown rays that enter and exit the inner core.



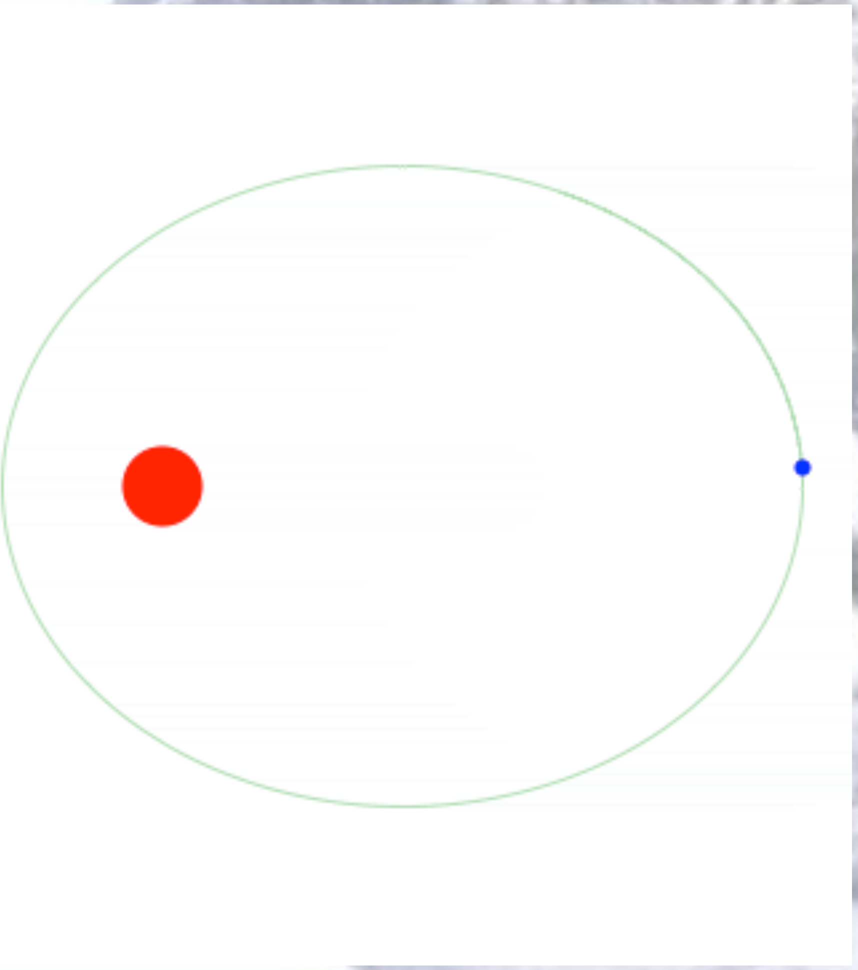
The animation shows the waves appearing one by one.
But they of course travel simultaneously.

The structure of the Earth



The planetary motion of the Earth

Kepler’s law: in the absence of other planets, the Earth moves on an ellipse with a focus at the Sun



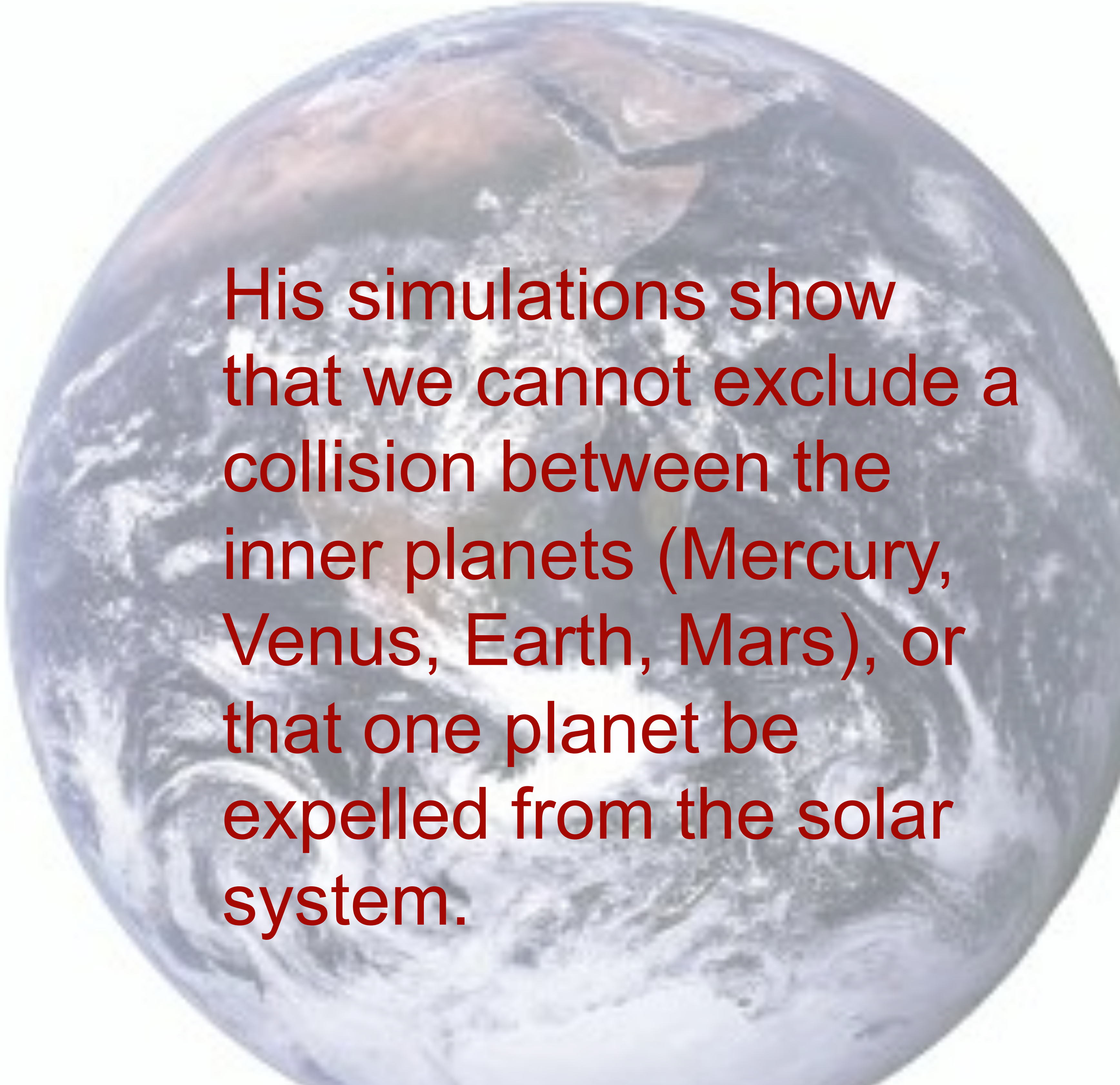
But what happens when we take into account the interactions of the planets within themselves?

Jacques Laskar showed in 1994 that the inner planets are chaotic.

This means that we cannot predict the position of the planets over periods of several billions years.

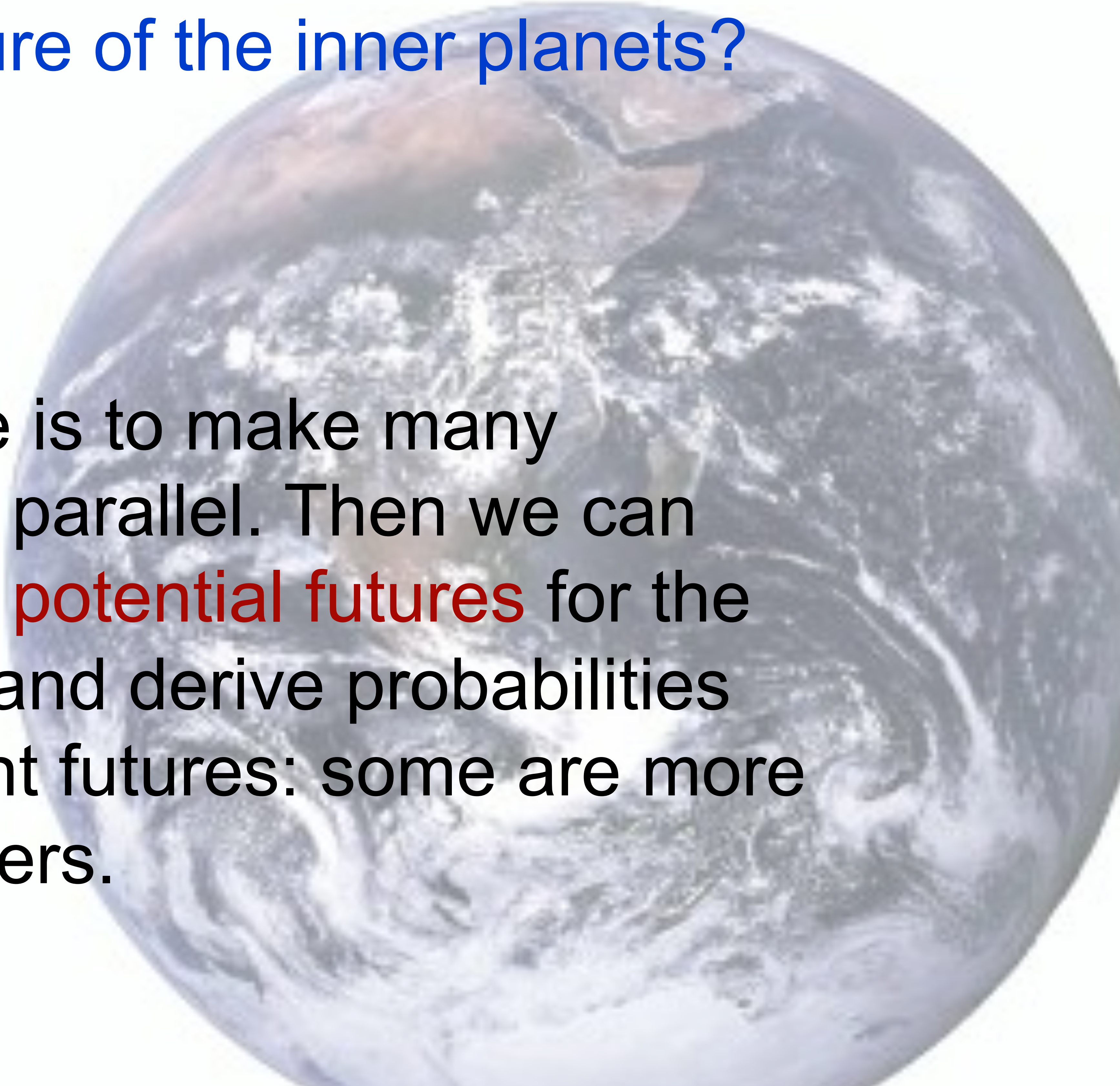
In a chaotic regime a small error in the initial conditions leads to very large error in simulating the system over billions of years

Jacques Laskar refined his results in 2009 by studying 2000 scenarios in parallel.

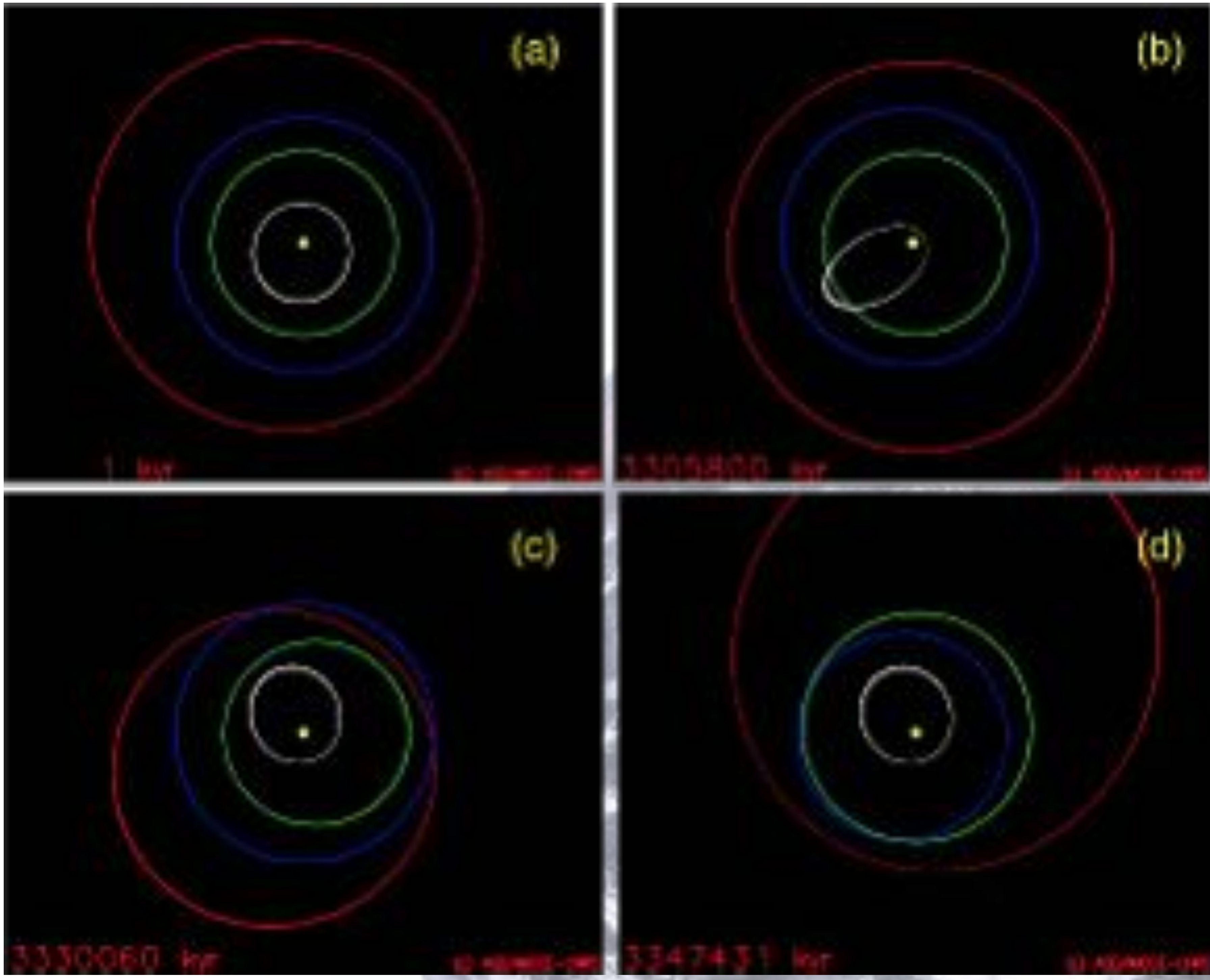


His simulations show that we cannot exclude a collision between the inner planets (Mercury, Venus, Earth, Mars), or that one planet be expelled from the solar system.

So, is there any hope to say something of the future of the inner planets?



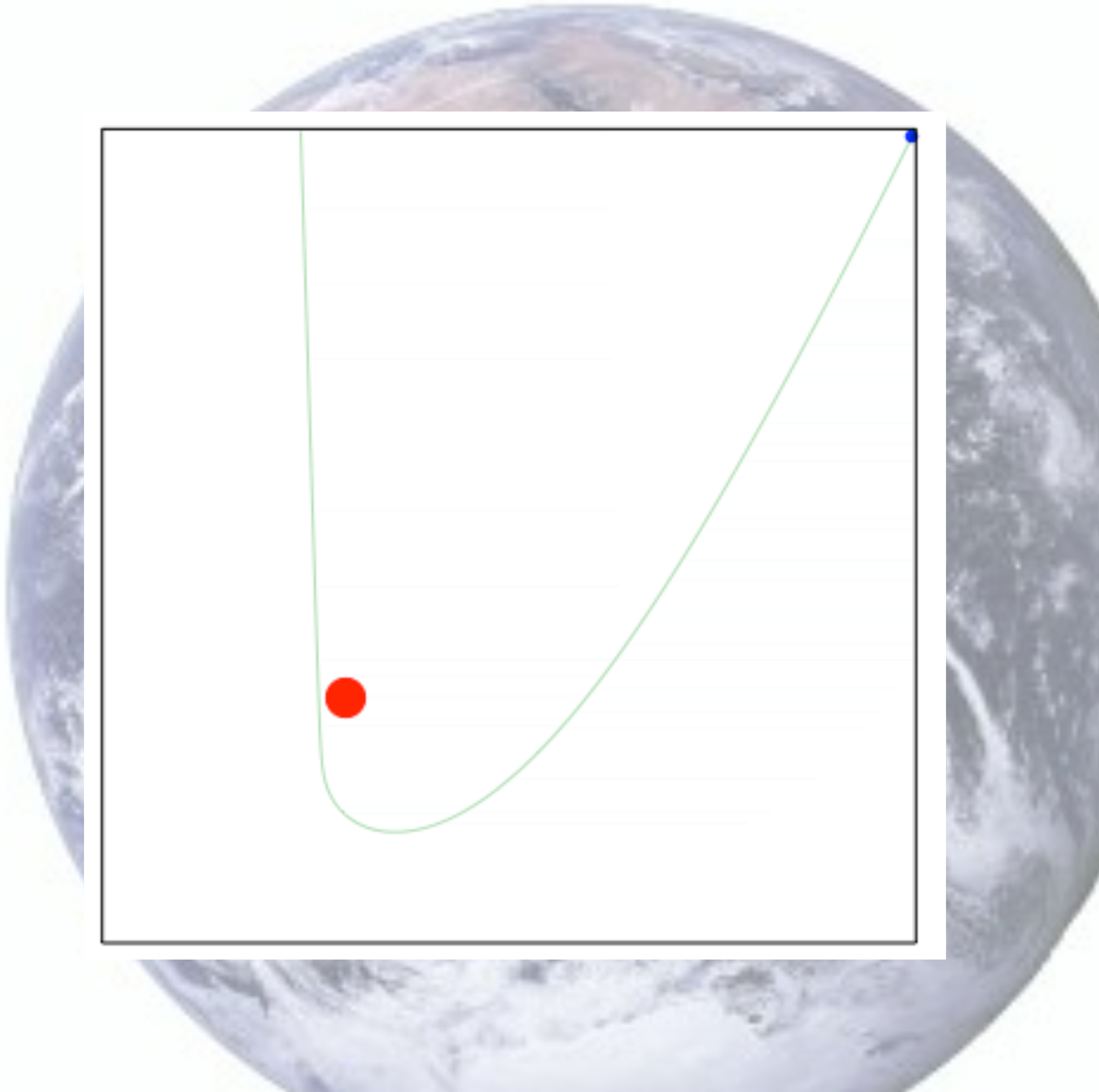
The technique is to make many simulations in parallel. Then we can learn of many potential futures for the inner planets and derive probabilities for the different futures: some are more likely than others.



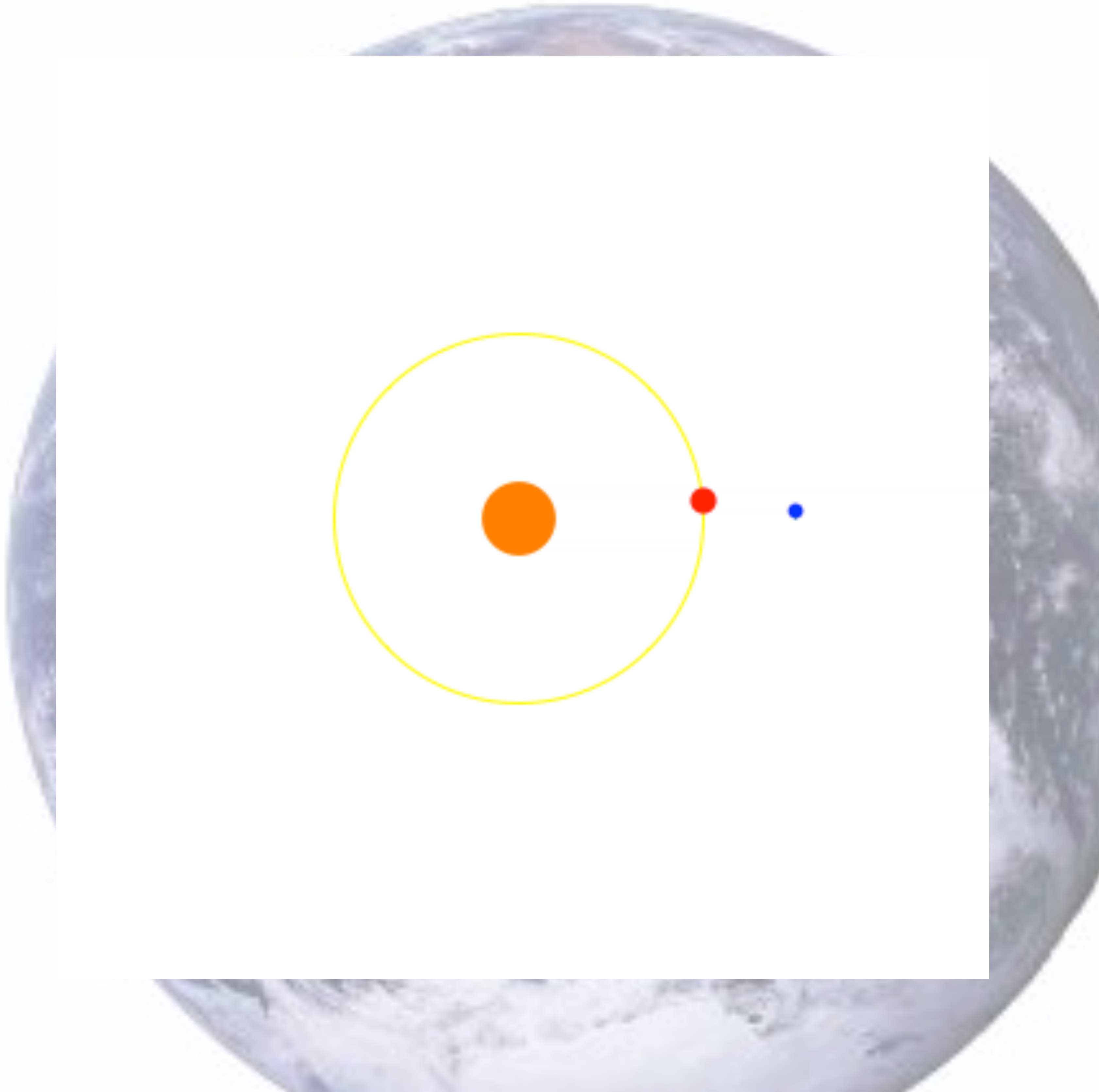
In (b), the orbit of Mercury crosses that of Venus
In (c), the orbit of Mars crosses the orbit of the Earth
In (d), the orbit of Venus crosses the orbit of the Earth, and Mars is destabilized

Can we explain why the movement of inner planets is chaotic?

A planet is disturbed by the attraction of the other planets. The attraction is stronger if a planet passes behind another one.



If the planets have almost the same periods, then the attraction of the large planet destabilizes the small one.



The variations of the orbit of the Earth around the Sun allow explaining the past climates, including the glaciation periods.

Several or these variations are periodic and called the

Milankovitch cycles

The Milankovitch cycles include

- Precession of the Earth's axis: this changes the time of the seasons (period: 21,000 years)
- Oscillation of the major axis of Earth's elliptical orbit
- Oscillation of the obliquity of Earth's axis: this is linked to glaciations (period: 41,000 years)
- Variations of the eccentricity of the Earth's elliptical orbit

Life could appear on Earth because we have a relatively stable system of seasons

Why?

Indeed, the oscillations of the obliquity of the axis of Venus and Mars are very large!

The Moon protects us!

Jacques Laskar showed in 1993 that, without the Moon, the Earth's axis would have very large oscillations, similar to those of Mars and Venus axis.

The shapes of Earth

The loss of equilibrium creates regular patterns:
- dunes
- waves
- vegetation patterns



The loss of equilibrium creating patterns is a recurrent theme in science

It is a very powerful idea that was introduced by Turing to explain the morphogenesis

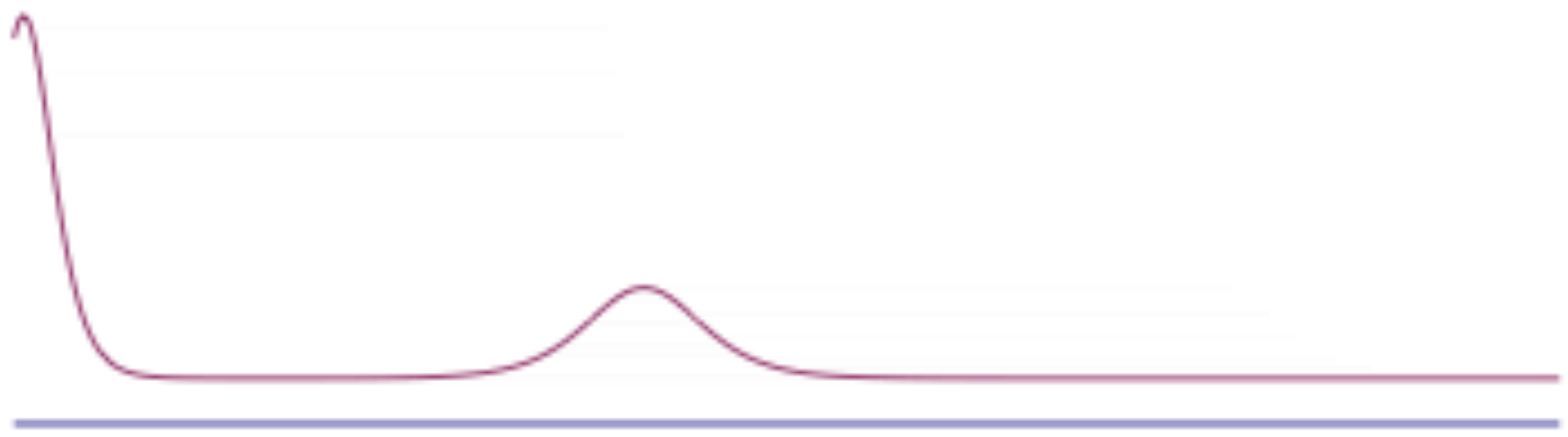
Some isolated waves (solitons) are large and travel without loss of energy

This is the case of tsunamis and rogue waves

A tsunami has a very large wavelength: up to 200 km. It could be only 2 meters high. Its speed is proportional to the square root of the depth of the ocean, thus allowing to forecast its arrival.

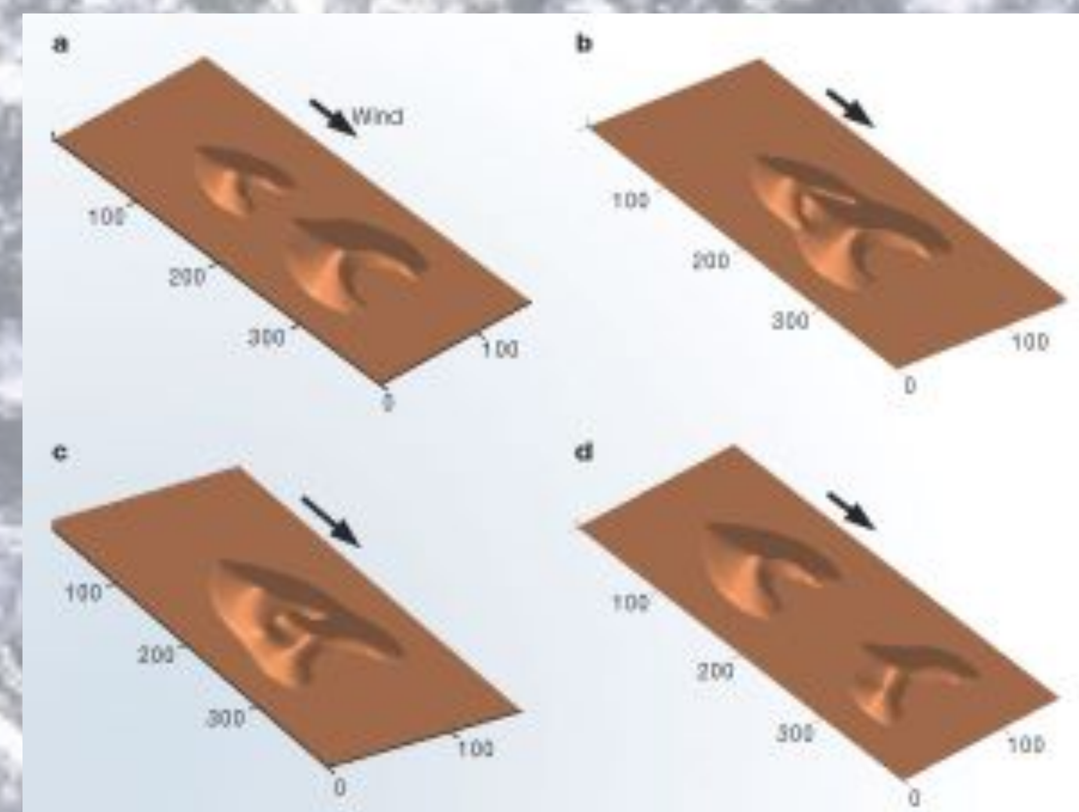
A rogue wave is very steep and has a short wavelength.

Solitary waves have elastic shocks



Hence, there is no use trying sending a counterwave to stop a tsunami.

Solitary wave behavior in sand dunes is observed from space, as well as the elastic passing of dunes (over 45 years)!



Work of Schwämmle & Herrmann

A biologically diverse planet

Earth is inhabited by millions of living species

Where does all this biodiversity come from?

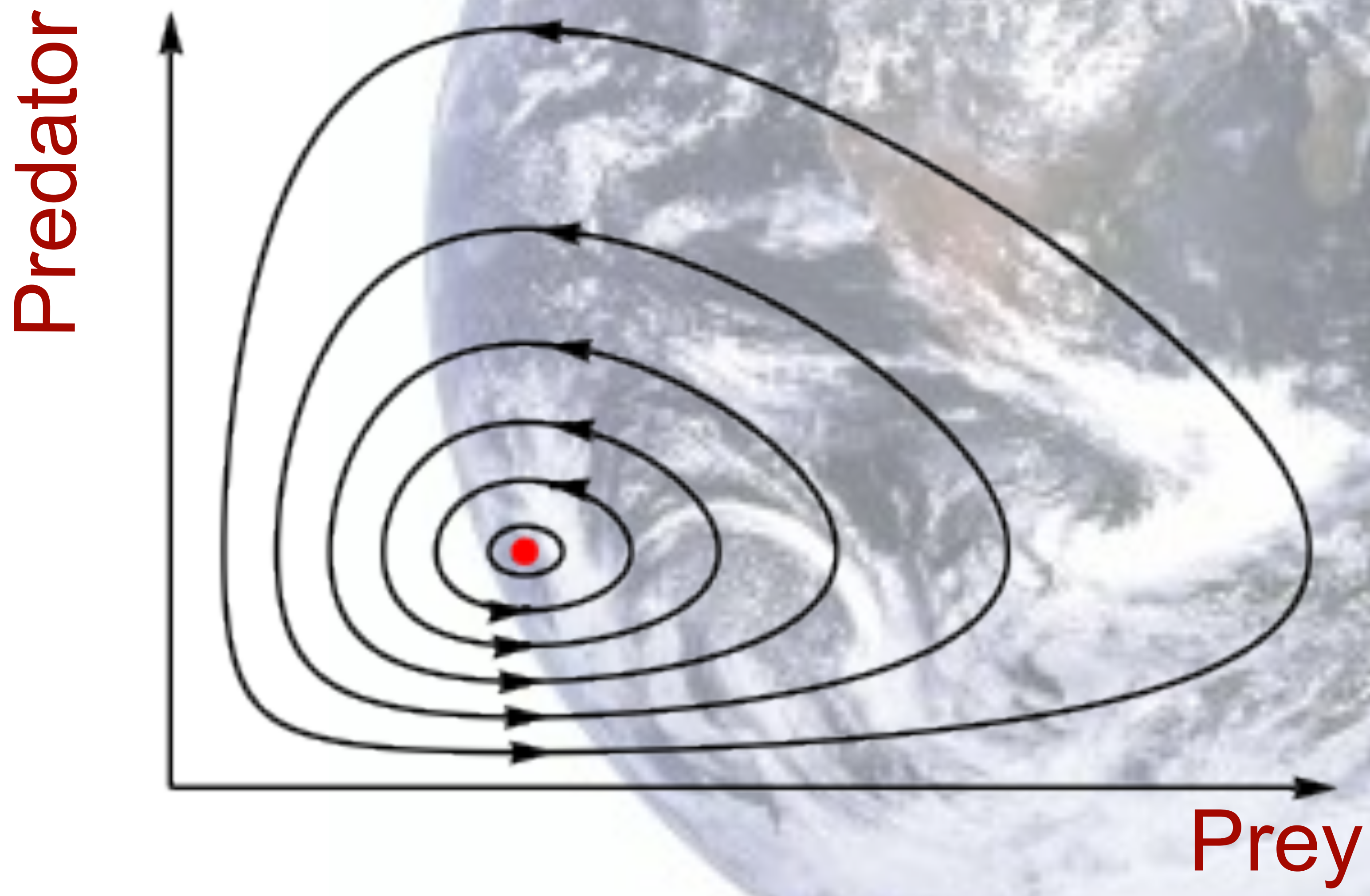
Mutations (randomness) create new species

These species interact to survive

How?

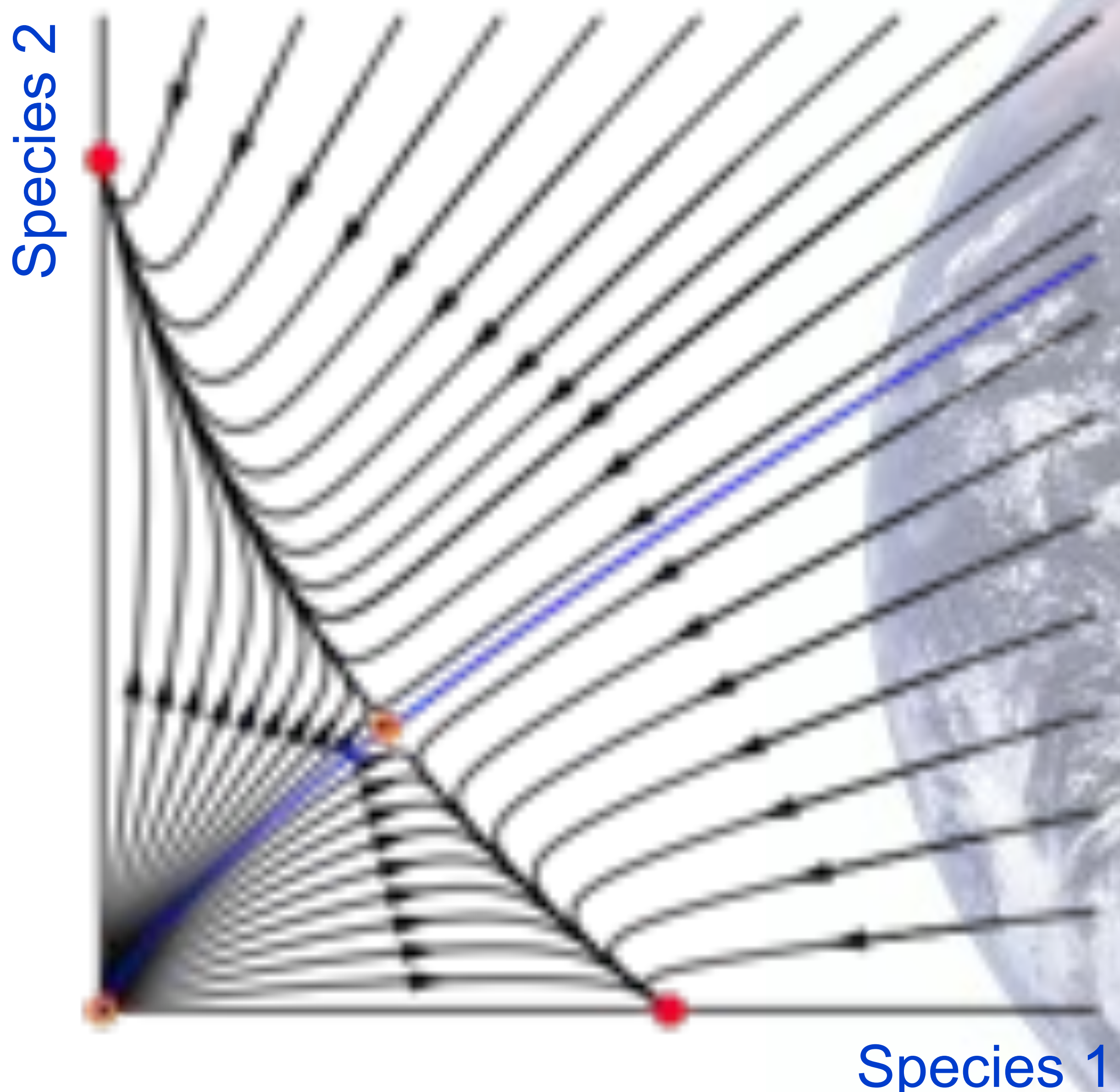
Predators and preys

Mathematicians represent their interaction by a geometric model

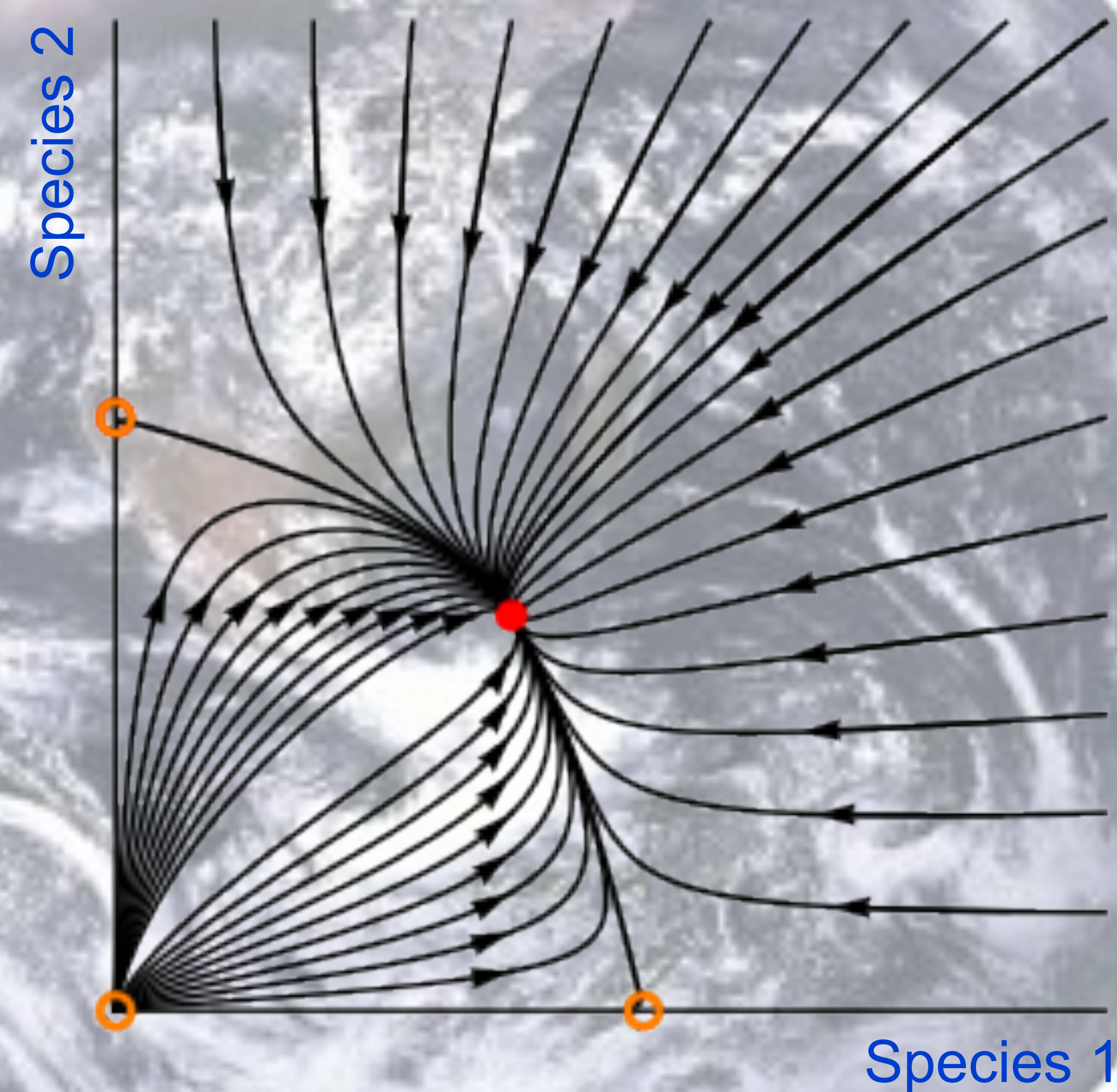


Competing species

We use the same type of models



Strong competition



Weak competition

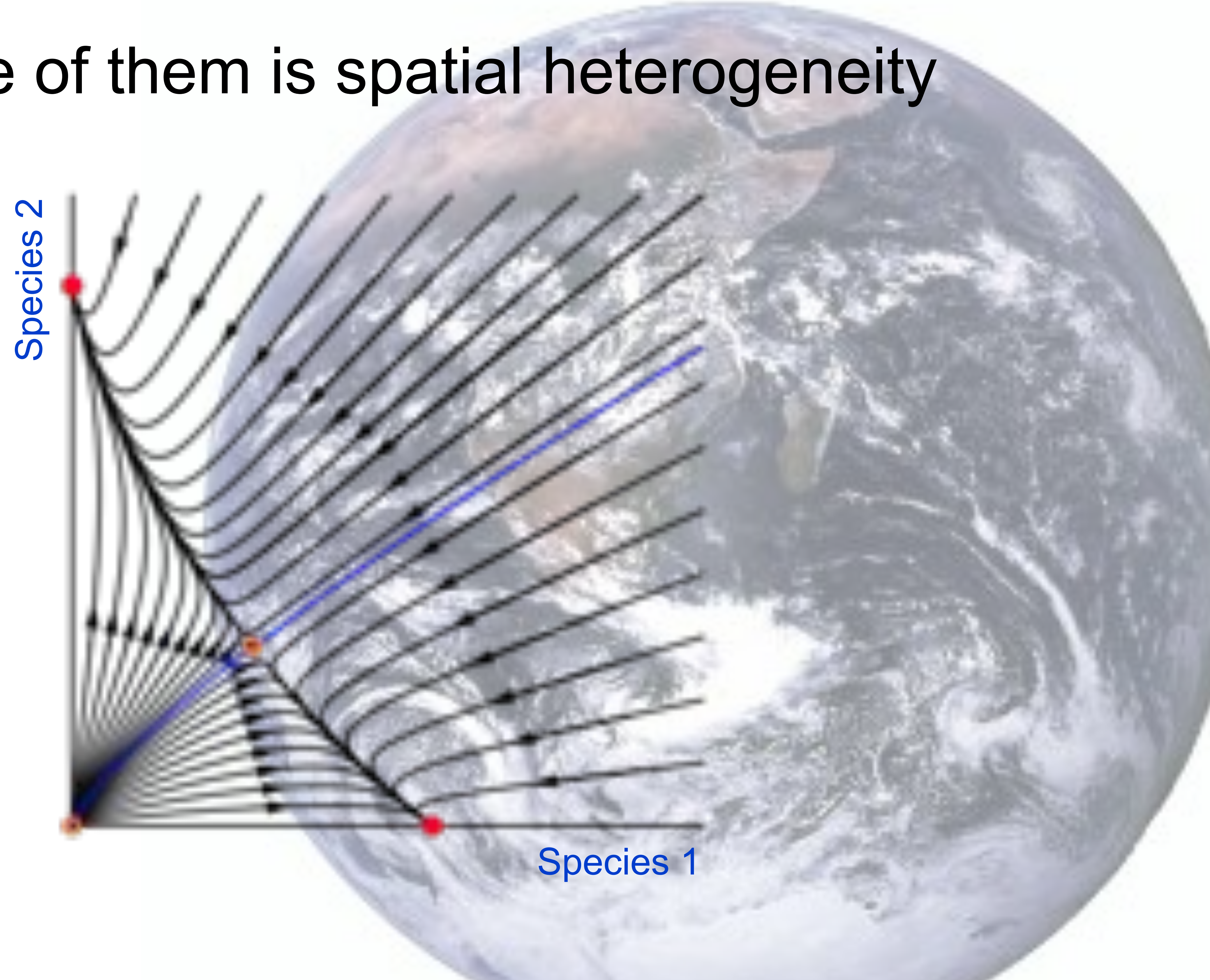
Strong competition for one resource leads to the extinction of one species

This has been generalized by Simon Levin: if $n > k$ species compete strongly for k resources, then no more than k species will survive.

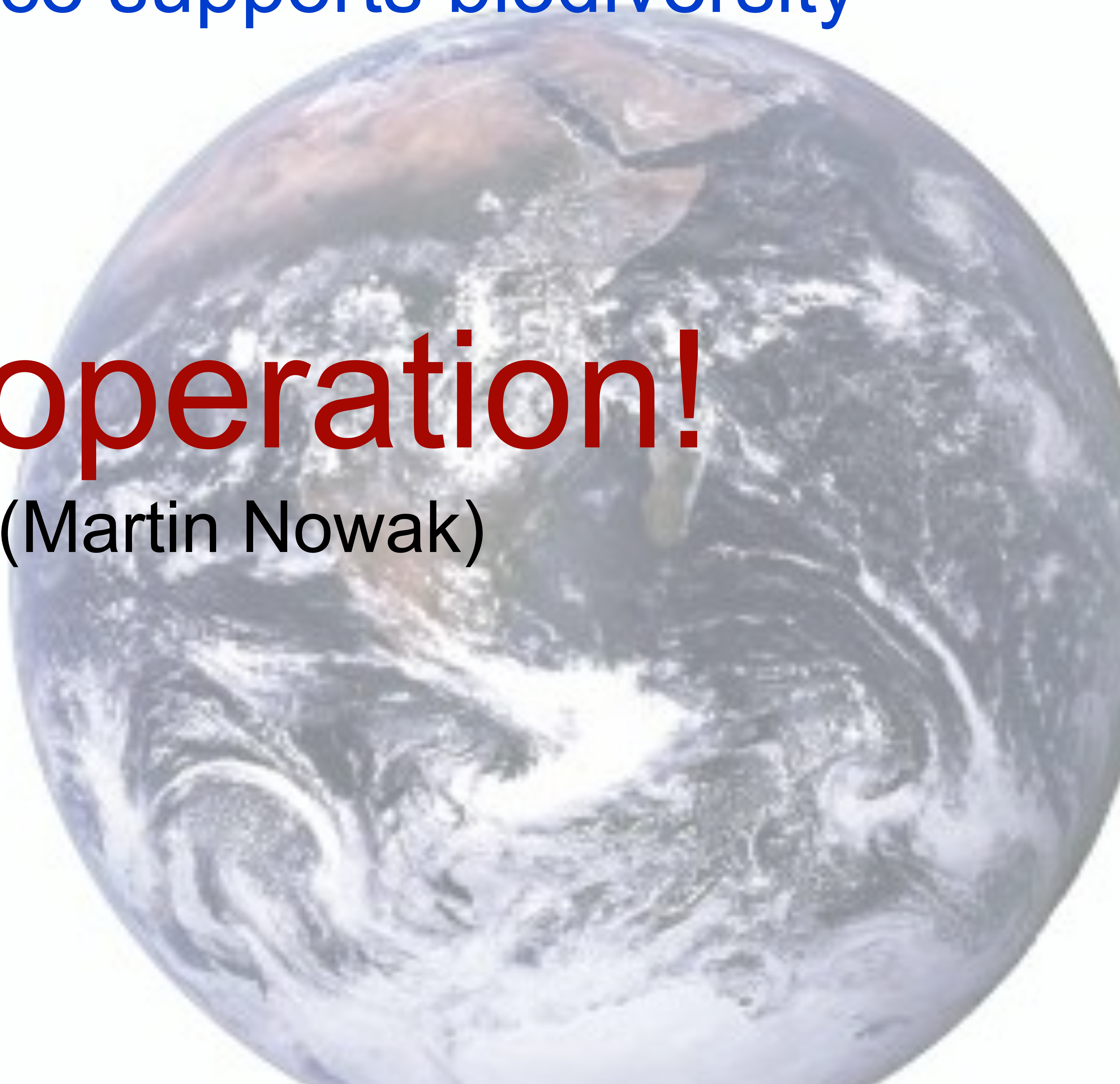
Hence, competition goes against biodiversity!

Other forces allow to maintain biodiversity

One of them is spatial heterogeneity



Another force supports biodiversity



Cooperation!

(Martin Nowak)

The Prisoner's Dilemma

Individual 2

Individual 1

	COOPERATE(remain silent)	DEFECT (confess)
COOPERATE(remain silent)	2 years in jail 2 years in jail	4 years in jail 1 year in jail
DEFECT (confess	1 year in jail 4 years in jail	3 years in jail 3 years in jail

The experiment

- Random distribution of defectors and cooperators

- The winners produce offspring who participate in the next round

- Within a few generations, all individuals were defecting

- Then a new strategy suddenly emerges: players would start cooperating, and then mirror their opponents' moves: tit for tat

The change led to communities dominated by cooperators

Five mechanisms

1. Direct reciprocity: vampire bats share with the bat who found no blood

2. Spatial selection when cooperators and defectors are not uniformly distributed, leading to patches of cooperators and defectors: yeast cells

3. Kin selection: cooperation (including sacrifice) between genetically related individuals

The change led to communities dominated by cooperators

Five mechanisms

4. Indirect reciprocity: help of another based on the needy’s individual reputation: Japanese macaques

5. Group selection: employees competing among themselves, but cooperating for their company

Cooperation has modeled the world as we know it

It explains the preservation of biodiversity

It is everywhere present in the human organization of the planet

The planetary challenges

The problems are very complex since all systems are intertwined

Stern Review on the Economics of Climate Change in 2006: the benefits of strong, early action on climate change far outweigh the costs for not acting.

But how to convince governments to act?

We have to pay now and benefits will only be felt in the period between 50 years and 200 years from now!

Also, the problems will be irregularly felt around the world.

Some countries might benefit from the climate warming (new areas opening to agriculture, trees growing faster), while others will be destroyed or ruined.

It is not necessarily the same countries that are contributing to the increase of green house gas and that will suffer from the consequences.

The real consequences are unknown

We could expect ecosystems to disappear and be replaced by others. How will the transition take place?

Smoothly with new species of plants and animals installing themselves among the old ones?

Or abruptly will all trees dying and an intermediate period with no forest before new forests develop?

A model only contains what you put in it

Arctic warming: now, we know that we must take into account the albedo effect

We realize that methane, a very strong green house gas, is released by the oceans

The permafrost is melting in Northern regions and releases methane

What have we forgotten?

Moving to an economy of sustainability

But what means sustainable development?

It is development that meets the needs of the present without compromising the needs of the future (Bruntland Commission, 1987)

Is this definition sufficiently precise to guide our actions?

The case of fisheries

We have a model for the evolution of a population of fish which is harvested.

Fisherman go fishing if they get more money for their catches than what they spend to go fishing. Otherwise, they stop fishing.

Hence, with free access to the resource and with prices increasing when the resource is rare, then we can run short of fish.

So let’s control the access. How do we control?

The answer is obvious

The quantity that fishermen are allowed to catch is chosen so as to maximize the revenue obtained from the resource over the years.

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This revenue depends of course on the discount factor

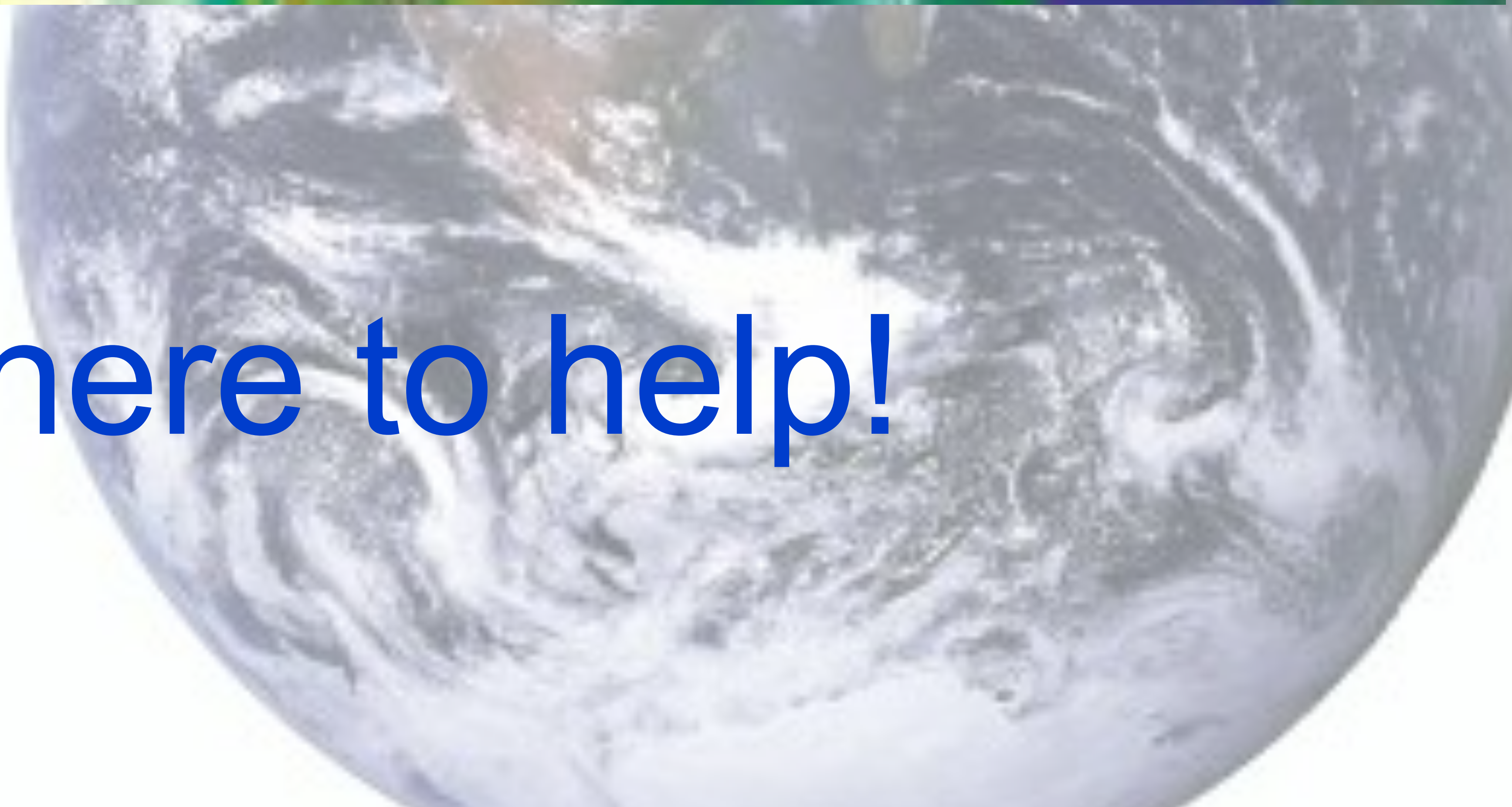
Colin Clark (1973): If the discount factor is at least twice the reproduction rate, then we maximize the revenue by fishing the whole population now and putting the money in the bank!

This is the case for populations of fish that reproduce very slowly (deep sea fish)

More questions than answers

We discover new pressing problems faster than the known ones are solved

Mathematics has an essential role to play in these issues



is here to help!

Thank you!