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Control Theory: History, Mathematical Achievements and Perspectives *

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Abstract

These notes are devoted to present some of the mathematical milestones of Control Theory. To do that, we first overview its origins and some of the main mathematical achievements. Then, we discuss the main domains of Sciences and Technologies where Control Theory arises and applies. This forces us to address modelling issues and to distinguish between the two main control theoretical approaches, controllability and optimal control, discussing the advantages and drawbacks of each of them.

In order to give adequate formulations of the main questions, we have introduced some of the most elementary mathematical material, avoiding unnecessary technical difficulties and trying to make the paper accessible to a large class of readers.

The subjects we address range from the basic concepts related to the dynamical systems approach to (linear and nonlinear) Mathematical Programming and Calculus of Variations. We also present a simplified version of the outstanding results by Kalman on the controllability of linear finite dimensional dynamical systems, Pontryaguin's maximum principle and the principle of dynamical programming.

Some aspects related to the complexity of modern control systems, the discrete versus continuous modelling, the numerical approximation of control problems and its control theoretical consequences are also discussed.

Finally, we describe some of the major challenging applications in Control Theory for the XXI Century. They will probably influence strongly the development of this discipline in the near future.

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1. Introduction

This article is devoted to present some of the mathematical milestones of Control Theory. We will focus on systems described in terms of ordinary differential equations. The control of (deterministic and stochastic) partial differential equations remains out of our scope. However, it must be underlined that most ideas, methods and results presented here do extend to this more general setting, which leads to very important technical developments.

The underlying idea that motivated this article is that Control Theory is certainly, at present, one of the most interdisciplinary areas of research. Control Theory arises in most modern applications. The same could be said about the very first technological discoveries of the industrial revolution. On the other hand, Control Theory has been a discipline where many mathematical ideas and methods have melt to produce a new body of important Mathematics. Accordingly, it is nowadays a rich crossing point of Engineering and Mathematics.

Along this paper, we have tried to avoid unnecessary technical difficulties, to make the text accessible to a large class of readers. However, in order to introduce some of the main achievements in Control Theory, a minimal body of basic mathematical concepts and results is needed. We develop this material to make the text self-contained.

These notes contain information not only on the main mathematical results in Control Theory, but also about its origins, history and the way applications and interactions of Control Theory with other Sciences and Technologies have conducted the development of the discipline.

The plan of the paper is the following. Section 2 is concerned with the origins and most basic concepts. In Section 3 we study a simple but very interesting example: the *pendulum*. As we shall see, an elementary analysis of this simple but important mechanical system indicates that the fundamental ideas of Control Theory are extremely meaningful from a physical viewpoint.

In Section 4 we describe some relevant historical facts and also some important contemporary applications. There, it will be shown that Control Theory is in fact an interdisciplinary subject that has been strongly involved in the development of the contemporary society.

In Section 5 we describe the two main approaches that allow to give rigorous formulations of control problems: controllability and optimal control. We also discuss their mutual relations, advantages and drawbacks.

In Sections 6 and 7 we present some basic results on the controllability of linear and nonlinear finite dimensional systems. In particular, we revisit the Kalman approach to the controllability of linear systems, and we recall the use of Lie brackets in the control of nonlinear systems, discussing a simple example of a planar moving square car.

In Section 8 we discuss how the complexity of the systems arising in modern technologies affects Control Theory and the impact of numerical approximations and discrete modelling, when compared to the classical modelling in the context of Continuum Mechanics.

In Section 9 we describe briefly two beautiful and extremely important challenging applications for Control Theory in which, from a mathematical viewpoint, almost all remains to be done: laser molecular control and the control of floods.

In Section 10 we present a list of possible future applications and lines of development of Control Theory: large space structures, Robotics, biomedical research, etc.

Finally, we have included two Appendices, where we recall briefly two of the main principles of modern Control Theory, namely *Pontryagin's maximum principle* and *Bellman's dynamical programming principle*.

2. Origins and basic ideas, concepts and ingredients

The word *control* has a double meaning. First, controlling a system can be understood simply as testing or checking that its behavior is satisfactory. In a deeper sense, to control is also to act, to put things in order to guarantee that the system behaves as desired.

S. Bennet starts the first volume of his book [2] on the history of *Control Engineering* quoting the following sentence of Chapter 3, Book 1, of the monograph "Politics" by Aristotle:

> "... if every instrument could accomplish its own work, obeying or anticipating the will of others ... if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves."

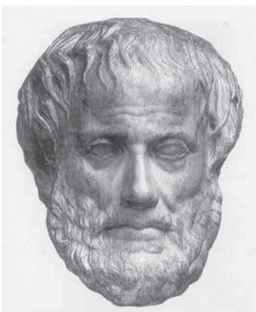

Figure 1: Aristotle (384–322 B.C.).

This sentence by Aristotle describes in a rather transparent way the guiding goal of *Control Theory:* the need of automatizing processes to let the human being gain in liberty, freedom, and quality of life.

Let us indicate briefly how control problems are stated nowadays in mathematical terms. To fix ideas, assume we want to get a good behavior of a physical system governed by the *state equation*

$$A(y) = f(v). \tag{1}$$

Here, y is the *state*, the unknown of the system that we are willing to control. It belongs to a vector space Y. On the other hand, v is the *control*. It belongs to the *set of admissible controls* $\mathcal{U}_{\rm ad}$. This is the variable that we can choose freely in $\mathcal{U}_{\rm ad}$ to act on the system.

Let us assume that $A : D(A) \subset Y \mapsto Y$ and $f : \mathcal{U}_{\rm ad} \mapsto Y$ are two given (linear or nonlinear) mappings. The operator A determines the equation that must be satisfied by the state variable y, according to the laws of Physics. The function f indicates the way the control v acts on the system governing the state. For simplicity, let us assume that, for each $v \in \mathcal{U}_{\rm ad}$, the state equation (1) possesses exactly one solution $y = y(v)$ in Y. Then, roughly speaking, to control (1) is to find $v \in \mathcal{U}_{\rm ad}$ such that the solution to (1) gets close to the desired prescribed state. The "best" among all the existing controls achieving the desired goal is frequently referred to as the *optimal control*.

This mathematical formulation might seem sophisticated or even obscure for readers not familiar with this topic. However, it is by now standard and it has been originated naturally along the history of this rich discipline. One of the main advantages of such a general setting is that many problems of very different nature may fit in it, as we shall see along this work.

As many other fields of human activities, the discipline of Control existed much earlier than it was given that name. Indeed, in the world of living species, organisms are endowed with sophisticated mechanisms that regulate the various tasks they develop. This is done to guarantee that the essential variables are kept in optimal regimes to keep the species alive allowing them to grow, develop and reproduce.

Thus, although the mathematical formulation of control problems is intrinsically complex, the key ideas in Control Theory can be found in Nature, in the evolution and behavior of living beings.

The first key idea is the *feedback* concept. This term was incorporated to Control Engineering in the twenties by the engineers of the "Bell Telephone Laboratory" but, at that time, it was already recognized and consolidated in other areas, such as Political Economics.

Essentially, a feedback process is the one in which the state of the system determines the way the control has to be exerted at any time. This is related to the notion of *real time control*, very important for applications. In the framework of (1), we say that the control u is given by a *feedback law* if we are able to provide a mapping $G : Y \mapsto \mathcal{U}_{\rm ad}$ such that

$$u = G(y), \quad \text{where } y = y(u), \tag{2}$$

i.e. y solves (1) with v replaced by u.

Nowadays, feedback processes are ubiquitous not only in Economics, but also in Biology, Psychology, etc. Accordingly, in many different related areas, the *cause-effect principle* is not understood as a static phenomenon any more, but it is rather being viewed from a dynamical perspective. Thus, we can speak of the *cause-effect-cause principle*. See [33] for a discussion on this and other related aspects.

The second key idea is clearly illustrated by the following sentence by H.R. Hall in [17] in 1907 and that we have taken from [2]:

> "It is a curious fact that, while political economists recognize that for the proper action of the law of supply and demand there must be fluctuations, it has not generally been recognized by mechanicians in this matter of the steam engine governor. The aim of the mechanical economist, as is that of the political economist, should be not to do away with these fluctuations all together (for then he does away with the principles of self-regulation), but to diminish them as much as possible, still leaving them large enough to have sufficient regulating power."

The need of having room for fluctuations that this paragraph evokes is related to a basic principle that we apply many times in our daily life. For instance, when driving a car at a high speed and needing to brake, we usually try to make it intermittently, in order to keep the vehicle under control at any moment. In the context of human relationships, it is also clear that insisting permanently in the same idea might not be precisely the most convincing strategy.

The same rule applies for the control of a system. Thus, to control a system arising in Nature or Technology, we do not have necessarily to stress the system and drive it to the desired state immediately and directly. Very often, it is much more efficient to control the system letting it fluctuate, trying to find a harmonic dynamics that will drive the system to the desired state without forcing it too much. An excess of control may indeed produce not only an inadmissible cost but also irreversible damages in the system under consideration.

Another important underlying notion in Control Theory is *Optimization*. This can be regarded as a branch of Mathematics whose goal is to improve a variable in order to maximize a benefit (or minimize a cost). This is applicable to a lot of practical situations (the variable can be a temperature, a velocity field, a measure of information, etc.). Optimization Theory and its related techniques are such a broad subject that it would be impossible to make a unified presentation. Furthermore, a lot of recent developments in *Informatics* and *Computer Science* have played a crucial role in Optimization. Indeed, the complexity of the systems we consider interesting nowadays makes it impossible to implement efficient control strategies without using appropriate (and sophisticated) software.

In order to understand why Optimization techniques and Control Theory are closely related, let us come back to (1). Assume that the set of admissible controls $\mathcal{U}_{\rm ad}$ is a subset of the Banach space $\mathcal{U}$ (with norm $\|\cdot\|_{\mathcal{U}}$) and the state space Y is another Banach space (with norm $\|\cdot\|_Y$). Also, assume that the state $y_d \in Y$ is the preferred state and is chosen as a target for the state of the system. Then, the control problem consists in finding controls v in $\mathcal{U}_{\rm ad}$ such that the associated solution coincides or gets close to y_d.

It is then reasonable to think that a fruitful way to choose a good control v is by minimizing a *cost function* of the form

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 \quad \forall v \in \mathcal{U}_{\rm ad} \tag{3}$$

or, more generally,

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 + \frac{\mu}{2}\|v\|_{\mathcal{U}}^2 \quad \forall v \in \mathcal{U}_{\mathrm{ad}}\,, \tag{4}$$

where $\mu \geq 0$.

These are (constrained) extremal problems whose analysis corresponds to Optimization Theory.

It is interesting to analyze the two terms arising in the functional J in (4) when $\mu > 0$ separately, since they play complementary roles. When minimizing the functional in (4), we are minimizing the balance between these two terms. The first one requires to get close to the target y_d while the second one penalizes using too much costly control. Thus, roughly speaking, when minimizing J we are trying to drive the system to a state close to the target y_d without too much effort.

We will give below more details of the connection of Control Theory and Optimization below.

So far, we have mentioned three main ingredients arising in Control Theory: the notion of feedback, the need of fluctuations and Optimization. But of course in the development of Control Theory many other concepts have been important.

One of them is *Cybernetics*. The word "cybernétique" was proposed by the French physicist A.-M. Ampère in the XIX Century to design the nonexistent science of process controlling. This was quickly forgotten until 1948, when N. Wiener chose "Cybernetics" as the title of his book.

Wiener defined Cybernetics as "the science of control and communication in animals and machines". In this way, he established the connection between Control Theory and Physiology and anticipated that, in a desirable future, engines would obey and imitate human beings.

Figure 2: Norbert Wiener (1894–1964).

At that time this was only a dream but now the situation is completely different, since recent developments have made possible a large number of new applications in *Robotics*, *Computer-Aided Design*, etc. (see [43] for an overview). Today, Cybernetics is not a dream any more but an ubiquitous reality. On the other hand, Cybernetics leads to many important questions that are relevant for the development of our society, very often in the borderline of *Ethics* and *Philosophy*. For instance,

Can we be inspired by Nature to create better engines and machines ?

Or

Is the animal behavior an acceptable criterium to judge the performance of an engine ?

Many movies of science fiction describe a world in which machines do not obey any more to humans and humans become their slaves. This is the opposite situation to the one Control Theory has been and is looking for. The development of Science and Technology is obeying very closely to the predictions made fifty years ago. Therefore, it seems desirable to deeply consider and revise our position towards Cybernetics from now on, many years ahead, as we do permanently in what concerns, for instance, Genetics and the possibilities it provides to intervene in human reproduction.

3. The pendulum

We will analyze in this Section a very simple and elementary control problem related to the dynamics of *the pendulum.*

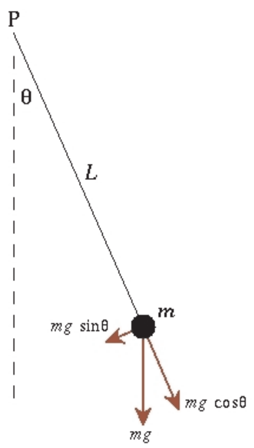

Figure 3: The simple pendulum.

The analysis of this model will allow us to present the most relevant ideas in the control of finite dimensional systems, that, as we said above, are essential for more sophisticated systems too. In our presentation, we will closely follow the book by E. Sontag [46].

The problem we discuss here, far from being purely academic, arises in many technological applications and in particular in Robotics, where the goal is to control a *gyratory arm* with a motor located at one extreme connecting the arm to the rest of the structure.

In order to model this system, we assume that the total mass m of the arm is located at the free extreme and the bar has unit length. Ignoring the effect of friction, we write

$$m\ddot{\theta}(t) = -mg\sin\theta(t) + v(t), \tag{5}$$

which is a direct consequence of *Newton's law.* Here, $\theta = \theta(t)$ is the angle of the arm with respect to the vertical axis measured counterclockwise, g is the acceleration due to gravity and u is the applied external *torsional momentum.* The state of the system is $(\theta, \dot{\theta})$, while $v = v(t)$ is the control (see Fig. 3).

To simplify our analysis, we also assume that $m = g = 1$. Then, (5) becomes:

$$\ddot{\theta}(t) + \sin\theta(t) = v(t). \tag{6}$$

The vertical stationary position ($\theta = \pi, \dot{\theta} = 0$) is an equilibrium configuration in the absence of control, i.e. with $v \equiv 0$. But, obviously, this is an *unstable* equilibrium. Let us analyze the system around this configuration, to understand how this instability can be compensated by means of the applied control force v.

Taking into account that $\sin\theta \sim \pi - \theta$ near $\theta = \pi$, at first approximation, the linearized system with respect to the variable $\varphi = \theta - \pi$ can be written in the form

$$\ddot{\varphi} - \varphi = v(t). \tag{7}$$

The goal is then to drive $(\varphi, \dot{\varphi})$ to the desired state $(0,0)$ for all small initial data, without making the angle and the velocity too large along the controlled trajectory.

The following control strategy is in agreement with common sense: when the system is to the left of the vertical line, i.e. when $\varphi = \theta - \pi > 0$, we push the system towards the right side, i.e. we apply a force v with negative sign; on the other hand, when $\varphi < 0$, it seems natural to choose $v > 0$.

This suggests the following *feedback law*, in which the control is proportional to the state:

$$v = -\alpha\varphi, \qquad \text{with } \alpha > 0. \tag{8}$$

In this way, we get the *closed loop system*

$$\ddot{\varphi} + (\alpha - 1)\varphi = 0. \tag{9}$$

It is important to understand that, solving (9), we simultaneously obtain the state $(\varphi, \dot{\varphi})$ and the control $v = -\alpha\varphi$. This justifies, at least in this case, the relevance of a feedback law like (8).

The roots of the characteristic polynomial of the linear equation (9) are $z = \pm\sqrt{1-\alpha}$. Hence, when $\alpha > 1$, the nontrivial solutions of this differential equation are *oscillatory*. When $\alpha < 1$, all solutions diverge to $\pm\infty$ as $t \to \pm\infty$, except those satisfying

$$\dot{\varphi}(0) = -\sqrt{1-\alpha}\;\varphi(0).$$

Finally, when $\alpha = 1$, all nontrivial solutions satisfying $\dot{\varphi}(0) = 0$ are constant.

Thus, the solutions to the linearized system (9) do not reach the desired configuration $(0,0)$ in general, independently of the constant α we put in (8).

This can be explained as follows. Let us first assume that $\alpha < 1$. When $\varphi(0)$ is positive and small and $\dot{\varphi}(0) = 0$, from equation (9) we deduce that $\ddot{\varphi}(0) > 0$. Thus, φ and $\dot{\varphi}$ grow and, consequently, the pendulum goes away from the vertical line. When $\alpha > 1$, the control acts on the correct direction but with too much inertia.

The same happens to be true for the nonlinear system (6).

The most natural solution is then to keep $\alpha > 1$, but introducing an additional term to diminish the oscillations and penalize the velocity. In this way, a new feedback law can be proposed in which the control is given as a linear combination of φ and $\dot{\varphi}$:

$$v = -\alpha\varphi - \beta\dot{\varphi}, \quad \text{with } \alpha > 1 \text{ and } \beta > 0. \tag{10}$$

The new closed loop system is

$$\ddot{\varphi} + \beta\dot{\varphi} + (\alpha - 1)\varphi = 0, \tag{11}$$

whose characteristic polynomial has the following roots

$$\frac{-\beta \pm \sqrt{\beta^2 - 4(\alpha - 1)}}{2}. \tag{12}$$

Now, the real part of the roots is negative and therefore, all solutions converge to zero as $t \to +\infty$. Moreover, if we impose the condition

$$\beta^2 > 4(\alpha - 1), \tag{13}$$

we see that solutions tend to zero monotonically, without oscillations.

This simple model is rich enough to illustrate some systematic properties of control systems:

- Linearizing the system is an useful tool to address its control, although the results that can be obtained this way are only of local nature.
- One can obtain feedback controls, but their effects on the system are not necessarily in agreement with the very first intuition. Certainly, the (asymptotic) stability properties of the system must be taken into account.
- Increasing dissipation one can eliminate the oscillations, as we have indicated in (13).

In connection with this last point, notice however that, as dissipation increases, trajectories converge to the equilibrium more slowly. Indeed, in (10), for fixed $\alpha > 1$, the value of β that minimizes the largest real part of a root of the characteristic polynomial (11) is

$$\beta = 2\sqrt{\alpha - 1}.$$

With this value of β, the associated real part is

$$\sigma^* = -\sqrt{\alpha - 1}$$

and, increasing β, the root corresponding to the plus sign increases and converges to zero:

$$\frac{-\beta + \sqrt{\beta^2 - 4(\alpha - 1)}}{2} > -\sqrt{\alpha - 1} \quad \forall \beta > 2\sqrt{\alpha - 1} \tag{14}$$

and

$$\frac{-\beta + \sqrt{\beta^2 - 4(\alpha - 1)}}{2} \to 0^- \quad \text{as } \beta \to +\infty. \tag{15}$$

This phenomenon is known as *overdamping* in Engineering and has to be taken into account systematically when designing feedback mechanisms.

At the practical level, implementing the control (10) is not so simple, since the computation of v requires knowing the position φ and the velocity $\dot{\varphi}$ at every time.

Let us now describe an interesting alternative. The key idea is to evaluate φ and $\dot{\varphi}$ only on a discrete set of times

$$0,\ \delta,\ 2\delta,\ \dots,\ k\delta,\ \dots$$

and modify the control at each of these values of t. The control we get this way is kept constant along each interval $[k\delta, (k+1)\delta]$.

Computing the solution to system (7), we see that the result of applying the constant control v_k in the time interval $[k\delta, (k+1)\delta]$ is as follows:

$$\begin{pmatrix} \varphi(k\delta+\delta) \\ \dot{\varphi}(k\delta+\delta) \end{pmatrix} = A \begin{pmatrix} \varphi(k\delta) \\ \dot{\varphi}(k\delta) \end{pmatrix} + v_k\, b,$$

where

$$A = \begin{pmatrix} \cos h\delta & \sin h\delta \\ \sin h\delta & \cos h\delta \end{pmatrix}, \quad b = \begin{pmatrix} \cos h\delta - 1 \\ \sin h\delta \end{pmatrix}.$$

Thus, we obtain a discrete system of the form

$$x_{k+1} = (A + bf^t)x_k\,,$$

where f is the vector such that

$$v_k = f^t x_k\,.$$

Observe that, if f is such that the matrix $A + bf^t$ is nilpotent, i.e.

$$[A + bf^t]^2 = 0,$$

then we reach the equilibrium in two steps. A simple computation shows that this property holds if $f^t = (f_1, f_2)$, with

$$f_1 = \frac{1 - 2\cos h\delta}{2(\cos h\delta - 1)}\,, \quad f_2 = -\frac{1 + 2\cos h\delta}{2\sin h\delta}\,. \tag{16}$$

The main advantage of using controllers of this form is that we get the stabilization of the trajectories in finite time and not only asymptotically, as $t \to +\infty$. The controller we have designed is a *digital control* and it is extremely useful because of its robustness and the ease of its implementation.

The digital controllers we have built are similar and closely related to the *bang-bang* controls we are going to describe now.

Once $\alpha > 1$ is fixed, for instance $\alpha = 2$, we can assume that

$$v = -2\varphi + w, \tag{17}$$

so that (7) can be written in the form

$$\ddot{\varphi} + \varphi = w. \tag{18}$$

This is Newton's law for the vibration of a spring.

This time, we look for controls below an admissible cost. For instance, we impose

$$|w(t)| \leq 1 \quad \forall t.$$

The function $w = w(t)$ that, satisfying this constraint, controls the system in minimal time, i.e. *the optimal control*, is necessarily of the form

$$w(t) = \mathrm{sgn}(p(t)),$$

where η is a solution of

$$\ddot{p} + p = 0.$$

This is a consequence of *Pontryagin's maximum principle* (see Appendix 1 for more details).

Therefore, the optimal control takes only the values ± 1 and, in practice, it is sufficient to determine the *switching times* at which the sign of the optimal control changes.

In order to compute the optimal control, let us first compute the solutions corresponding to the extremal controllers ± 1. Using the new variables x_1 and x_2 with $x_1 = \varphi$ and $x_2 = \dot{\varphi}$, this is equivalent to solve the systems

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + 1 \end{cases} \tag{19}$$

and

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 1. \end{cases} \tag{20}$$

The solutions can be identified to the circumferences in the plane (x_1, x_2) centered at $(1, 0)$ and $(-1, 0)$, respectively. Consequently, in order to drive (18) to the final state $(\varphi, \dot{\varphi})(T) = (0, 0)$, we must follow these circumferences, starting from the prescribed initial state and switching from one to another appropriately.

For instance, assume that we start from the initial state $(\varphi, \dot{\varphi})(0) = (\varphi^0, \varphi^1)$, where φ^0 and φ^1 are positive and small (see Fig. 4). Then, we first take $w(t) = 1$ and solve (19) for $t \in [0, T_1]$, where T_1 is such that $x_2(T_1) = 0$, i.e. we follow counterclockwise the arc connecting the points (φ^0, φ^1) and $(x_1(T_1), 0)$ in the (x_1, x_2) plane. In a second step, we take $w(t) = -1$ and solve (20) for $t \in [T_1, T_2]$, where T_2 is such that $(1 - x_1(T_2))^2 + x_2(T_2)^2 = 1$. We thus follow (again counterclockwise) the arc connecting the points $(x_1(T_1), 0)$ and $(x_1(T_2), x_2(T_2))$. Finally, we take $w(t) = 1$ and solve (19) for $t \in [T_2, T_3]$, with T_3 such that $x_1(T_3 = x_2(T_3) = 0$.

Similar constructions of the control can be done when $\varphi^0 \leq 1$ or $\varphi^1 \leq 0$.

In this way, we reach the equilibrium $(0, 0)$ in finite time and we obtain a feedback mechanism

$$\ddot{\varphi} + \varphi = F(\varphi, \dot{\varphi}),$$

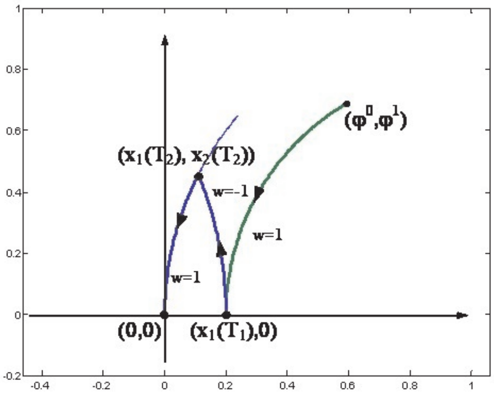

Figure 4: The action of a bang-bang control.

where F is the function taking the value -1 above the switching curve and $+1$ below. In what concerns the original system (7), we have

$$\ddot{\varphi} - \varphi = -2\varphi + F(\varphi, \dot{\varphi}).$$

The action of the control in this example shows clearly the suitability of self-regulation mechanisms. If we want to lead the system to rest in a minimal time, it is advisable to do it following a somewhat indirect path, allowing the system to evolve naturally and avoiding any excessive forcing.

Bang-bang controllers are of high interest for practical purposes. Although they might seem irregular and unnatural, they have the advantages of providing minimal time control and being easy to compute.

As we said above, although the problem we have considered is very simple, it leads naturally to some of the most relevant ideas of Control Theory: feedback laws, overdamping, digital and bang-bang controls, etc.

4. History and contemporary applications

In this paper, we do not intend to make a complete overview of the history of Control Theory, nor to address its connections with the philosophical questions we have just mentioned. Without any doubt, this would need much more space. Our intention is simply to recall some classical and well known results that have to some extent influenced the development of this discipline, pointing out several facts that, in our opinion, have been relevant for the recent achievements of Control Theory.

Let us go back to the origins of Control Engineering and Control Theory and let us describe the role this discipline has played in History.

Figure 5: A Roman aqueduct.

Going backwards in time, we will easily conclude that Romans did use some elements of Control Theory in their aqueducts. Indeed, ingenious systems of regulating valves were used in these constructions in order to keep the water level constant.

Some people claim that, in the ancient Mesopotamia, more than 2000 years B.C., the control of the irrigation systems was also a well known art.

On the other hand, in the ancient Egypt the "harpenodaptai" (string stretchers), were specialized in stretching very long strings leading to long straight segments to help in large constructions. Somehow, this is an evidence of the fact that in the ancient Egypt the following two assertions were already well understood:

- The shortest distance between two points is the straight line (which can be considered to be the most classical assertion in Optimization and Calculus of Variations);
- This is equivalent to the following dual property: among all the paths of a given length the one that produces the longest distance between its extremes is the straight line as well.

The task of the "harpenodaptai" was precisely to build these "optimal curves".

The work by Ch. Huygens and R. Hooke at the end of the XVII Century on the *oscillations of the pendulum* is a more modern example of development in Control Theory. Their goal was to achieve a precise measurement of time and location, so precious in navigation.

These works were later adapted to regulate the velocity of windmills. The main mechanism was based on a system of balls rotating around an axis, with a velocity proportional to the velocity of the windmill. When the rotational velocity increased, the balls got farther from the axis, acting on the wings of the mill through appropriate mechanisms.

J. Watt adapted these ideas when he invented the *steam engine* and this constituted a magnificent step in the industrial revolution. In this mechanism, when the velocity of the balls increases, one or several valves open to let the vapor scape. This makes the pressure diminish. When this happens, i.e. when the pressure inside the boiler becomes weaker, the velocity begins to go down. The goal of introducing and using this mechanism is of course to keep the velocity as close as possible to a constant.

Figure 6: J. Watt (1736–1819).

The British astronomer G. Airy was the first scientist to analyze mathematically the regulating system invented by Watt. But the first definitive mathematical description was given only in the works by J.C. Maxwell, in 1868, where some of the erratic behaviors encountered in the steam engine were described and some control mechanisms were proposed.

The central ideas of Control Theory gained soon a remarkable impact and, in the twenties, engineers were already preferring the continuous processing and using semi-automatic or automatic control techniques. In this way, Control Engineering germinated and got the recognition of a distinguished discipline.

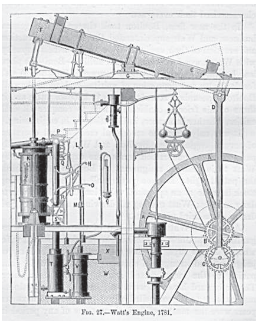

Figure 7: Watt's 1781 steam engine (taken from [50]).

In the thirties important progresses were made on automatic control and design and analysis techniques. The number of applications increased covering *amplifiers* in telephone systems, distribution systems in electrical plants, stabilization of aeroplanes, electrical mechanisms in paper production, Chemistry, petroleum and steel Industry, etc.

By the end of that decade, two emerging and clearly different methods or approaches were available: a first method based on the use of differential equations and a second one, of frequential nature, based on the analysis of amplitudes and phases of "inputs" and "outputs".

By that time, many institutions took conscience of the relevance of automatic control. This happened for instance in the American ASME (American Society of Mechanical Engineers) and the British IEE (Institution of Electrical Engineers). During the Second World War and the following years, engineers and scientists improved their experience on the control mechanisms of plane tracking and ballistic missiles and other designs of anti-aircraft batteries. This produced an important development of frequential methods.

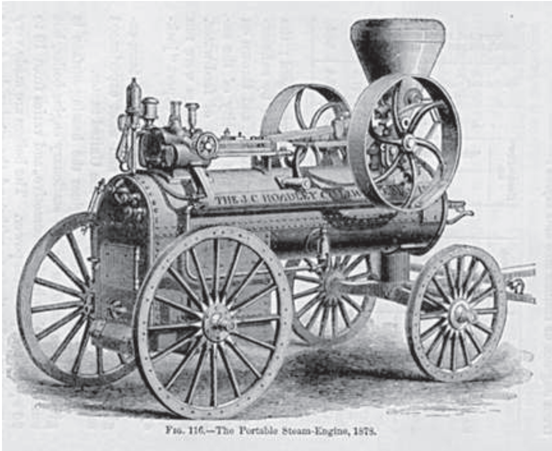

Figure 8: A primitive steam engine (taken from [50]).

After 1960, the methods and ideas mentioned above began to be considered as part of “classical” Control Theory. The war made clear that the models considered up to that moment were not accurate enough to describe the complexity of the real word. Indeed, by that time it was clear that *true systems* are often *nonlinear* and *nondeterministic*, since they are affected by “noise”. This generated important new efforts in this field.

The contributions of the U.S. scientist R. Bellman in the context of *dynamic programming*, R. Kalman in *filtering techniques* and the algebraic approach to linear systems and the Russian L. Pontryagin with the *maximum principle* for nonlinear optimal control problems established the foundations of modern Control Theory.

We shall describe in Section 6 the approach by Kalman to the controllability of linear finite dimensional systems. Furthermore, at the end of this paper we give two short Appendices where we have tried to present, as simply as possible, the central ideas of Bellman’s and Pontryagin’s works.

As we have explained, the developments of Industry and Technology had a tremendous impact in the history of Control Engineering. But the development of Mathematics had a similar effect.

Indeed, we hav already mentioned that, in the late thirties, two emerging strategies were already established. The first one was based on the use of differential equations and, therefore, the contributions made by the most celebrated mathematicians between the XVIIth and the XIXth Centuries played a fundamental role in that approach. The second one, based on a frequential approach, was greatly influenced by the works of J. Fourier.

Accordingly, Control Theory may be regarded nowadays from two different and complementary points of view: as a theoretical support to *Control Engineering* (a part of *System Engineering*) and also as a mathematical discipline. In practice, the frontiers between these two subworlds are extremely vague. In fact, Control Theory is one of the most interdisciplinary areas of Science nowadays, where Engineering and Mathematics melt perfectly and enrich each other.

Mathematics is currently playing an increasing role in Control Theory. Indeed, the degree of sophistication of the systems that Control Theory has to deal with increases permanently and this produces also an increasing demand of Mathematics in the field.

Along these notes, it will become clear that Control Theory and Calculus of Variations have also common roots. In fact, these two disciplines are very often hard to distinguish.

The history of the Calculus of Variations is also full of mathematical achievements. We shall now mention some of them.

As we said above, one can consider that the starting point of the Calculus of Variations is the understanding that the straight line is the shortest path between two given points. In the first Century, Heron of Alexandria showed in his work "La Catoptrique" that the law of reflection of light (the fact that the incidence and reflection angles are identical) may be obtained as a consequence of the variational principle that light minimizes distance along the preferred path.

In the XVII Century, P. De Fermat generalized this remark by Heron and formulated the following minimum principle:

> *Light in a medium with variable velocity prefers the path that guarantees the minimal time.*

Later Leibnitz and Huygens proved that the law of refraction of light may be obtained as a consequence of Fermat's principle. Although this had been discovered by G. Snell in 1621, it remained unpublished until 1703, as Huygens published his *Dioptrica*.

In 1691, J. Bernoulli proved that the *catenary* is the curve which provides the shape of a string of a given length and constant density with fixed ends under the action of gravity. Let us also mention that the problem of the *bachistocrone*, formulated by Bernoulli in 1696, is equivalent to finding the rays of light in the upper half-plane $y \geq 0$ corresponding to a light velocity c given by the formula $c(x, y) = \sqrt{y}$ (Newton proved in 1697 that the solution is the *cycloid*). The reader interested in these questions may consult the paper by H. Sussmann [48].

R. Kalman, one of the greatest protagonists of modern Control Theory, said in 1974 that, in the future, the main advances in Control and Optimization of systems would come more from mathematical progress than from the technological development. Today, the state of the art and the possibilities that Technology offers are so impressive that maintaining that statement is probably very risky. But, without any doubt, the development of Control Theory will require deep contributions coming from both fields.

In view of the rich history of Control Theory and all the mathematical achievements that have been undertaken in its domain of influence, one could

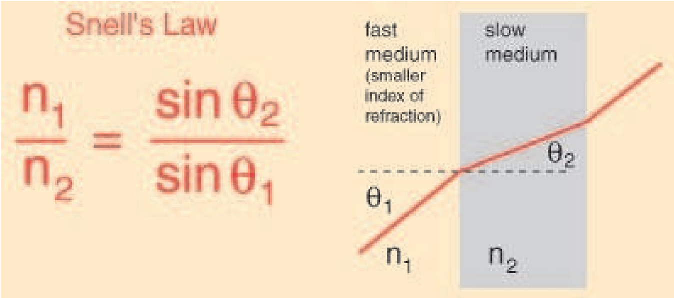

Figure 9: Snell's law of refraction.

ask whether the field has reached its end. But this is far from reality. Our society provides every day new problems to Control Theory and this fact is stimulating the creation of new Mathematics.

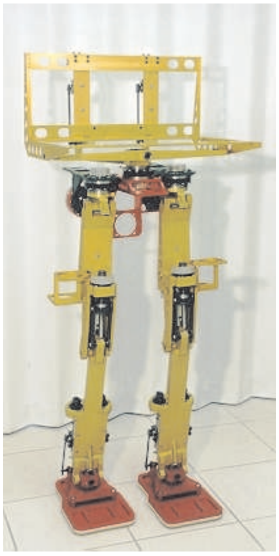

Figure 10: The biped BIP2000.

Indeed, the range of applications of Control Theory goes from the simplest mechanisms we manipulate in everyday life to the most sophisticated ones, emerging in new technologies.

The book edited by W.S. Levine [26] provides a rather complete description of this variety of applications.

One of the simplest applications of Control Theory appears in such an apparently simple machine as the tank of our bathroom. There are many variants of tanks and some of the licences go back to 1886 and can be found in [25]. But all them work under the same basic principles: the tank is supplied of regulating valves, security mechanisms that start the control process, feedback mechanisms that provide more or less water to the tank depending of the level of water in its interior and, finally, mechanisms that avoid the unpleasant flooding in case that some of the other components fail.

The systems of heating, ventilation and air conditioning in big buildings are also very efficient large scale control systems composed of interconnected thermo-fluid and electro-mechanical subsystems. The main goal of these systems is to keep a comfortable and good quality air under any circumstance, with a low operational cost and a high degree of reliability. The relevance of a proper and efficient functioning of these systems is crucial from the viewpoint of the impact in Economical and Environmental Sciences. The predecessor of these sophisticated systems is the classical *thermostat* that we all know and regulates temperature at home.

The list of applications of Control Theory in Industry is endless. We can mention, for instance, the *pH* control in chemical reactions, the paper and automobile industries, nuclear security, defense, etc.

The control of *chaos* is also being considered by many researchers nowadays. The chaotic behavior of a system may be an obstacle for its control; but it may also be of help. For instance, the control along unstable trajectories is of great use in controlling the dynamics of fight aircrafts. We refer to [35] for a description of the state of the art of *active control* in this area.

Space structures, optical reflectors of large dimensions, satellite communication systems, etc. are also examples of modern and complex control systems. The control of *robots*, ranging from the most simple engines to the *bipeds* that simulate the locomotive ability of humans is also another emerging area of Control Theory.

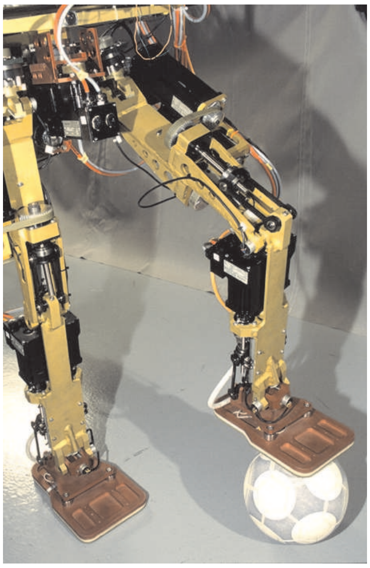

Figure 11: Another view of the biped BIP2000.

For instance, see the web page `http://www.inrialpes.fr/bipop/` of the French Institute I.N.R.I.A. (Institut National de Recherche en Informatique et Automatique), where illustrating images and movies of the antropomorphic biped BIP2000 can be found.

Compact disk players is another area of application of modern control systems. A CD player is endowed with an optical mechanism allowing to interpret the registered code and produce an acoustic signal. The main goal when designing CD players is to reach higher velocities of rotation, permitting a faster reading, without affecting the stability of the disk. The control mechanisms have to be even more robust when dealing with portable equipments.

Electrical plants and distribution networks are other modern applications of Control Theory that influence significantly our daily life. There are also many relevant applications in Medicine ranging from artificial organs to mechanisms for insulin supply, for instance.

We could keep quoting other relevant applications. But those we have mentioned and some others that will appear later suffice to prove the ubiquity of control mechanisms in the real world. The underlying mathematical theory is also impressive. The reader interested in an introduction to the classical and basic mathematical techniques in Control Engineering is referred to [8] and [36].

5. Controllability versus optimization

As already mentioned, for systems of the form (1), the main goal of Control Theory is to find *controls* v leading the *associated states* $y(v)$, i.e. the solutions of the corresponding controlled systems, to a desired situation.

There are however (at least) two ways of specifying a "desired prescribed situation":

- To fix a desired state y_d and require

$$y(v) = y_d \tag{21}$$

or, at least,

$$y(v) \sim y_d \tag{22}$$

in some sense. This is the *controllability* viewpoint.

The main question is then the existence of an admissible control v so that the corresponding state $y(v)$ satisfies (21) or (22). Once the existence of such a control v is established, it is meaningful to look for an optimal control, for instance, a control of *minimal size.* Other important questions arise in this context too. For instance, the existence of "bang-bang" controls, the minimal time of control, etc.

As we shall see, this problem may be difficult (or even very difficult) to solve. In recent years, an important body of beautiful Mathematics has been developed in connection with these questions.

- To fix a *cost function* $J = J(v)$ like for instance (3) or (4) and to look for a *minimizer* u of J. This is the *optimization* or *optimal control* viewpoint.

As in (3) and (4), J is typically related to the "distance" to a prescribed state. Both approaches have the same ultimate goal, to bring the state close to the desired target but, in some sense, the second one is more realistic and easier to implement.

The optimization viewpoint is, at least apparently, humble in comparison with the controllability approach. But it is many times much more realistic. In practice, it provides satisfactory results in many situations and, at the same time, it requires simpler mathematical tools.

To illustrate this, we will discuss now a very simple example. It is trivial in the context of Linear Algebra but it is of great help to introduce some of the basic tools of Control Theory.

We will assume that the state equation is

$$Ay = b, \tag{23}$$

where A is a $n \times n$ real matrix and the state is a column vector $y = (y_1, y_2, \ldots, y_n)^t \in \mathbb{R}^n$. To simplify the situation, let us assume that A is nonsingular. The control vector is $b \in \mathbb{R}^n$. Obviously, we can rewrite (23) in the form $y = A^{-1}b$, but we do not want to do this. In fact, we are mainly interested in those cases in which (23) can be difficult to solve.

Let us first adopt the controllability viewpoint. To be specific, let us impose as an objective to make the first component y_1 of y coincide with a prescribed value y_1^* :

$$y_1 = y_1^* . \tag{24}$$

This is the sense we are giving to (22) in this particular case. So, we are consider the following controllability problem:

PROBLEM 0: *To find* $b \in \mathbb{R}^n$ *such that the solution of* (23) *satisfies* (24).

Roughly speaking, we are addressing here a *partial controllability* problem, in the sense that we are controlling only one component, y_1 , of the state.

Obviously, such controls b exist. For instance, it suffices to take $y^* = (y_1^*, 0, \cdots, 0)^t$ and then choose $b = Ay^*$. But this argument, by means of which we find the state directly without previously determining the control, is frequently impossible to implement in practice. Indeed, in most real problems, we have first to find the control and, only then, we can compute the state by solving the state equation.

The number of control parameters (the n components of b) is greater or equal than the number of state components we have to control. But, what happens if we stress our own possibilities ? What happens if, for instance, $b_1, \dots, b_{n-1}$ are fixed and we only have at our disposal b_n to control the system ?

From a mathematical viewpoint, the question can be formulated as follows. In this case,

$$Ay = c + be \tag{25}$$

where $c \in \mathbb{R}^n$ is a prescribed column vector, e is the unit vector $(0, \dots, 0, 1)^t$ and b is a scalar control parameter. The corresponding controllability problem is now the following:

PROBLEM 1: *To find* $b \in \mathbb{R}$ *such that the solution of* (25) *satisfies* (24).

This is a less obvious question. However, it is not too difficult to solve. Note that the solution y to (25) can be decomposed in the following way:

$$y = x + z, \tag{26}$$

where

$$x = A^{-1}c \tag{27}$$

and z satisfies

$$Az = be, \quad \text{i.e.} \quad z = bz^* \quad\quad z^* = A^{-1}e. \tag{28}$$

To guarantee that y_1 can take *any* value in $\mathbb{R}$, as we have required in (24), it is necessary and sufficient to have $z_1^* \neq 0$, z_1^* being the first component of $z^* = A^{-1}e$.

In this way, we have a precise answer to this second controllability problem:

The problem above can be solved for any y_1^* if and only if the first component of $A^{-1}e$ does not vanish.

Notice that, when the first component of $A^{-1}e$ vanishes, whatever the control b is, we always have $y_1 = x_1$, x_1 being the first component of the fixed vector x in (27). In other words, y_1 *is not sensitive* to the control b_n . In this degenerate case, the set of values taken by y_1 is a singleton, a 0-dimensional manifold. Thus, we see that the state is confined in a "space" of low dimension and controllability is lost in general.

But, is it really frequent in practice to meet degenerate situations like the previous one, where some components of the system are insensitive to the control ?

Roughly speaking, it can be said that systems are generically not degenerate. In other words, in examples like the one above, it is actually rare that z_1^* vanishes.

There are however a few remarks to do. When z_1^* does not vanish but is very small, even though controllability holds, the control process is very unstable in the sense that one needs very large controls in order to get very small variations of the state. In practice, this is very important and must be taken into account (one needs the system not only to be controllable but this to happen with realistic and feasible controls).

On the other hand, it can be easily imagined that, when systems under consideration are complex, i.e. many parameters are involved, it is difficult to know *a priori* whether or not there are components of the state that are insensitive to the control[1].

Let us now turn to the optimization approach. Let us see that the difficulties we have encountered related to the possible degeneracy of the system disappear (which confirms the fact that this strategy leads to easier questions).

For example, let us assume that $k > 0$ is a reasonable bound of the control b that we can apply. Let us put

$$J(b_n) = \frac{1}{2}|y_1 - y_1^*|^2 \qquad \forall b_n \in \mathbb{R}, \tag{29}$$

where y_1 is the first component of the solution to (25). Then, it is reasonable to admit that the best response is given by the solution to the following problem:

PROBLEM 1′: *To find $b_n^k \in [-k, k]$ such that*

$$J(b_n^k) \leq J(b_n) \quad \forall b_n \in [-k, k]. \tag{30}$$

Since $b_n \mapsto J(b_n)$ is a continuous function, it is clear that this problem possesses a solution $b_n^k \in I_k$ for each $k > 0$. This confirms that the considered optimal control problem is simpler.

[1] In fact, it is a very interesting and non trivial task to design strategies guaranteeing that we do not fall in a degenerate situation.

On the other hand, this point of view is completely natural and agrees with common sense. According to our intuition, most systems arising in real life should possess an optimal strategy or configuration. At this respect L. Euler said:

> "Universe is the most perfect system, designed by the most wise Creator. Nothing will happen without emerging, at some extent, a maximum or minimum principle".

Let us analyze more closely the similarities and differences arising in the two previous formulations of the control problem.

- Assume the controllability property holds, that is, PROBLEM 1 is solvable for any y_1^*. Then, if the target y_1^* is given and k is sufficiently large, the solution to PROBLEM 1′ coincides with the solution to PROBLEM 1.
- On the other hand, when there is no possibility to attain y_1^* exactly, the optimization viewpoint, i.e. PROBLEM 1′, furnishes the best response.
- To investigate whether the controllability property is satisfied, it can be appropriate to solve PROBLEM 1′ for each $k > 0$ and analyze the behavior of the *cost*
$$J_k = \min_{b_n \in [-k,k]} J(b_n) \tag{31}$$
as k grows to infinity. If J_k stabilizes near a positive constant as k grows, we can suspect that y_1^* cannot be attained exactly, i.e. that PROBLEM 1 does not have a solution for this value of y_1^*.

In view of these considerations, it is natural to address the question of whether it is actually necessary to solve controllability problems like PROBLEM 1 or, by the contrary, whether solving a related optimal control problem (like PROBLEM 1′) suffices.

There is not a generic and systematic answer to this question. It depends on the level of precision we require to the control process and this depends heavily on the particular application one has in mind. For instance, when thinking of technologies used to stabilize buildings, or when controlling space vehicles, etc., the efficiency of the control that is required demands much more than simply choosing the best one with respect to a given criterion. In those cases, it is relevant to know how close the control will drive the state to the prescribed target. There are, consequently, a lot of examples for which simple optimization arguments as those developed here are insufficient.

In order to choose the appropriate control we need first to develop a rigorous modelling (in other words, we have to put equations to the real life system). The choice of the control problem is then a second relevant step in modelling.

Let us now recall and discuss some mathematical techniques allowing to handle the minimization problems arising in the optimization approach (in fact, we shall see that these techniques are also relevant when the controllability point of view is adopted).

These problems are closely related to the Calculus of Variations. Here, we do not intend to provide a survey of the techniques in this field but simply to mention some of the most common ideas.

For clarity, we shall start discussing *Mathematical Programming.* In the context of Optimization, *Programming* is not the art of writing computer codes. It was originated by the attempt to *optimize* the planning of the various tasks or activities in an organized system (a plant, a company, etc.). The goal is then to find what is known as an *optimal planning* or *optimal programme.*

The simplest problem of *assignment* suffices to exhibit the need of a mathematical theory to address these issues.

Assume that we have 70 workers in a plant. They have different qualifications and we have to assign them 70 different tasks. The total number of possible distributions is $70\,!$, which is of the order of 10^{100}. Obviously, in order to be able to solve rapidly a problem like this, we need a mathematical theory to provide a good strategy.

This is an example of assignment problem. Needless to say, problems of this kind are not only of academic nature, since they appear in most human activities.

In the context of *Mathematical Programming*, we first find *linear programming techniques.* As their name indicates, these are concerned with those optimization problems in which the involved functional is linear.

Linear Programming was essentially unknown before 1947, even though Joseph Fourier had already observed in 1823 the relevance of the questions it deals with. L.V. Kantorovich, in a monograph published in 1939, was the first to indicate that a large class of different planning problems could be covered with the same formulation. The *method of simplex,* that we will recall below, was introduced in 1947 and its efficiency turned out to be so impressive that very rapidly it became a common tool in Industry.

There has been a very intense research in these topics that goes beyond Linear Programming and the method of simplex. We can mention for instance *nonlinear programming methods*, inspired by the *method of descent.* This was formally introduced by the French mathematician A.L. Cauchy in the XIX Century. It relies on the idea of solving a nonlinear equation by searching the critical points of the corresponding primitive function.

Let us now give more details on Linear Programming. At this point, we will follow a presentation similar to the one by G. Strang in [47].

The problems that one can address by means of linear programming involve the minimization of linear functions subject to linear constraints. Although they seem extremely simple, they are ubiquitous and can be applied in a large variety of areas such as the control of traffic, Game Theory, Economics, etc. Furthermore, they involve in practice a huge quantity of unknowns, as in the case of the optimal planning problems we have presented before.

The simplest problem in this field can be formulated in the following way:

> *Given a real matrix A of order $M \times N$ (with $M \leq N$), and given a column vector b of M components and a column vector c with N*

components, to minimize the linear function

$$\langle c, x\rangle = c_1 x_1 + \cdots + c_N x_N$$

under the restrictions

$$Ax = b, \quad x \geq 0.$$

Here and in the sequel, we use $\langle \cdot\,, \cdot\rangle$ to denote the usual Euclidean scalar products in $\mathbb{R}^N$ and $\mathbb{R}^M$. The associated norm will be denoted by $|\cdot|$.

Of course, the second restriction has to be understood in the following way:

$$x_j \geq 0, \quad j = 1, \ldots, N.$$

In general, the solution to this problem is given by a unique vector x with the property that $N - M$ components vanish. Accordingly, the problem consists in finding out which are the $N - M$ components that vanish and, then, computing the values of the remaining M components.

The method of simplex leads to the correct answer after a finite number of steps. The procedure is as follows:

- Step 1: We look for a vector x with $N-M$ zero components and satisfying $Ax = b$, in addition to the unilateral restriction $x \geq 0$. Obviously, this first choice of x will not provide the optimal answer in general.

- Step 2: We modify appropriately this first choice of x allowing one of the zero components to become positive and vanishing one of the positive components and this in such a way that the restrictions $Ax = b$ and $x \geq 0$ are kept.

After a finite number of steps like Step 2, the value of $\langle c, x\rangle$ will have been tested at all possible minimal points. Obviously, the solution to the problem is obtained by choosing, among these points x, that one at which the minimum of $\langle c, x\rangle$ is attained.

Let us analyze the geometric meaning of the simplex method with an example.

Let us consider the problem of minimizing the function

$$10x_1 + 4x_2 + 7x_3$$

under the constraints

$$2x_1 + x_2 + x_3 = 1, \quad x_1\,, x_2\,, x_3 \geq 0.$$

In this case, the set of admissible triplets $(x_1\,, x_2\,, x_3)$, i.e. those satisfying the constraints is the triangle in $\mathbb{R}^3$ of vertices $(0, 0, 1)$, $(0, 1, 0)$ and $(1/2, 0, 0)$ (a face of a tetrahedron). It is easy to see that the minimum is achieved at $(0, 1, 0)$, where the value is 4.

Let us try to give a geometrical explanation to this fact. Since $x_1, x_2, x_3 \geq 0$ for any admissible triplet, the minimum of the function $10x_1 + 4x_2 + 7x_3$ has necessarily to be nonnegative. Moreover, the minimum cannot be zero since the hyperplane

$$10x_1 + 4x_2 + 7x_3 = 0$$

has an empty intersection with the triangle of admissible states. When increasing the cost $10x_1 + 4x_2 + 7x_3$, i.e. when considering level sets of the form $10x_1 + 4x_2 + 7x_3 = c$ with increasing $c > 0$, we are considering planes parallel to $10x_1 + 4x_2 + 7x_3 = 0$ that are getting away from the origin and closer to the triangle of admissible states. The first value of c for which the level set intersects the admissible triangle provides the minimum of the cost function and the point of contact is the minimizer.

It is immediate that this point is the vertex $(0, 1, 0)$.

These geometrical considerations indicate the relevance of the convexity of the set where the minimum is being searched. Recall that, in a linear space E, a set K is *convex* if it satisfies the following property:

$$x, y \in K, \quad \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda) y \in K.$$

The crucial role played by convexity will be also observed below, when considering more sophisticated problems.

The method of simplex, despite its simplicity, is very efficient. There are many variants, adapted to deal with particular problems. In some of them, when looking for the minimum, one runs across the convex set and not only along its boundary. For instance, this is the case of *Karmakar's method*, see [47]. For more information on Linear Programming, the method of simplex and its variants, see for instance [40].

As the reader can easily figure out, many problems of interest in Mathematical Programming concern the minimization of *nonlinear* functions. At this respect, let us recall the following fundamental result whose proof is the basis of the so called *Direct Method of the Calculus of Variations* (DMCV):

Theorem 1 *If H is a Hilbert space with norm $\|\cdot\|_H$ and the function $J : H \mapsto \mathbb{R}$ is continuous, convex and coercive in H, i.e. it satisfies*

$$J(v) \to +\infty \quad \text{as} \quad \|v\|_H \to +\infty, \tag{32}$$

then J attains its minimum at some point $u \in H$. If, moreover, J is strictly convex, this point is unique.

If, in the previous result, J is a C^1 function, any minimizer u necessarily satisfies

$$J'(u) = 0, \quad u \in H. \tag{33}$$

Usually, (33) is known as the *Euler equation* of the minimization problem

$$\text{Minimize } J(v) \text{ subject to } v \in H. \tag{34}$$

Consequently, if J is C^1, Theorem 1 serves to prove that the (generally nonlinear) Euler equation (33) possesses at least one solution.

Many systems arising in Continuum Mechanics can be viewed as the Euler equation of a minimization problem. Conversely, one can associate Euler equations to many minimization problems. This mutual relation can be used in both directions: either to solve differential equations by means of minimization techniques, or to solve minimization problems through the corresponding Euler equations.

In particular, this allows proving existence results of equilibrium configurations for many problems in Continuum Mechanics.

Furthermore, combining these ideas with the approximation of the space H where the minimization problem is formulated by means of finite dimensional spaces and increasing the dimension to cover in the limit the whole space H, one obtains *Galerkin's approximation method*. Suitable choices of the approximating subspaces lead to the *finite element methods*.

In order to illustrate these statements and connect them to Control Theory, let us consider the example

$$\left\{\begin{array}{l} \dot{x} = Ax + Bv, \quad t \in [0,T], \\ x(0) = x^0, \end{array}\right. \tag{35}$$

in which the *state* $x = (x_1(t), \ldots, x_N(t))^t$ is a vector in $\mathbb{R}^N$ depending on t (the time variable) and the *control* $v = (v_1(t), \ldots, v_M(t))^t$ is a vector with M components that also depends on time.

In (35), we will assume that A is a square, constant coefficient matrix of dimension $N \times N$, so that the underlying system is *autonomous*, i.e. invariant with respect to translations in time. The matrix B has also constant coefficients and dimension $N \times M$.

Let us set

$$J(v) = \frac{1}{2}|x(T) - x^1|^2 + \frac{\mu}{2}\int_0^T |v(t)|^2\,dt \quad \forall v \in L^2(0,T;\mathbb{R}^M), \tag{36}$$

where $x^1 \in \mathbb{R}^N$ is given, $x(T)$ is the final value of the solution of (35) and $\mu > 0$.

It is not hard to prove that $J : L^2(0,T;\mathbb{R}^M) \mapsto \mathbb{R}$ is well defined, continuous, coercive and strictly convex. Consequently, J has a unique minimizer in $L^2(0,T;\mathbb{R}^M)$. This shows that the control problem (35)–(36) has a unique solution.

With the DMCV, the existence of minimizers for a large class of problems can be proved. But there are many other interesting problems that do not enter in this simple framework, for which minimizers do not exist.

Indeed, let us consider the simplest and most classical problem in the Calculus of Variations: to show that the shortest path between two given points is the straight line segment.

Of course, it is very easy to show this by means of geometric arguments. However, *What happens if we try to use the DMCV ?*

The question is now to minimize the functional

$$\int_0^1 |\dot{x}(t)|\, dt$$

in the class of curves $x : [0,1] \mapsto \mathbb{R}^2$ such that $x(0) = P$ and $x(1) = Q$, where P and Q are two given points in the plane.

The natural functional space for this problem is not a Hilbert space. It can be the Sobolev space $W^{1,1}(0,1)$ constituted by all functions $x = x(t)$ such that x and its time derivative $\dot{x}$ belong to $L^1(0,1)$. It can also be the more sophisticated space $BV(0,1)$ of functions of bounded variation. But these are not Hilbert spaces and solving the problem in any of them, preferably in $BV(0,1)$, becomes much more subtle.

We have described the DMCV in the context of problems without constraints. Indeed, up to now, the functional has been minimized in the whole space. But in most realistic situations the nature of the problem imposes restrictions on the control and/or the state. This is the case for instance for the linear programming problems we have considered above.

As we mentioned above, convexity plays a key role in this context too:

Theorem 2 *Let H be a Hilbert space, $K \subset H$ a closed convex set and $J : K \mapsto \mathbb{R}$ a convex continuous function. Let us also assume that either K is bounded or J is coercive in K, i.e.*

$$J(v) \to +\infty \quad as v \in K, \quad \|v\|_H \to +\infty.$$

Then, there exists a point $u \in K$ where J reaches its minimum over K.

Furthermore, if J is strictly convex, the minimizer is unique.

In order to illustrate this result, let us consider again the system (35) and the functional

$$J(v) = \frac{1}{2}|x(T) - x^1|^2 + \frac{\mu}{2}\int_0^T |v(t)|^2\, dt \quad \forall v \in K, \tag{37}$$

where $\mu \geq 0$ and $K \subset L^2(0,T;\mathbb{R}^M)$ is a closed convex set. In view of Theorem 2, we see that, if $\mu > 0$, the optimal control problem determined by (35) and (37) has a unique solution. If $\mu = 0$ and K is bounded, this problem possesses at least one solution.

Let us discuss more deeply the application of these techniques to the analysis of the control properties of the linear finite dimensional system (35).

Let $J : H \mapsto \mathbb{R}$ be, for instance, a functional of class C^1. Recall again that, at each point u where J reaches its minimum, one has

$$J'(u) = 0, \quad u \in H. \tag{38}$$

It is also true that, when J is convex and C^1, if u solves (38) then u is a global minimizer of J in H. Equation (38) is the *Euler equation* of the corresponding minimization problem.

More generally, in a convex minimization problem, if the function to be minimized is of class C^1, an *Euler inequality* is satisfied by each minimizer. Thus, u is a minimizer of the convex functional J in the convex set K of the Hilbert space H if and only if

$$(J'(u), v-u)_H \geq 0 \quad \forall v \in K, \quad u \in K. \tag{39}$$

Here, $(\cdot\,,\cdot)_H$ stands for the scalar product in H.

In the context of Optimal Control, this characterization of u can be used to deduce the corresponding *optimality conditions*, also called the *optimality system*.

For instance, this can be made in the case of problem (35),(37). Indeed, it is easy to see that in this case (39) reduces to

$$\left\{ \begin{array}{l} \mu \displaystyle\int_0^T \langle u(t), v(t)-u(t)\rangle\, dt + \langle x(T)-x^1, z_v(T)-z_u(T)\rangle \geq 0 \\ \forall v \in K, \quad u \in K, \end{array} \right. \tag{40}$$

where, for each $v \in L^2(0,T;\mathbb{R}^M)$, $z_v = z_v(t)$ is the solution of

$$\left\{ \begin{array}{l} \dot z_v = Az_v + Bv, \quad t \in [0,T], \\ z_v(0) = 0 \end{array} \right.$$

(recall that $\langle\cdot\,,\cdot\rangle$ stands for the Euclidean scalar products in $\mathbb{R}^M$ and $\mathbb{R}^N$).

Now, let $p = p(t)$ be the solution of the backward in time differential problem

$$\left\{ \begin{array}{ll} -\dot p = A^t p, & t \in [0,T], \\ p(T) = x(T) - x^1. & \end{array} \right. \tag{41}$$

Then

$$\langle x(T)-x^1, z_v(T)-z_u(T)\rangle = \langle p(T), z_v(T)-z_u(T)\rangle = \int_0^T \langle p(t), B(v(t)-u(t))\rangle\, dt$$

and (40) can also be written in the form:

$$\left\{ \begin{array}{l} \displaystyle\int_0^T \langle \mu u(t) + B^t p(t), v(t)-u(t)\rangle\, dt \geq 0 \\ \forall v \in K, \quad u \in K. \end{array} \right. \tag{42}$$

The system constituted by the state equation (35) for $v = u$, i.e.

$$\left\{ \begin{array}{l} \dot x = Ax + Bu, \quad t \in [0,T], \\ x(0) = x^0, \end{array} \right. \tag{43}$$

the *adjoint state equation* (41) and the inequalities (42) is referred to as the *optimality system.* This system provides, in the case under consideration, a characterization of the optimal control.

The function $p = p(t)$ is the *adjoint state.* As we have seen, the introduction of p leads to a rewriting of (40) that is more explicit and easier to handle.

Very often, when addressing optimization problems, we have to deal with *restrictions* or *constraints* on the controls and/or state. *Lagrange multipliers* then play a fundamental role and are needed in order to write the equations satisfied by the minimizers: the so called *Euler-Lagrange equations.*

To do that, we must introduce the associated *Lagrangian* and, then, we must analyze its *saddle points.* The determination of saddle points leads to two equivalent extremal problems of dual nature.

This is a surprising fact in this theory that can be often used with efficiency: the original minimization problem being difficult to solve, one may often write a *dual minimization problem* (passing through the Lagrangian); it may well happen to the second problem to be simpler than the original one.

Saddle points arise naturally in many optimization problems. But they can also be viewed as the solutions of *minimax problems.* Minimax problems arise in many contexts, for instance:

- In *Differential Game Theory*, where two or more players *compete* trying to maximize their profit and minimize the one of the others.

- In the characterization of the proper vibrations of elastic bodies. Indeed, very often these can be characterized as eigenvalues of a self-adjoint compact operator in a Hilbert space through a minimax principle related to the *Rayleigh quotient.*

One of the most relevant contributions in this field was the one by J. Von Neumann in the middle of the XX Century, proving that the existence of a minimax is guaranteed under very weak conditions.

In the last three decades, these results have been used systematically for solving nonlinear differential problems, in particular with the help of the *Mountain Pass Lemma* (for instance, see [20]). At this respect, it is worth mentioning that a mountain pass is indeed a beautiful example of saddle point provided by Nature. A mountain pass is the location one chooses to cross a mountain chain: this point must be of minimal height along the mountain chain but, on the contrary, it is of maximal height along the crossing path we follow.

The reader interested in learning more about Convex Analysis and the related duality theory is referred to the books [9] and [41], by I. Ekeland and R. Temam and R.T. Rockafellar, respectively. The lecture notes by B. Larrouturou and P.L. Lions [23] contain interesting introductions to these and other related topics, like mathematical modelling, the theory of partial differential equations and numerical approximation techniques.

6. Controllability of linear finite dimensional systems

We will now be concerned with the controllability of ordinary differential equations. We will start by considering linear systems.

As we said above, Control Theory is full of interesting mathematical results that have had a tremendous impact in the world of applications (most of them are too complex to be reproduced in these notes). One of these important results, simple at the same time, is a theorem by R.E. Kalman which characterizes the linear systems that are controllable.

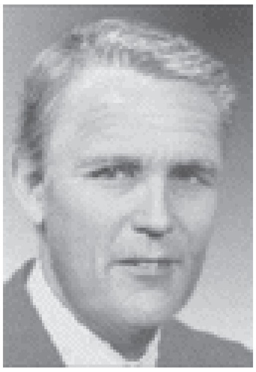

Figure 12: Rudolph E. Kalman (1930).

Let us consider again the linear system

$$\begin{cases} \dot{x} = Ax + Bv, \quad t > 0, \\ x(0) = x^0, \end{cases} \tag{44}$$

with state $x = (x_1(t), \dots, x_N(t))^t$ and control $v = (v_1(t), \dots, v_M(t))^t$. The matrices A and B have constant coefficients and dimensions $N \times N$ and $N \times M$, respectively.

Assume that $N \geq M \geq 1$. In practice, the cases where M is much smaller than N are especially significant. Of course, the most interesting case is that in which $M = 1$ and, simultaneously, N is very large. We then dispose of a single scalar control to govern the behavior of a very large number N of components of the state.

System (44) is said to be controllable at time $T > 0$ if, for every initial state $x^0 \in \mathbb{R}^N$ and every final state $x^1 \in \mathbb{R}^N$, there exists at least one control $u \in C^0([0,T]; \mathbb{R}^M)$ such that the associated solution satisfies

$$x(T) = x^1. \tag{45}$$

The following result, due to Kalman, characterizes the controllability of (44) (see for instance [25]):

Theorem 3 *A necessary and sufficient condition for system* (44) *to be controllable at some time $T > 0$ is that*

$$\text{rank } \left[B \,|\, AB \,|\, \cdots \,|\, A^{N-1}B \right] = N. \tag{46}$$

Moreover, if this is satisfied, the system is controllable for all $T > 0$.

When the rank of this matrix is k, with $1 \leq k \leq N-1$, the system is not controllable and, for each $x^0 \in \mathbb{R}^N$ and each $T > 0$, the set of solutions of (44) *at time $T > 0$ covers an affine subspace of $\mathbb{R}^N$ of dimension k.*

The following remarks are now in order:

- The degree of controllability of a system like (44) is completely determined by the rank of the corresponding matrix in (46). This rank indicates how many components of the system are sensitive to the action of the control.

- The matrix in (46) is of dimension $(N \times M) \times N$ so that, when we only have one control at our disposal (i.e. $M = 1$), this is a $N \times N$ matrix. It is obviously in this case when it is harder to the rank of this matrix to be N. This is in agreement with common sense, since the system should be easier to control when the number of controllers is larger.

- The system is controllable at some time if and only if it is controllable at any positive time. In some sense, this means that, in (44), *information propagates at infinite speed.* Of course, this property is not true in general in the context of partial differential equations.

As we mentioned above, the concept of *adjoint system* plays an important role in Control Theory. In the present context, the adjoint system of (44) is the following:

$$\left\{ \begin{array}{l} -\dot{\varphi} = A^t \varphi, \quad t < T, \\ \varphi(T) = \varphi^0. \end{array} \right. \tag{47}$$

Let us emphasize the fact that (47) is a backward (in time) system. Indeed, in (47) the sense of time has been reversed and the differential system has been completed with a *final condition* at time $t = T$.

The following result holds:

Theorem 4 *The rank of the matrix in* (46) *is N if and only if, for every $T > 0$, there exists a constant $C(T) > 0$ such that*

$$|\varphi^0|^2 \leq C(T) \int_0^T |B^t \varphi|^2 \, dt \tag{48}$$

for every solution of (47).

The inequality (48) is called an *observability inequality.* It can be viewed as the *dual* version of the controllability property of system (44).

This inequality guarantees that the adjoint system can be "observed" through $B^t\varphi$, which provides M linear combinations of the adjoint state. When (48) is satisfied, we can affirm that, from the controllability viewpoint, B^t captures appropriately all the components of the adjoint state φ. This turns out to be equivalent to the controllability of (44) since, in this case, the control u acts efficiently through the matrix B on all the components of the state x.

Inequalities of this kind play also a central role in *inverse problems*, where the goal is to reconstruct the properties of an unknown (or only partially known) medium or system by means of partial measurements. The observability inequality guarantees that the measurements $B^t\varphi$ are sufficient to detect all the components of the system.

The proof of Theorem 4 is quite simple. Actually, it suffices to write the solutions of (44) and (47) using the *variation of constants formula* and, then, to apply the *Cayley-Hamilton theorem*, that guarantees that any matrix is a root of its own characteristic polynomial.

Thus, to prove that (46) implies (48), it is sufficient to show that, when (46) is true, the mapping

$$\varphi^0 \mapsto \left(\int_0^T |B^t \varphi|^2 \, dt \right)^{1/2}$$

is a norm in $\mathbb{R}^N$. To do that, it suffices to check that the following *uniqueness* or *unique continuation* result holds:

If $B^t \varphi = 0$ *for* $0 \leq t \leq T$ *then, necessarily,* $\varphi \equiv 0$.

It is in the proof of this result that the rank condition is needed.

Let us now see how, using (48), we can build controls such that the associated solutions to (44) satisfy (45). This will provide another idea of how controllability and optimal control problems are related.

Given initial and final states x^0 and x^1 and a control time $T > 0$, let us consider the quadratic functional I, with

$$I(\varphi^0) = \frac{1}{2} \int_0^T |B^t \varphi|^2 \, dt - \langle x^1, \varphi^0 \rangle + \langle x^0, \varphi(0) \rangle \quad \forall \varphi^0 \in \mathbb{R}^N, \tag{49}$$

where φ is the solution of the adjoint system (47) associated to the final state φ^0.

The function $\varphi^0 \mapsto I(\varphi^0)$ is strictly convex and continuous in $\mathbb{R}^N$. In view of (48), it is also coercive, that is,

$$\lim_{|\varphi^0| \to \infty} I(\varphi^0) = +\infty. \tag{50}$$

Therefore, I has a unique minimizer in $\mathbb{R}^N$, that we shall denote by $\hat{\varphi}^0$. Let us write the Euler equation associated to the minimization of the functional (49):

$$\int_0^T \langle B^t \hat{\varphi}, B^t \varphi \rangle \, dt - \langle x^1, \varphi^0 \rangle + \langle x^0, \varphi(0) \rangle = 0 \quad \forall \varphi^0 \in \mathbb{R}^N, \quad \hat{\varphi}^0 \in \mathbb{R}^N. \tag{51}$$

Here, $\hat{\varphi}$ is the solution of the adjoint system (47) associated to the final state $\hat{\varphi}^0$.

From (51), we deduce that $\hat{u} = B^t \hat{\varphi}$ is a control for (44) that guarantees that (45) is satisfied. Indeed, if we denote by $\hat{x}$ the solution of (44) associated to $\hat{u}$, we have that

$$\int_0^T \langle B^t \hat{\varphi}, B^t \varphi \rangle \, dt = \langle \hat{x}(T), \varphi^0 \rangle - \langle x^0, \varphi(0) \rangle \qquad \forall \varphi^0 \in \mathbb{R}^N. \tag{52}$$

Comparing (51) and (52), we see that the previous assertion is true.

It is interesting to observe that, from the rank condition, we can deduce several variants of the observability inequality (48). In particular,

$$|\varphi^0| \leq C(T) \int_0^T |B^t \varphi| \, dt \tag{53}$$

This allows us to build controllers of different kinds.

Indeed, consider for instance the functional J_{bb}, given by

$$J_{bb}(\varphi^0) = \frac{1}{2} \left(\int_0^T |B^t \varphi| \, dt \right)^2 - \langle x^1, \varphi^0 \rangle + \langle x^0, \varphi(0) \rangle \quad \forall \varphi^0 \in \mathbb{R}^N. \tag{54}$$

This is again strictly convex, continuous and coercive. Thus, it possesses exactly one minimizer $\hat{\varphi}^0_{bb}$. Let us denote by $\hat{\varphi}_{bb}$ the solution of the corresponding adjoint system. Arguing as above, it can be seen that the new control $\hat{u}_{bb}$, with

$$\hat{u}_{bb} = \left(\int_0^T |B^t \hat{\varphi}_{bb}| \, dt \right) \operatorname{sgn}(B^t \hat{\varphi}_{bb}), \tag{55}$$

makes the solution of (44) satisfy (45). This time, we have built a *bang-bang* control, whose components can only take two values:

$$\pm \int_0^T |B^t \hat{\varphi}_{bb}| \, dt.$$

The control $\hat{u}$ that we have obtained minimizing J is the one of minimal norm in $L^2(0, T; \mathbb{R}^M)$ among all controls guaranteeing (45). On the other hand, $\hat{u}_{bb}$ is the control of minimal L^∞ norm. The first one is smooth and the second one is piecewise constant and, therefore, discontinuous in general. However, the bang-bang control is easier to compute and apply since, as we saw explicitly in the case of the pendulum, we only need to determine its amplitude and the location of the switching points. Both controls $\hat{u}$ and $\hat{u}_{bb}$ are optimal with respect to some optimality criterium.

We have seen that, in the context of linear control systems, when controllability holds, the control may be computed by solving a minimization problem. This is also relevant from a computational viewpoint since it provides useful ideas to design efficient approximation methods.

7. Controllability of nonlinear finite dimensional systems

Let us now discuss the controllability of some nonlinear control systems. This is a very complex topic and it would be impossible to describe in a few pages all the significant results in this field. We will just recall some basic ideas.

When the goal is to produce small variations or deformations of the state, it might be sufficient to proceed using linearization arguments. More precisely, let us consider the system

$$\begin{cases} \dot{x} = f(x, u), \quad t > 0, \\ x(0) = x^0, \end{cases} \tag{56}$$

where $f : \mathbb{R}^N \times \mathbb{R}^M \mapsto \mathbb{R}^N$ is smooth and $f(0,0) = 0$. The linearized system at $u = 0$, $x = 0$ is the following:

$$\begin{cases} \dot{x} = \dfrac{\partial f}{\partial x}(0,0)x + \dfrac{\partial f}{\partial u}(0,0)u, \quad t > 0, \\ x(0) = 0. \end{cases} \tag{57}$$

Obviously, (57) is of the form (44), with

$$A = \frac{\partial f}{\partial x}(0,0), \quad B = \frac{\partial f}{\partial u}(0,0), \quad x^0 = 0. \tag{58}$$

Therefore, the rank condition

$$\text{rank } [B|AB|\cdots|A^{N-1}B] = N \tag{59}$$

is the one that guarantees the controllability of (57).

Based on the *inverse function theorem*, it is not difficult to see that, if condition (59) is satisfied, then (56) is *locally controllable* in the following sense:

For every $T > 0$, there exists a neighborhood $\mathcal{B}_T$ of the origin in $\mathbb{R}^N$ such that, for any initial and final states $x_0, x_1 \in \mathcal{B}_T$, there exist controls u such that the associated solutions of the system (56) *satisfy*

$$x(T) = x^1\,. \tag{60}$$

However, this analysis is not sufficient to obtain results of global nature.

A *natural* condition that can be imposed on the system (56) in order to guarantee global controllability is that, at each point $x^0 \in \mathbb{R}^N$, by choosing all admissible controls $u \in \mathcal{U}_{\rm ad}$, we can recover deformations of the state in all the directions of $\mathbb{R}^N$. But,

> Which are the directions in which the state x can be deformed starting from x^0 ?

Obviously, the state can be deformed in all directions $f(x_0, u)$ with $u \in \mathcal{U}_{\rm ad}$. But these are not all the directions of $\mathbb{R}^N$ when $M < N$. On the other hand, as we have seen in the linear case, there exist situations in which $M < N$ and, at the same time, controllability holds thanks to the rank condition (59).

In the nonlinear framework, the directions in which the state may be deformed around x^0 are actually those belonging to the *Lie algebra* generated by the vector fields $f(x^0, u)$, when u varies in the set of admissible controls $\mathcal{U}_{\rm ad}$. Recall that the Lie algebra $\mathcal{A}$ generated by a family $\mathcal{F}$ of regular vector fields is the set of *Lie brackets* $[f, g]$ with $f, g \in \mathcal{F}$, where

$$[f, g] = (\nabla g)f - (\nabla f)g$$

and all the fields that can be obtained iterating this process of computing Lie brackets.

The following result can be proved (see [46]):

Theorem 5 *Assume that, for each x^0, the Lie algebra generated by $f(x^0, u)$ with $u \in \mathcal{U}_{\rm ad}$ coincides with $\mathbb{R}^N$. Then* (56) *is controllable, i.e. it can be driven from any initial state to any final state in a sufficiently large time.*

The following simple model of driving a car provides a good example to apply these ideas.

Thus, let us consider a state with four components $x = (x_1\,, x_2\,, x_3\,, x_4)$ in which the first two, x_1 and x_2 , provide the coordinates of the center of the axis $x_2 = 0$ of the vehicle, the third one, $x_3 = \varphi$, is the counterclockwise angle of the car with respect to the half axis $x_1 > 0$ and the fourth one, $x_4 = \theta$, is the angle of the front wheels with respect to the axis of the car. For simplicity, we will assume that the distance from the front to the rear wheels is $\ell = 1$.

The front wheels are then parallel to the vector $(\cos(\theta + \varphi), \sin(\theta + \varphi))$, so that the instantaneous velocity of the center of the front axis is parallel to this vector. Accordingly,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = u_2(t)\begin{pmatrix} \cos(\theta+\varphi) \\ \sin(\theta+\varphi) \end{pmatrix}$$

for some scalar function $u_2 = u_2(t)$.

The center of the rear axis is the point $(x_1 - \cos\varphi,\ x_2 - \sin\varphi)$. The velocity of this point has to be parallel to the orientation of the rear wheels $(\cos\varphi,\ \sin\varphi)$, so that

$$(\sin\varphi)\,\frac{d}{dt}(x_1 - \cos\varphi) - (\cos\varphi)\,\frac{d}{dt}(x_2 - \sin\varphi) = 0.$$

In this way, we deduce that

$$\dot\varphi = u_2 \sin\theta.$$

On the other hand, we set

$$\dot\theta = u_1$$

and this reflects the fact that the velocity at which the angle of the wheels varies is the second variable that we can control. We obtain the following reversible system:

$$\dot x = u_1(t)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2(t)\begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{pmatrix}. \tag{61}$$

According to the previous analysis, in order to guarantee the controllability of (61), it is sufficient to check that the Lie algebra of the directions in which the control may be deformed coincides with $\mathbb{R}^4$ at each point.

With $(u_1, u_2) = (0, 1)$ and $(u_1, u_2) = (1, 0)$, we obtain the directions

$$\begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{62}$$

respectively. The corresponding Lie bracket provides the direction

$$\begin{pmatrix} -\sin(\varphi+\theta) \\ \cos(\varphi+\theta) \\ \cos\theta \\ 0 \end{pmatrix}, \tag{63}$$

whose Lie bracket with the first one in (62) provides the new direction

$$\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \\ 0 \end{pmatrix}. \tag{64}$$

Taking into account that the determinant of the matrix formed by the four column vectors in (62), (63) and (64) is identically equal to 1, we deduce that, at each point, the set of directions in which the state may be deformed is the whole $\mathbb{R}^4$.

Thus, system (61) is controllable.

It is an interesting exercise to think on how one uses in practice the four vectors (62) – (64) to park a car. The reader interested in getting more deeply into this subject may consult the book by E. Sontag [46].

The analysis of the controllability of systems governed by partial differential equations has been the objective of a very intensive research the last decades. However, the subject is older than that.

In 1978, D.L. Russell [42] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper, the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects concerning partial differential equations: multipliers, moment problems, nonharmonic Fourier series, etc. More recently, J.L. Lions introduced the so called *Hilbert Uniqueness Method* (H.U.M.; for instance, see [29],[30]) and this was the starting point of a fruitful period on the subject.

In this context, which is the usual for modelling problems from Continuum Mechanics, one needs to deal with infinite dimensional dynamical systems and this introduces a lot of nontrivial difficulties to the theory and raises many relevant and mathematically interesting questions. Furthermore, the solvability of the problem depends very much on the nature of the precise question under consideration and, in particular, the following features may play a crucial role: linearity or nonlinearity of the system, time reversibility, the structure of the set of admissible controls, etc.

For more details, the reader is referred to the books [21] and [27] and the survey papers [13], [54] and [55].

8. Control, complexity and numerical simulation

Real life systems are genuinely complex. *Internet*, the large quantity of components entering in the fabrication of a car or the decoding of human genoma are good examples of this fact.

The algebraic system (23) considered in Section 5 is of course academic but it suffices by itself to show that not all the components of the state are always sensitive to the chosen control. One can easily imagine how dramatic can the situation be when dealing with complex (industrial) systems. Indeed, determining whether a given controller allows to act on all the components of a system may be a very difficult task.

But complexity does not only arise for systems in Technology and Industry. It is also present in Nature. At this respect, it is worth recalling the following anecdote. In 1526, the Spanish King "Alfonso X El Sabio" got into the *Alcázar of Segovia* after a violent storm and exclaimed:

> "If God had consulted me when He was creating the world, I would have recommended a simpler system."

Recently we have learned about a great news, a historical achievement of Science: the complete decoding of human *genoma*. The genoma code is a good proof of the complexity which is intrinsic to life. And, however, one has not to forget that, although the decoding has been achieved, there will still be a lot to do before being able to use efficiently all this information for medical purposes.

Complexity is also closely related to numerical simulation. In practice, any efficient control strategy, in order to be implemented, has to go through numerical simulation. This requires discretizing the control system, which very often increases its already high complexity.

The recent advances produced in Informatics allow nowadays to use numerical simulation at any step of an industrial project: conception, development and qualification. This relative success of numerical methods in Engineering versus other traditional methods relies on the facts that the associated experimental costs are considerably lower and, also, that numerical simulation allows testing at the realistic scale, without the technical restrictions motivated by instrumentation.

This new scientific method, based on a combination of Mathematics and Informatics, is being seriously consolidated. Other Sciences are also closely involved in this melting, since many mathematical models stem from them: Mechanics, Physics, Chemistry, Biology, Economics, etc. Thus, we are now able to solve more sophisticated problems than before and the complexity of the systems we will be able to solve in the near future will keep increasing. Thanks in particular to parallelization techniques, the description and numerical simulation of complex systems in an acceptable time is more and more feasible.

However, this panorama leads to significant and challenging difficulties that we are now going to discuss.

The first one is that, in practice, the systems under consideration are in fact the coupling of several complex subsystems. Each of them has its own dynamics but the coupling may produce new and unexpected phenomena due to their interaction.

An example of this situation is found in the mathematical description of reactive fluids which are used, for instance, for the propulsion of spatial vehicles.

For these systems, one has to perform a modular analysis, separating and simulating numerically each single element and, then, assembling the results. But this is a major task and much has still to be done[2].

There are many relevant examples of complex systems for which coupling can be the origin of important difficulties. In the context of *Aerospatial Technology*, besides the combustion of reactive fluids, we find fluid-structure interactions which are extremely important when driving the craft, because of the vibrations originated by combustion. Other significant examples are weather prediction and Climatology, where the interactions of atmosphere, ocean, earth, etc. play a crucial role. A more detailed description of the present situation of research and perspectives at this respect can be found in the paper [1], by J. Achache and A. Bensoussan.

In our context, the following must be taken into account:

- Only complex systems are actually relevant from the viewpoint of applications.
- Furthermore, in order to solve a relevant problem, we must first identify the various subsystems and the way they interact.

Let us now indicate some of the mathematical techniques that have been recently developed (and to some extent re-visited) to deal with complexity and perform the appropriate decomposition of large systems that we have mentioned as a need:

- **The solution of linear systems.**

  When the linear system we have to solve presents a block-sparse structure, it is convenient to apply methods combining appropriately the *local* solution of the subsystems corresponding to the individual blocks. This is a frequent situation when dealing with finite difference or finite element discretizations of a differential system.

  The most usual way to proceed is to introduce *preconditioners*, determined by the solutions to the subsystems, each of them being computed with one processor and, then, to perform the global solution with parallelized iterative methods.

- **Multigrid methods.**

  These are very popular today. Assume we are considering a linear system originated by the discretization of a differential equation. The main idea of a multigrid method is to "separate" the low and the high frequencies of the solution in the computation procedure. Thus we compute approximations of the solution at different levels, for instance working alternately with a coarse and a fine grid and incorporating adequate coupling mechanisms.

[2] To understand the level of difficulty, it is sufficient to consider a hybrid parabolic-hyperbolic system and try to match the numerical methods obtained with a finite difference method in the hyperbolic component and a finite element method in the parabolic one.

The underlying reason is that any grid, even if it is very fine, is unable to capture sufficiently high frequency oscillations, just as an ordinary watch is unable to measure microseconds.

- **Domain decomposition methods.**

  Now, assume that (1) is a boundary value problem for a partial differential equation in the N-dimensional domain Ω. If Ω has a complex geometrical structure, it is very natural to decompose (1) in several similar systems written in simpler domains.

  This can be achieved with domain decomposition techniques. The main idea is to split $\overline{\Omega}$ in the form

  $$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \cdots \cup \overline{\Omega_m}\,, \tag{65}$$

  and to introduce then an iterative scheme based on computations on each Ω_i separately.

  Actually, this is not new. Some seminal ideas can be found in Volume II of the book [7] by R. Courant and D. Hilbert. Since then, there have been lots of works on domain decomposition methods applied to partial differential systems (see for instance [24]). However, the role of these methods in the solution of control problems has not still been analyzed completely.

- **Alternating direction methods.**

  Frequently, we have to consider models involving time-dependent partial differential equations in several space dimensions. After standard time discretization, one is led at each time step to a set of (stationary) partial differential problems whose solution, in many cases, is difficult to achieve.

  This is again connected to the need of decomposing complex systems in more simple subsystems. These ideas lead to the methods of alternating directions, of great use in this context. A complete analysis can be found in [51]. In the particular, but very illustrating context of the Navier-Stokes equations, these methods have been described for instance in [19] and [38].

  However, from the viewpoint of Control Theory, alternating direction methods have not been, up to now, sufficiently explored.

The interaction of the various components of a complex system is also a difficulty of major importance in control problems. As we mentioned above, for real life control problems, we have first to choose an appropriate model and then we have also to make a choice of the control property. But necessarily one ends up introducing numerical discretization algorithms to make all this computable. Essentially, we will have to be able to compute an accurate approximation of the control and this will be made only if we solve numerically a *discretized control problem.*

At this point, let us observe that, as mentioned in [53], some models obtained after discretization (for instance via the finite element method) are not only relevant regarded as approximations of the underlying continuous models but also by themselves, as genuine models of the real physical world[3].

Let us consider a simple example in order to illustrate some extra, somehow unexpected, difficulties that the discretization may bring to the control process.

Consider again the state equation (1). To fix ideas, we will assume that our control problem is as follows

To find $u \in \mathcal{U}_{\rm ad}$ such that

$$\Phi(u, y(u)) \leq \Phi(v, y(v)) \quad \forall v \in \mathcal{U}_{\rm ad}\,, \tag{66}$$

where $\Phi = \Phi(v, y)$ is a given function.

Then, we are led to the following crucial question:

What is an appropriate discretized control problem ?

There are at least two reasonable possible answers:

- **First approximation method.**

  We first discretize $\mathcal{U}_{\rm ad}$ and (1) and obtain $\mathcal{U}_{{\rm ad},h}$ and the new (discrete) state equation
  $$A_h(y_h) = f(v_h). \tag{67}$$

  Here, h stands for a small parameter that measures the *characteristic size* of the "numerical mesh". Later, we let $h \to 0$ to make the discrete problem converge to the continuous one. If $\mathcal{U}_{{\rm ad},h}$ and (67) are introduced the right way, we can expect to obtain a "discrete state" $y_h(v_h)$ for each "discrete admissible" control $v_h \in \mathcal{U}_{{\rm ad},h}$.

  Then, we search for an optimal control at the discrete level, i.e. a control $u_h \in \mathcal{U}_{{\rm ad},h}$ such that

  $$\Phi(u_h, y_h(u_h)) \leq \Phi(v_h, y_h(v_h)) \quad \forall v_h \in \mathcal{U}_{{\rm ad},h}\,. \tag{68}$$

  This corresponds to the following scheme:

  $$\text{MODEL} \longrightarrow \text{DISCRETIZATION} \longrightarrow \text{CONTROL}.$$

  Indeed, starting from the continuous control problem, we first discretize it and we then compute the control of the discretized model. This provides a first natural method for solving in practice the control problem.

[3]The reader can find in [53] details on how the finite element method was born around 1960. In this article it is also explained that, since its origins, finite elements have been viewed as a tool to build legitimate discrete models for the mechanical systems arising in Nature and Engineering, as well as a method to approximate partial differential systems.

- **Second approximation method.**

  However, we can also do as follows. We analyze the original control problem (1),(66) and we characterize the optimal solution and control in terms of an *optimality system.* We have already seen that, in practice, this is just to write the Euler or Euler-Lagrange equations associated to the minimization problem we are dealing with. We have already described how optimality systems can be found for some particular control problems.

  The optimality systems are of the form

  $$A(y) = f(u), \quad B(y)p = g(u, y) \tag{69}$$

  (where $B(y)$ is a linear operator), together with an additional equation relating u, y and p. To simplify our exposition, let us assume that the latter can be written in the form

  $$Q(u, y, p) = 0 \tag{70}$$

  for some mapping Q. The key point is that, if u, y and p solve the optimality system (69) − (70), then u is an optimal control and y is the associate state. Of course, p is the *adjoint state* associated to u and y.

  Then, we can discretize and solve numerically (69),(70). This corresponds to a different approach:

  $$\text{MODEL} \longrightarrow \text{CONTROL} \longrightarrow \text{DISCRETIZATION.}$$

  Notice that, in this second approach, we have interchanged the control and discretization steps. Now, we first analyze the continuous control problem and, only later, we proceed to the numerical discretization.

It is not always true that these two methods provide the same results.

For example, it is shown in [18] that, with a finite element approximation, the first one may give erroneous results in vibration problems. This is connected to the lack of accuracy of finite elements in the computation of high frequency solutions to the wave equation, see [53][4].

On the other hand, it has been observed that, for the solution of a lot of *optimal design problems*, the first strategy is preferable; see for instance [34] and [37].

The commutativity of the DISCRETIZATION/CONTROL scheme is at present a subject that is not well understood and requires further investigation. We do not still have a significant set of results allowing to determine when these two approaches provide similar results and when they do not. Certainly, the answer depends heavily on the nature of the model under consideration.

[4]Nevertheless, the disagreement of these two methods may be relevant not only as a purely numerical phenomenon but also at the level of modelling since, as we said above, in many engineering applications discrete models are often directly chosen.

In this sense, control problems for elliptic and parabolic partial differential equations, because of their intrinsic dissipative feature, will be better behaved than hyperbolic systems. We refer the interested reader to [56] for a complete account of this fact. It is however expected that much progress will be made in this context in the near future.

9. Two challenging applications

In this Section, we will mention two control problems whose solution will probably play an important role in the context of applications in the near future.

9.1. Molecular control via laser technology

We have already said that there are many technological contexts where Control Theory plays a crucial role. One of them, which has had a very recent development and announces very promising perspectives, is the *laser control of chemical reactions*.

The basic principles used for the control of industrial processes in Chemistry have traditionally been the same for many years. Essentially, the strategies have been (a) to introduce changes in the temperature or pressure in the reactions and (b) to use *catalyzers*.

Laser technology, developed in the last four decades, is now playing an increasingly important role in molecular design. Indeed, the basic principles in *Quantum Mechanics* rely on the wave nature of both light and matter. Accordingly, it is reasonable to believe that the use of laser will be an efficient mechanism for the control of chemical reactions.

The experimental results we have at our disposal at present allow us to expect that this approach will reach high levels of precision in the near future. However, there are still many important technical difficulties to overcome.

For instance, one of the greatest drawbacks is found when the molecules are "not very isolated". In this case, collisions make it difficult to define their phases and, as a consequence, it is very hard to choose an appropriate choice of the control. A second limitation, of a much more technological nature, is related to the design of lasers with well defined phases, not too sensitive to the instabilities of instruments.

For more details on the modelling and technological aspects, the reader is referred to the expository paper [4] by P. Brumer and M. Shapiro.

The goal of this subsection is to provide a brief introduction to the mathematical problems one finds when addressing the control of chemical reactions.

Laser control is a subject of high interest where Mathematics are not sufficiently developed. The models needed to describe these phenomena lead to complex (nonlinear) *Schrödinger equations* for which the results we are able to deduce are really poor at present. Thus,

- We do not dispose at this moment of a complete theory for the corresponding initial or iniial/boundary value problems.
- Standard numerical methods are not sufficiently efficient and, accordingly, it is difficult to test the accuracy of the models that are by now available.

The control problems arising in this context are *bilinear*. This adds fundamental difficulties from a mathematical viewpoint and makes these problems extremely challenging. Indeed, we find here genuine nonlinear problems for which, apparently, the existing linear theory is insufficient to provide an answer in a first approach.

In fact, it suffices to analyze the most simple bilinear control problems where wave phenomena appear to understand the complexity of this topic. Thus, let us illustrate this situation with a model concerning the linear one-dimensional Schrödinger equation. It is clear that this is insufficient by itself to describe all the complex phenomena arising in molecular control via laser technology. But it suffices to present the main mathematical problem and difficulties arising in this context.

The system is the following:

$$\begin{cases} i\phi_t + \phi_{xx} + p(t)x\,\phi = 0 & 0 < x < 1, \quad 0 < t < T, \\ \phi(0,t) = \phi(1,t) = 0, & 0 < t < T, \\ \phi(x,0) = \phi^0(x), & 0 < x < 1. \end{cases} \tag{71}$$

In (71), $\phi = \phi(x,t)$ is the *state* and $p = p(t)$ is the control. Although ϕ is complex-valued, $p(t)$ is real for all t. The control p can be interpreted as the intensity of an applied electrical field and x is the (prescribed) direction of the laser.

The state $\phi = \phi(x,t)$ is the wave function of the molecular system. It can be regarded as a function that furnishes information on the location of an elementary particle: for arbitrary a and b with $0 \leq a < b \leq 1$, the quantity

$$P(a,b;t) = \int_a^b |\phi(x,t)|^2 \, dx$$

can be viewed as the probability that the particle is located in (a,b) at time t.

The controllability problem for (71) is to find the set of attainable states $\phi(\cdot,T)$ at a final time T as p runs over the whole space $L^2(0,T)$.

It is worth mentioning that, contrarily to what happens to many other control problems, the set of attainable states at time T depends strongly on the initial data ϕ^0. In particular, when $\phi^0 = 0$ the unique solution of (71) is $\phi \equiv 0$ whatever p is and, therefore, the unique attainable state is $\phi(\cdot,T) \equiv 0$. It is thus clear that, if we want to consider a nontrivial situation, we must suppose that $\phi^0 \neq 0$.

We say that this is a *bilinear control problem*, since the unique nonlinearity in the model is the term $p(t)x\,\phi$, which is essentially the product of the control and the state. Although the nonlinearity might seem simple, this control problem becomes rather complex and out of the scope of the existing methods in the literature.

For an overview on the present state of the art of the control of systems governed by the Schrödinger equation, we refer to the survey article [57] and the references therein.

9.2. An environmental control problem

For those who live and work on the seaside or next to a river, the relevance of being able to predict drastic changes of weather or on the state of the sea is obvious. In particular, it is vital to predict whether flooding may arise, in order to be prepared in time.

Floodings are one of the most common environmental catastrophic events and cause regularly important damages in several regions of our planet. They are produced as the consequence of very complex interactions of tides, waves and storms. The varying wind and the fluctuations of the atmospherical pressure produced by a storm can be the origin of an elevation or descent of several meters of the sea level in a time period that can change from several hours to two or three days. The wind can cause waves of a period of 20 seconds and a wavelenght of 20 or 30 meters. The simultaneous combination of these two phenomena leads to a great risk of destruction and flooding.

The amplitude of the disaster depends frequently on the possible accumulation of factors or events with high tides. Indeed, when this exceptional elevation of water occurs during a high tide, the risk of flooding increases dangerously.

This problem is being considered increasingly as a priority by the authorities of many cities and countries. Indeed, the increase of temperature of the planet and the melting of polar ice are making these issues more and more relevant for an increasing population in all the continents.

For instance, it is well known that, since the Middle Age, regular floods in the Thames river cover important pieces of land in the city of London and cause tremendous damages to buildings and population.

When floods occur in the Thames river, the increase on the level of water can reach a height of 2 meters. On the other hand, the average level of water at the London bridge increases at a rate of about 75 centimeters per century due to melting of polar ice. Obviously, this makes the problem increasingly dangerous.

Before explaining how the British authorities have handled this problem, it is important to analyze the process that lead to these important floods.

It is rather complex. Indeed, low atmospheric pressures on the Canadian coast may produce an increase of about 30 centimeters in the average sea level in an area of about 1 600 square kilometers approximately. On the other hand, due to the north wind and ocean currents, this tremendous mass of water may move across the Atlantic Ocean at a velocity of about 80 to 90 kilometers per day to reach the coast of Great Britain. Occasionally, the north wind may even push this mass of water down along the coast of England to reach the

Thames Estuary. Then, this mass of water is sent back along the Thames and the conditions for a disaster arise.

In 1953, a tremendous flooding happened killing over 300 people while 64 000 hectares of land were covered by water. After that, the British Government decided to create a Committee to analyze the problem and the possibilities of building defense mechanisms. There was consensus on the Committee about the need of some defense mechanism but not about which one should be implemented. Finally, in 1970 the decision of building a barrier, the *Thames Barrier*, was taken.

Figure 13: The Thames Barrier.

Obviously, the main goal of the barrier is to close the river when a dangerous increase of water level is detected. The barrier was built during 8 years and 4 000 workers participated on that gigantic engineering programme. The barrier was finally opened in 1984. It consists of 10 enormous steel gates built over the basement of reinforced concrete structures and endowed with sophisticated mechanisms that allow normal traffic on the river when the barrier is open but that allows closing and cutting the traffic and the flux of water when needed. Since its opening, the barrier has been closed three times up to now.

Obviously, as for other many control mechanisms, it is a priority to close the barrier a minimal number of times. Every time the barrier is closed, important economic losses are produced due to the suppression of river traffic. Furthermore, once the barrier is closed, it has to remain closed at least for 8 hours until the water level stabilizes at both sides. On the other hand, the process of closing the barrier takes two hours and, therefore, it is not possible to wait and see at place the flood arriving but, rather, one has to take the decision of closing on the basis of *predictions*. Consequently, extremely efficient methods of prediction are needed.

At present, the predictions are made by means of mathematical models that combine or match two different subsystems: the first one concerns the tides around the British Islands and the second one deals with weather prediction. In this way, every hour, predictions are made 30 hours ahead on several selected points of the coast.

The numerical simulation and solution of this model is performed on the supercomputer of the British Meteorological Office and the results are transferred to the computer of the Thames Barrier. The data are then introduced in another model, at a bigger scale, including the North Sea, the Thames Estuary and the low part of the river where the effect of tides is important. The models that are being used at present reduce to systems of partial differential equations and are solved by finite difference methods. The results obtained this way are compared to the average predictions and, in view of this analysis, the authorities have the responsibility of taking the decision of

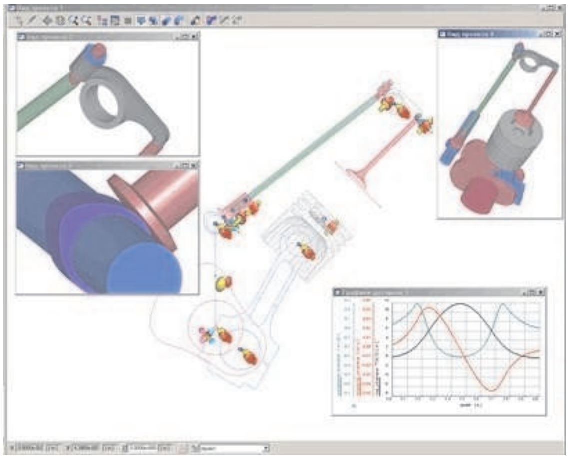

Figure 14: Design of a combustion controller valve.

closing the barrier or keeping it opened.

The Thames Barrier provides, at present, a satisfactory solution to the problem of flooding in the London area. But this is not a long term solution since, as we said above, the average water level increases of approximately 75 centimeters per century and, consequently, in the future, this method of prevention will not suffice anymore.

We have mentioned here the main task that the Thames Barrier carries out: the prevention of flooding. But it also serves of course to prevent the water level to go down beyond some limits that put in danger the traffic along the river.

The Thames Barrier is surely one of the greatest achievements of Control Theory in the context of the environmental protection. Here, the combination of mathematical modelling, numerical simulation and Engineering has allowed to provide a satisfactory solution to an environmental problem of first magnitude.

The reader interested in learning more about the Thames Barrier is referred to [14].

10. The future

At present, there are many branches of Science and Technology in which Control Theory plays a central role and faces fascinating challenges. In some cases, one expects to solve the problems by means of technological developments that will make possible to implement more sophisticated control mechanisms. To some extent, this is the case for instance of the laser control of chemical reactions we have discussed above. But, in many other areas, important theoretical developments will also be required to solve the complex control problems that arise. In this Section, we will briefly mention some of the fields in which these challenges are present. The reader interested in learning more about these topics is referred to the SIAM Report [44].

• **Large space structures -** Quite frequently, we learn about the difficulties found while deploying an antenna by a satellite, or on getting the precise orientation of a telescope. In some cases, this may cause huge losses and damages and may even be a reason to render the whole space mission useless. The importance of space structures is increasing rapidly, both for communications and research within our planet and also in the space adventure. These structures are built coupling several components, rigid and flexible ones. The problem of stabilizing these structures so that they remain oriented in the right direction without too large deformations is therefore complex and relevant. Designing robust control mechanisms for these structures is a challenging problem that requires important cooperative developments in Control Theory, computational issues and Engineering.

• **Robotics -** This is a branch of Technology of primary importance, where the scientific challenges are diverse and numerous. These include, for instance, computer vision. Control Theory is also at the heart of this area and its development relies to a large extent on robust computational algorithms for controlling. It is not hard to imagine how difficult it is to get a robot "walking" along a stable dynamics or catching an objet with its "hands" (see other related comments and Figures in Section 4).

• **Information and energy networks -** The globalization of our planet is an irreversible process. This is valid in an increasing number of human activities as air traffic, generation and distribution of energy, informatic networks, etc. The dimensions and complexity of the networks one has to manage are so large that, very often, one has to take decisions locally, without having a complete global information, but taking into account that local decisions will have global effects. Therefore, there is a tremendous need of developing methods and techniques for the control of large interconnected systems.

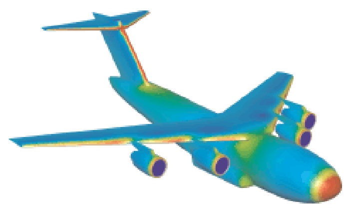

Figure 15: Numerical approximation of the pressure distribution on the surface of an aircraft.

• **Control of combustion -** This is an extremely important problem in Aerospatial and Aeronautical Industry. Indeed, the control of the instabilities that combustion produces is a great challenge. In the past, the emphasis has been put on design aspects, modifying the geometry of the system to interfere on the acoustic-combustion interaction or incorporating dissipative elements. The active control of combustion by means of thermal or acoustic mechanisms is also a subject in which almost everything is to be done.

Figure 16: An aerodynamic obstacle: a Delta Wing.

• **Control of fluids -** The interaction between Control Theory and *Fluid Mechanics* is also very rich nowadays. This is an important topic in *Aeronautics*, for instance, since the structural dynamics of a plane in flight interacts with the flux of the neighboring air. In conventional planes, this fact can be ignored but, for the new generations, it will have to be taken into account, to avoid turbulent flow around the wings.

From a mathematical point of view, almost everything remains to be done in what concerns modelling, computational and control issues. A crucial contribution was made by J.L. Lions in [31], where the approximate controllability of the Navier-Stokes equations was conjectured. For an overview of the main existing results, see [12].

• **Solidification processes and steel industry -** The increasingly important development in *Material Sciences* has produced intensive research in solidification processes. The form and the stability of the liquid-solid interface are central aspects of this field, since an irregular interface may produce undesired products. The sources of instabilities can be of different nature: convection, surface tension, ... The *Free Boundary Problems* area has experienced important developments in the near past, but very little has been done from a control theoretical viewpoint. There are very interesting problems like, for instance, *building interfaces* by various indirect measurements, or its control by means of heating mechanisms, or applying electric or magnetic currents or rotations of the alloy in the furnace. Essentially, there is no mathematical theory to address these problems.

• **Control of plasma -** In order to solve the energetic needs of our planet, one of the main projects is the obtention of fusion reactions under control. At present, *Tokomak machines* provide one of the most promising approaches to this problem. Plasma is confined in a Tokomak machine by means of electromagnetic fields. The main problem consists then in keeping the plasma at high density and temperature on a desired configuration along long time intervals despite its instabilities. This may be done placing *sensors* that provide the information one needs to modify the currents rapidly to compensate the perturbations in the plasma. Still today there is a lot to be done in this area.

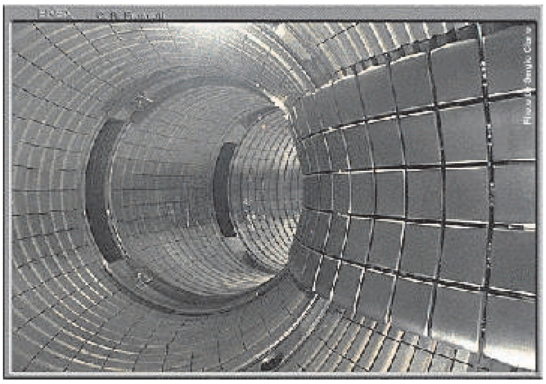

Figure 17: A Tokamak machine.

There are also important identification problems arising due to the difficulties to get precise measurements. Therefore, this is a field that provides many challenging topics in the areas of Control Theory and *Inverse Problems Theory*.

• **Biomedical research -** The design of medical therapies depends very strongly on the understanding of the dynamics of Physiology. This is a very active topic nowadays in which almost everything is still to be done from a mathematical viewpoint. Control Theory will also play an important role in this field. As an example, we can mention the design of mechanisms for insulin supply endowed with control chips.

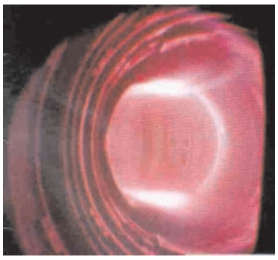

Figure 18: The plasma in a Tokamak machine.

• **Hydrology -** The problem of governing water resources is extremely relevant nowadays. Sometimes this is because there are little resources, some others because they are polluted and, in general, because of the complexity of the network of supply to all consumers (domestic, agricultural, industrial, ...). The control problems arising in this context are also of different nature. For instance, the *parameter identification problem*, in which the goal is to determine the location of sensors that provide sufficient information for an efficient extraction and supply and, on the other hand, the design of efficient management strategies.

• **Recovery of natural resources -** Important efforts are being made on the modelling and theoretical and numerical analysis in the area of simulation of reservoirs of water, oil, minerals, etc. One of the main goals is to optimize the extraction strategies. Again, inverse problems arise and, also, issues related to the control of the interface between the injected and the extracted fluid.

• **Economics -** The increasingly important role that Mathematics are playing in the world of *Economics* and *Finances* is well known. Indeed, nowadays, it is very frequent to use Mathematics to predict the fluctuations in financial markets. The models are frequently stochastic and the existing *Stochastic Control Theory* may be of great help to design optimal strategies of investment and consumption.

• **Manufacturing systems -** Large automatic manufacturing systems are designed as flexible systems that allow rapid changes of the production planning as a function of demand. But this increasing flexibility is obtained at the price of an increasing complexity. In this context, Control Theory faces also the need of designing efficient computerized control systems.

• **Evaluation of efficiency on computerized systems -** The existing software packages to evaluate the efficiency of computer systems are based on its representation by means of the *Theory of Networks*. The development of parallel and synchronized computer systems makes them insufficient. Thus, it is necessary to develop new models and, at this level, the Stochastic Control Theory of *discrete systems* may play an important role.

• **Control of computer aided systems -** As we mentioned above, the complexity of the control problems we are facing nowadays is extremely high. Therefore, it is impossible to design efficient control strategies without the aid of computers and this has to be taken into account when designing these strategies. This is a multidisciplinary research field concerned with Control Theory, Computer Sciences, Numerical Analysis and Optimization, among other areas.

Appendix 1: Pontryagin's maximum principle

As we said in Section 3, one of the main contributions to Control Theory in the sixties was made by L. Pontryagin by means of *the maximum principle*. In this Appendix, we shall briefly recall the main underlying ideas.

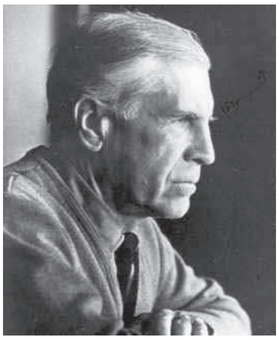

Figure 19: Lev S. Pontryagin (1908–1988).

In order to clarify the situation and show how powerful is this approach, we will consider a *minimal time control* problem. Thus, let us consider again the differential system

$$\begin{cases} \dot{x} = f(x,u), \quad t > 0, \\ x(0) = x^0, \end{cases} \tag{72}$$

with state $x = (x_1(t), \dots, x_N(t))$ and control $u = (u_1(t), \dots, u_M(t))$.

For simplicity, we will assume that the function $f : \mathbb{R}^N \times \mathbb{R}^M \mapsto \mathbb{R}^N$ is well defined and smooth, although this is not strictly necessary (actually, this is one of the main contributions of Pontryagin's principle). We will also assume that a nonempty closed set $G \subset \mathbb{R}^M$ is given and that the family of admissible controls is

$$\mathcal{U}_{\rm ad} = \{\, u \in L^2(0,+\infty;\mathbb{R}^M) : u(t) \in G \ \text{ a.e.}\,\}. \tag{73}$$

Let us introduce a manifold $\mathcal{M}$ of $\mathbb{R}^N$, with

$$\mathcal{M} = \{\, x \in \mathbb{R}^N : \, \mu(x) = 0 \,\},$$

where $\mu : \mathbb{R}^N \mapsto \mathbb{R}^q$ is a regular map ($q \le N$), so that the matrix $\nabla\mu(x)$ is of rank q at each point $x \in \mathcal{M}$ (thus, $\mathcal{M}$ is a smooth differential manifold of dimension $N - q$). Recall that the tangent space to $\mathcal{M}$ at a point $x \in \mathcal{M}$ is given by:

$$T_x\mathcal{M} = \{\, v \in \mathbb{R}^N : \nabla\mu(x) \cdot v = 0 \,\}.$$

Let us fix the initial state x^0 in $\mathbb{R}^N \setminus \mathcal{M}$. Then, to each control $u = u(t)$ we can associate a trajectory, defined by the solution of (72). Our minimal time control problem consists in finding a control in the admissible set $\mathcal{U}_{\rm ad}$ driving the corresponding trajectory to the manifold $\mathcal{M}$ in a time as short as possible.

In other words, we intend to minimize the quantity T subject to the following constraints:

- $T > 0$,
- For some $u \in \mathcal{U}_{\rm ad}$, the associated solution to (72) satisfies $x(T) \in \mathcal{M}$.

Obviously, the difficulty of the problem increases when the dimension of $\mathcal{M}$ decreases.

The following result holds (Pontryagin's maximum principle):

Theorem 6 *Assume that $\hat{T}$ is the minimal time and $\hat{u}$, defined for $t \in [0,\hat{T}]$, is an optimal control for this problem. Let $\hat{x}$ be the corresponding trajectory. Then there exists $\hat{p} = \hat{p}(t)$ such that the following identities hold almost everywhere in $[0,\hat{T}]$:*

$$\dot{\hat{x}} = f(\hat{x},\hat{u}), \quad -\dot{\hat{p}} = \left(\frac{\partial f}{\partial x}(\hat{x},\hat{u})\right)^t \cdot \hat{p} \tag{74}$$

and

$$H(\hat{x}(t),\hat{p}(t),\hat{u}) = \max_{v \in G} H(\hat{x}(t),\hat{p}(t),v), \tag{75}$$

where

$$H(x,p,v) = \langle f(x,v), p\rangle \quad \forall (x,p,v) \in \mathbb{R}^N \times \mathbb{R}^N \times G. \tag{76}$$

Furthermore, the quantity

$$H^*(\hat{x},\hat{p}) = \max_{v \in G} H(\hat{x},\hat{p},v) \tag{77}$$

is constant and nonnegative (maximum condition) and we have

$$\hat{x}(\hat{T}) = x^0, \quad \hat{x}(\hat{T}) \in \mathcal{M} \tag{78}$$

and

$$\hat{p}(\hat{T}) \perp T_{\hat{x}(\hat{T})}\mathcal{M} \tag{79}$$

(transversality condition).

The function H is referred to as the *Hamiltonian* of (72) and the solutions $(\hat{x}, \hat{p}, \hat{u})$ of the equations (74)–(79) are called *extremal points*. Of course, $\hat{p}$ is the extremal *adjoint state*.

Very frequently in practice, in order to compute the minimal time $\hat{T}$ and the optimal control $\hat{u}$, system (74)–(79) is used as follows. First, assuming that $\hat{x}$ and $\hat{p}$ are known, we determine $\hat{u}(t)$ for each t from (75). Then, with $\hat{u}$ being determined in terms of $\hat{x}$ and $\hat{p}$, we solve (74) with the initial and final conditions (78) and (79).

Observe that this is a well posed boundary-value problem for the couple $(\hat{x}, \hat{p})$ in the time interval $(0, \hat{T})$.

From (74), the initial and final conditions and (75), provide the control in terms of the state. Consequently, the maximum principle can be viewed as a feedback law for determining a good control $\hat{u}$.

In order to clarify the statement in Theorem 6, we will now present a heuristic proof.

We introduce the Hilbert space $X \times \mathcal{U}$, where $\mathcal{U} = L^2(0, +\infty; \mathbb{R}^M)$ and X is the space of functions $x = x(t)$ satisfying $x \in L^2(0, +\infty; \mathbb{R}^N)$ and $\dot{x} \in L^2(0, +\infty; \mathbb{R}^N)$[5].

Let us consider the functional

$$F(T, x, u) = T \quad \forall (T, x, u) \in \mathbb{R} \times X \times \mathcal{U}.$$

Then, the problem under consideration is

$$\text{To minimize } F(T, x, u), \tag{80}$$

subject to the inequality constraint

$$T \geq 0, \tag{81}$$

the pointwise control constraints

$$u(t) \in G \quad \text{a.e. in } (0, T) \tag{82}$$

(that is to say $u \in \mathcal{U}_{\mathrm{ad}}$) and the equality constraints

$$\dot{x} - f(x, u) = 0 \quad \text{a.e. in } (0, T), \tag{83}$$

$$x(0) - x^0 = 0 \tag{84}$$

and

$$\mu(x(T)) = 0. \tag{85}$$

[5]This is the Sobolev space $H^1(0, +\infty; \mathbb{R}^N)$. More details can be found, for instance, in [3].

Let us assume that $(\hat{T}, \hat{x}, \hat{u})$ is a solution to this constrained extremal problem. One can then prove the existence of *Lagrange multipliers* $(\hat{p}, \hat{z}, \hat{w}) \in X \times \mathbb{R}^N \times \mathbb{R}^N$ such that $(\hat{T}, \hat{x}, \hat{u})$ is, together with $(\hat{p}, \hat{z}, \hat{w})$, a saddle point of the *Lagrangian*

$$\mathcal{L}(T, x, u; p, z, w) = T + \int_0^T \langle p, \dot{x} - f(x, u)\rangle \, dt + \langle z, x(0) - x^0\rangle + \langle w, \mu(x(T))\rangle$$

in $\mathbb{R}_+ \times X \times \mathcal{U}_{\text{ad}} \times X \times \mathbb{R}^N \times \mathbb{R}^N$.

In other words, we have

$$\left\{ \begin{array}{l} \mathcal{L}(\hat{T}, \hat{x}, \hat{u}; p, z, w) \le \mathcal{L}(\hat{T}, \hat{x}, \hat{u}; \hat{p}, \hat{z}, \hat{w}) \le \mathcal{L}(T, x, u; \hat{p}, \hat{z}, \hat{w}) \\ \forall (T, x, u) \in \mathbb{R}_+ \times X \times \mathcal{U}_{\text{ad}}\,, \quad \forall (p, z, w) \in X \times \mathbb{R}^N \times \mathbb{R}^N. \end{array} \right. \tag{86}$$

The first inequalities in (86) indicate that the equality constraints $(83)-(85)$ are satisfied for $\hat{T}$, $\hat{x}$ and $\hat{u}$. Let us now see what is implied by the second inequalities in (86).

First, taking $T = \hat{T}$ and $x = \hat{x}$ and choosing u arbitrarily in $\mathcal{U}_{\text{ad}}$, we find that

$$\int_0^{\hat{T}} \langle \hat{p}, f(\hat{x}, u)\rangle \, dt \le \int_0^{\hat{T}} \langle \hat{p}, f(\hat{x}, \hat{u})\rangle \, dt \quad \forall u \in \mathcal{U}_{\text{ad}}\,.$$

It is not difficult to see that this is equivalent to (75), in view of the definition of $\mathcal{U}_{\text{ad}}$.

Secondly, taking $T = \hat{T}$ and $u = \hat{u}$, we see that

$$\int_0^{\hat{T}} \langle p, \dot{x} - f(x, \hat{u})\rangle \, dt + \langle z, x(0) - x^0\rangle + \langle w, \mu(x(\hat{T}))\rangle \ge 0 \quad \forall x \in X. \tag{87}$$

From (87) written for $x = \hat{x} \pm \varepsilon y$, taking into account that $(83) - (85)$ are satisfied for $\hat{T}$, $\hat{x}$ and $\hat{u}$, after passing to the limit as $\varepsilon \to 0$, we easily find that

$$\int_0^{\hat{T}} \langle \hat{p}, \dot{y} - \frac{\partial f}{\partial x}(\hat{x}, \hat{u}) \cdot y\rangle \, dt + \langle \hat{z}, y(0)\rangle + \langle \hat{w}, \nabla\mu(\hat{x}(\hat{T})) \cdot y(\hat{T})\rangle = 0 \quad \forall y \in X. \tag{88}$$

Taking $y \in X$ such that $y(0) = y(\hat{T}) = 0$, we can deduce at once the differential system satisfied by $\hat{p}$ in $(0, \hat{T})$. Indeed, after integration by parts, we have from (88) that

$$\int_0^{\hat{T}} \langle -\dot{\hat{p}} - \left(\frac{\partial f}{\partial x}(\hat{x}, \hat{u})\right)^t \cdot \hat{p}, y\rangle \, dt = 0$$

for all such y. This leads to the second differential system in (74).

Finally, let us fix λ in $\mathbb{R}^N$ an let us take in (88) a function $y \in X$ such that $y(0) = 0$ and $y(\hat{T}) = \lambda$. Integrating again by parts, in view of (74), we find that

$$\langle \hat{p}(\hat{T}), \lambda\rangle + \langle \hat{w}, \nabla\mu(\hat{x}(\hat{T})) \cdot \lambda\rangle = 0$$

and, since λ is arbitrary, this implies

$$\hat{p}(\hat{T}) = -\left(\nabla\mu(\hat{x}(\hat{T}))\right)^t \hat{w}.$$

This yields the transversality condition (79).

We have presented here the maximum principle for a minimal time control problem, but there are many variants and generalizations.

For instance, let the final time $T > 0$ and a non-empty closed convex set $S \subset \mathbb{R}^N$ be fixed and let $\mathcal{U}_{\rm ad}$ be now the family of controls $u \in L^2(0,T;\mathbb{R}^M)$ with values in the closed set $G \subset \mathbb{R}^M$ such that the associated states $x = x(t)$ satisfy

$$x(0) = x^0, \quad x(T) \in S. \tag{89}$$

Let $f^0 : \mathbb{R}^N \times \mathbb{R}^M \mapsto \mathbb{R}$ be a smooth bounded function and let us put

$$F(u) = \int_0^T f^0(x(t), u(t))\, dt \quad \forall u \in \mathcal{U}_{\rm ad}\,, \tag{90}$$

where x is the state associated to u through (72). In this case, the Hamiltonian H is given by

$$H(x,p,u) = \langle f(x,u), p\rangle + f^0(x,u) \quad \forall (x,p,u) \in \mathbb{R}^N \times \mathbb{R}^N \times G. \tag{91}$$

Then, if $\hat{u}$ minimizes F over $\mathcal{U}_{\rm ad}$ and $\hat{x}$ is the associated state, the maximum principle guarantees the existence of a function $\hat{p}$ such that the following system holds:

$$\dot{\hat{x}} = f(\hat{x},\hat{u}), \quad -\dot{\hat{p}} = \left(\frac{\partial f}{\partial x}(\hat{x},\hat{u})\right)^t \cdot \hat{p} + \frac{\partial f^0}{\partial x}(\hat{x},\hat{u}) \quad \text{a.e. in (0,T)}, \tag{92}$$

$$H(\hat{x}(t),\hat{p}(t),\hat{u}) = \max_{v\in G} H(\hat{x}(t),\hat{p}(t),v), \tag{93}$$

$$\hat{x}(0) = x^0, \quad \hat{x}(T) \in S \tag{94}$$

and

$$\langle \hat{p}(T), y - \hat{x}(T)\rangle \geq 0 \quad \forall y \in S. \tag{95}$$

For general nonlinear systems, the optimality conditions that the Pontryagin maximum principle provides may be difficult to analyze. In fact, in many cases, these conditions do not yield a complete information of the optimal trajectories. Very often, this requires appropriate geometrical tools, as the Lie brackets mentioned in Section 4. The interested reader is referred to H. Sussmann [48] for a more careful discussion of these issues.

In this context, the work by J.A. Reeds and L.A. Shepp [39] is worth mentioning. This paper is devoted to analyze a dynamical system for a vehicle, similar to the one considered at the end of Section 4, but allowing both backwards and forwards motion. As an example of the complexity of the dynamics of this system, it is interesting to point out that an optimal trajectory consists of, at most, five pieces. Each piece is either a segment or an arc of circumference, so that the whole set of possible optimal trajectories may be classified in 48 three-parameters families. More recently, an exhaustive analysis carried out in [49] by means of geometric tools allowed the authors to reduce the number of families actually to 46.

The extension of the maximum principle to control problems for partial differential equations has also been the objective of intensive research. As usual, when extending this principle, technical conditions are required to take into account the intrinsic difficulties of the infinite dimensional character of the system. The interested reader is referred to the books by H.O. Fattorini [15] and X. Li and J. Yong [28].

Appendix 2: Dynamical programming

We have already said in Section 3 that the *dynamical programming principle*, introduced by R. Bellman in the sixties, is another historical contribution to Control Theory.

Figure 20: Richard Bellman (1920).

The main goal of this principle is the same as of Pontryagin's main result: to characterize the optimal control by means of a system that may be viewed as a feedback law.

Bellman's central idea was to do it through the *value function* (also called the Bellman function) and, more precisely, to benefit from the fact that this function satisfies a *Hamilton-Jacobi equation*.

In order to give an introduction to this theory, let us consider for each $t \in [0,T]$ the following differential problem:

$$\left\{ \begin{array}{l} \dot{x}(s) = f(x(s),\, u(s)), \quad s \in [t,T], \\ x(t) = x^0 \,. \end{array} \right. \tag{96}$$

Again, $x = x(s)$ plays the role of the state and takes values in $\mathbb{R}^N$ and $u = u(s)$ is the control and takes values in $\mathbb{R}^M$. The solution to (96) will be denoted by $x(\cdot\,; t, x^0)$.

We will assume that u can be any measurable function in $[0,T]$ with values in a compact set $G \subset \mathbb{R}^M$. The family of admissible controls will be denoted, as usual, by $\mathcal{U}_{\rm ad}$.

The final goal is to solve a control problem for the state equation in (96) in the whole interval $[0,T]$. But it will be also useful to consider (96) for each $t \in [0,T]$, with the "initial" data prescribed at time t.

Thus, for any $t \in [0,T]$, let us consider the problem of minimizing the cost $C(\cdot\,; t, x^0)$, with

$$C(u; t, x^0) = \int_t^T f^0(x(\tau; t, x^0), u(\tau))\, d\tau + f^1(T, x(T; t, x^0)) \quad \forall u \in \mathcal{U}_{\rm ad} \tag{97}$$

(the final goal is to minimize $C(\cdot;0,x^0)$ in the set of admissible controls $\mathcal{U}_{\rm ad}$). To simplify our exposition, we will assume that the functions f, f^0 and f^1 are regular and bounded with bounded derivatives.

The main idea in *dynamical programming* is to introduce and analyze the so called *value function* $V = V(x^0,t)$, where

$$V(x^0,t) = \inf_{u\in\mathcal{U}_{\rm ad}} C(u;t,x^0) \quad \forall x^0 \in \mathbb{R}^N, \quad \forall t \in [0,T]. \tag{98}$$

This function provides the minimal cost obtained when the system starts from x^0 at time t and evolves for $s \in [t,T]$. The main property of V is that it satisfies a *Hamilton-Jacobi* equation. This fact can be used to characterize and even compute the optimal control.

Before writing the Hamilton-Jacobi equation satisfied by V, it is convenient to state the following fundamental result:

Theorem 7 *The value function $V = V(x^0,t)$ satisfies the Bellman optimality principle, or dynamical programming principle. According to it, for any $x^0 \in \mathbb{R}^N$ and any $t \in [0,T]$, the following identity is satisfied:*

$$V(x^0,t) = \inf_{u\in\mathcal{U}_{\rm ad}} \left[V(x(s;t,x^0),s) + \int_t^s f^0(x(\tau;t,x^0),u(\tau))\,d\tau \right] \quad \forall s \in [t,T]. \tag{99}$$

In other words, the minimal cost that is produced starting from x^0 at time t coincides with the minimal cost generated starting from $x(s;t,x^0)$ at time s plus the "energy" lost during the time interval $[t,s]$. The underlying idea is that a control, to be optimal in the whole time interval $[0,T]$, has also to be optimal in every interval of the form $[t,T]$.

A consequence of (99) is the following:

Theorem 8 *The value function $V = V(x,t)$ is globally Lipschitz-continuous. Furthermore, it is the unique viscosity solution of the following Cauchy problem for the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} V_t + \inf\limits_{v\in G} \left\{ \langle f(x,v), \nabla V\rangle + f^0(x,v) \right\} = 0, & (x,t) \in \mathbb{R}^N \times (0,T), \\ V(x,T) = f^1(T,x), & x \in \mathbb{R}^N. \end{cases} \tag{100}$$

The equation in (100) is, indeed, a Hamilton-Jacobi equation, i.e. an equation of the form

$$V_t + H(x,\nabla V) = 0,$$

with Hamiltonian

$$H(x,p) = \inf_{v\in G} \left\{ \langle f(x,v), p\rangle + f^0(x,v) \right\} \tag{101}$$

(recall (91)).

The notion of *viscosity solution* of a Hamilton-Jacobi equation was introduced to compensate the absence of existence and uniqueness of classical solution, two phenomena that can be easily observed using the method of characteristics. Let us briefly recall it.

Assume that $H = H(x,p)$ is a continuous function (defined for $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$) and $g = g(x)$ is a continuous bounded function in $\mathbb{R}^N$. Consider the following initial-value problem:

$$\left\{ \begin{array}{ll} y_t + H(x, \nabla y) = 0, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ y(x,0) = g(x), & x \in \mathbb{R}^N. \end{array} \right. \tag{102}$$

Let $y = y(x,t)$ be bounded and continuous. It will be said that y is a viscosity solution of (102) if the following holds:

- For each $v \in C^\infty(\mathbb{R}^N \times (0,\infty))$, one has

  $$\left\{ \begin{array}{l} \text{If } y - v \text{ has a local maximum at } (x^0,t^0) \in \mathbb{R}^N \times (0,\infty), \text{ then} \\ \qquad v_t(x^0,t^0) + H(x^0, \nabla v(x^0,t^0)) \leq 0 \end{array} \right.$$

  and

  $$\left\{ \begin{array}{l} \text{If } y - v \text{ has a local minimum at } (x^0,t^0) \in \mathbb{R}^N \times (0,\infty), \text{ then} \\ \qquad v_t(x^0,t^0) + H(x^0, \nabla v(x^0,t^0)) \geq 0. \end{array} \right.$$

- $y(x,0) = g(x)$ for all $x \in \mathbb{R}^N$.

This definition is justified by the following fact. Assume that y is a classical solution to (102). It is then easy to see that, whenever $v \in C^\infty(\mathbb{R}^N \times (0,\infty))$ and $v_t(x^0,t^0) + H(x^0, \nabla v(x^0,t^0)) > 0$ (resp. < 0), the function $y - v$ cannot have a local maximum (resp. a local minimum) at (x^0,t^0). Consequently, a classical solution is a viscosity solution and the previous definition makes sense.

On the other hand, it can be checked that the solutions to (102) obtained by the *vanishing viscosity method* satisfy these conditions and, therefore, are viscosity solutions. The vanishing viscosity method consists in solving, for each $\varepsilon > 0$, the parabolic problem

$$\left\{ \begin{array}{ll} y_t + H(x, \nabla y) = \varepsilon \Delta y, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ y(x,0) = g(x), & x \in \mathbb{R}^N \end{array} \right. \tag{103}$$

and, then, passing to the limit as $\varepsilon \to 0^+$.

A very interesting feature of viscosity solutions is that the two properties entering in its definition suffice to prove uniqueness. The proof of this uniqueness result is inspired on the pioneering work by N. Kruzhkov [22] on entropy solutions for hyperbolic equations. The most relevant contributions to this subject are due to M. Crandall and P.L. Lions and L.C. Evans, see [6], [10].

But let us return to the dynamical programming principle (the fact that the value function V satisfies (99)) and let us see how can it be used.

One may proceed as follows. First, we solve (100) and obtain in this way the value function V. Then, we try to compute $\hat{u}(t)$ at each time t using the identities

$$f(\hat{x}(t), \hat{u}(t)) \cdot \nabla V(\hat{x}(t), t) + f^0(\hat{x}(t), \hat{u}(t)) = H(\hat{x}(t), \nabla V(\hat{x}(t),\, t)), \tag{104}$$

i.e. we look for the values $\hat{u}(t)$ such that the minimum of the Hamiltonian in (101) is achieved. In this way, we can expect to obtain a function $\hat{u} = \hat{u}(t)$ which is the optimal control.

Recall that, for each $\hat{u}$, the state $\hat{x}$ is obtained as the solution of

$$\left\{ \begin{array}{l} \dot{\hat{x}}(s) = f\left(\hat{x}(s), \hat{u}(s)\right), \quad s \in [0, \hat{T}], \\ \hat{x}(0) = x^0 \, . \end{array} \right. \tag{105}$$

Therefore, $\hat{x}$ is determined by $\hat{u}$ and (104) is an equation in which $\hat{u}(t)$ is in fact the sole unknown.

In this way, one gets indeed an optimal control $\hat{u}$ in feedback form that provides an optimal trajectory $\hat{x}$ (however, at the points where V is not smooth, important difficulties arise; for instance, see [16]).

If we compare the results obtained by means of the maximum principle and the dynamical programming principle, we see that, in both approaches, the main conclusions are the same. It could not be otherwise, since the objective was to characterize optimal controls and trajectories.

However, it is important to underline that the points of view are completely different. While Pontryagin's principle extends the notion of Lagrange multiplier, Bellman's principle provides a dynamical viewpoint in which the value function and its time evolution play a crucial role.

The reader interested in a simple but relatively complete introduction to Hamilton-Jacobi equations and dynamical programming can consult the book by L.C. Evans [11]. For a more complete analysis of these questions see for instance W. Fleming and M. Soner [16] and P.-L. Lions [32]. For an extension of these methods to partial differential equations, the reader is referred to the book by X. Li and J. Yong [28].

References

[1] J. Achache and A. Bensoussan, *Assimilation, simulation et prévision*, MATAPLI, **60** (1999), 25–34.

[2] S. Bennet, *A History of Control Engineering 1800–1930*, IEE Control Engineering Series 8, Peter Peregrinus Ltd., London 1979.

[3] H. Brézis, *Analyse Fonctionnelle*, Masson, Paris 1996.

[4] P. Brumer and M. Shapiro, *Laser control of chemical reactions*, Scientific American, 1995, p. 34–39.

[5] J. Céa, *Optimisation: Théorie et Algorithmes*, Dunod, Gauthiers-Villars, Paris 1971.

[6] M. Crandall and P.L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. 277 (1983), no. 1, 1–42.

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II, Interscience Publishers, New York 1962.

[8] R.C. Dorf, *Sistemas Modernos de Control*, Addison-Wesley Iberoamericana, Madrid 1989.

[9] I. Ekeland and R. Temam, *Analyse Convexe et Problèmes Variationnels*, Dunnod, Paris 1974.

[10] L.C. Evans, *On solving certain nonlinear partial differential equations by accretive operator methods*, Israel J. Math. 36 (1980), no. 3-4, 225–247.

[11] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Graduate Texts in Mathematics, **19**, 1998.

[12] E. Fernández-Cara, *On the approximate and null controllability of the Navier-Stokes equations*, SIAM Rev. 41 (1999), no. 2, p. 269–277.

[13] E. Fernández-Cara and S. Guerrero *Global Carleman inequalities for parabolic systems and applications to null controllability*, to appear.

[14] D.G. Farmer and M.J. Rycroft, *Computer Modelling in the Environmental Sciences*, Clarendon Press, Oxford 1991.

[15] H.O. Fattorini, *Infinite Dimensional Optimization and Control Theory*, Encyclopedia of Mathematics and its Applications **62**, Cambridge University Press, 1999.

[16] W. Fleming and M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York 1993.

[17] H.R. Hall, *Governors and Governing Mechanisms*, The Technical Publishing Co., 2nd ed., Manchester 1907.

[18] J.A. Infante and E. Zuazua, *Boundary observability for the space semi-discretizations of the* $1-d$ *wave equation*, M^2AN, **33** (2) (1999), 407–438.

[19] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems* (2nd Edition), Springer-Verlag, New York 1984.

[20] O. Kavian, *Introduction a la Théorie des Points Critiques*, Mathématiques & Applications, **13**, Paris 1993.

[21] V. Komornik, *Exact controllability and stabilization. The multiplier method*, Masson, John Wiley & Sons, Chichester 1994.

[22] N.S. Kruzhkov, *First-order quasilinear equations in several independent variables*, Mat. USSR-Sb., **10** (1970), 217–243.

[23] B. Larrouturou and P.-L. Lions, *Méthodes mathématiques pour les sciences de l'ingénieur: Optimisation et analyse numérique*, Publications de l'Ecole Polytechnique de Paris, 1996.

[24] P. Le Tallec, *Domain decomposition methods in computational mechanics*, Comp. Mechanics Advances, **1** (1994), 121–220.

[25] E.B. Lee and L. Markus, *Foundations of Optimal Control Theory*, The SIAM Series in Applied Mathematics, John Wiley & Sons, New York 1967.

[26] W.S. Levine, *Control System and Applications*, CRC Press, 2000.

[27] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, Vol. I and II. Encyclopedia of Mathematics and its Applications, 74–75, Cambridge University Press, Cambridge 2000.

[28] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, Boston 1995.

[29] J.L. Lions, *Exact controllability, stabilizability and perturbations for distributed systems*, SIAM Review, **30** (1988), 1–68.

[30] J.L. Lions, *Contrôlabilité Exacte, Stabilisation et Perturbations de Systèmes Distribués, Tomes 1 & 2*. Masson, RMA **8** & **9**, Paris 1988.

[31] J.L. Lions, *Remarques sur la controlâbilite approchée*, in "Spanish-French Conference on Distributed-Systems Control", Univ. Málaga, 1990, p. 77–87.

[32] P.-L. Lions, *Generalized Solutions of Hamilton-Jacobi Equations.* Research Notes in Mathematics, 69, Pitman, New York 1982.

[33] O. Mayr, *The origins of Feedback Control*, MIT Press, Cambridge, MA, 1970.

[34] J. Mohammadi and O. Pironneau, *Applied Shape Optimization for Fluids*, The Clarendon Press, Oxford University Press, New York 2001.

[35] P. Moin and Th. Bewley, *Feedback control of turbulence*, in "Mechanics USA 1994", A.S. Kobayashi ed., Appl. Mech. Rev., **47** (6) (1994), S3–S13.

[36] K. Ogata, *Ingeniería de Control Moderna*, Prentice Hall Hispanoamericana, Madrid 1998.

[37] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, New York 1984.

[38] A. Prohl, *Projection and Quasi-Compressibility Methods for Solving the Incompressible Navier-Stokes Equations*, Advances in Numerical Mathematics, B.G. Teubner, Stuttgart 1997.

[39] J.A. Reeds and L.A. Shepp, *Optimal paths for a car that goes both forwards and backwards*, Pacific J. Math., **145** (1990), 367–393.

[40] S.S. Rao, *Optimization. Theory and Applications*, 2nd edition, John Wiley & Sons, New York 1984.

[41] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton 1970.

[42] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions*, SIAM Review **20** (1978), 639–739.

[43] M. Salomone, *Los humanoides ya están aquí*, El País Semanal, 12338, June 18th, 2000.

[44] S.I.A.M., *Future Directions in Control Theory*, Report of the Panel of Future Directions in Control Theory, SIAM Report on Issues in Mathematical Sciences, Philadelphia, 1988.

[45] D.R. Smith, *Variational Methods in Optimization*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

[46] E.D. Sontag, *Mathematical Control Theory. Deterministic Finite Dimensional Systems* (2nd Edition), Texts in Applied Mathematics, **6**, Springer-Verlag, New York 1998.

[47] G. Strang, *Introduction to Applied Mathematics*, Wellesley-Cambridge Press, Massachusetts, 1986.

[48] H.J. Sussmann, *Résultats récents sur les courbes optimales*, Journée Annuelle SMF, June 2000, (`http://www.math.rutgers.edu/~sussmann`).

[49] H.J. Sussmann and G. Tang, *Shortest paths for the Reeds-Shepp car*, in "Rutgers Center for Systems and Control (SYCON)", Report 91-10, September 1991.

[50] R.H. Thurston, *A History of the Growth of the Steam-Engine* (Web document), Stevens Institute of Technology, Hoboken, N.J. `http://www.history.rochester.edu/steam/thurston/1878/`.

[51] R. Varga, *Matrix Iterative Analysis*, 2nd edition, Springer-Verlag, Berlin, 2000.

[52] R.M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York 1980.

[53] O.C. Zienkiewicz, *Achievements and some unsolved problems of the finite element method*, Int. J. Numer. Meth. Eng. **47** (2000), 9–28.

[54] E. Zuazua, *Some Problems and Results on the controllability of Partial Differential Equations*, in "Proceedings of the Second European Congress of Mathematics", Budapest, July 1996. Progress in Mathematics, **169**, 1998, Birkhäuser Verlag Basel/Switzerland, pp. 276–311.

[55] E. Zuazua, *Controllability of Partial Differential Equations and its Semi-Discrete Approximation*, Discrete and Continuous Dynamical Systems, **8** (2) (2002), 469-513.

[56] E. Zuazua, *Propagation, Observation, Control and Numerical Approximation of Waves*, Bol. Soc. Esp. Mat. Apl. No. 25 (2003), 55–126.

[57] E. Zuazua, *Remarks on the controllability of the Schrödinger equation*, in "Quantum Control: Mathematical and Numerical Challenges", A. Bandrauk, M.C. Delfour and C. Le Bris eds., CRM Proc. Lecture Notes Series **33**, AMS Publications, Providence, R.I. 2003, pp. 181–199.