Bishop operators and Diophantine approximation

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Rutgers Number Theory Seminar

December 3, 2019

1. Invariant subspaces
2. Davie theorem
3. Atzmon theorem
4. Results
5. Diophantine approx.
6. Proofs

Short abstract. One of the oldest unsolved problems in functional analysis is the invariant subspace problem. E. Bishop proposed a family of potential counterexamples depending on an irrational parameter and 20 years later A. M. Davie strongly contradicted Bishop's intuition proving that it only could be a counterexample for Liouville numbers. The purpose of this talk is to illustrate this noticeable interplay between number theory and functional analysis and to present a recent joint work with E. Gallardo, M. Monsalve and A. Ubis.

Warning. The emphasis here is in the *application* of number theory not in the strength of the number theoretical results proven with this purpose.

The invariant subspace problem

Does every bounded linear operator on a (complex, ∞-dim., separable) Hilbert space have a nontrivial closed invariant subspace?

These are natural hypotheses

complex

∞-dim.

separable

Recall: invariant means $T(M) \subset M$. We focus on closed subspaces M to avoid silly examples "erasing" the boundary of the full Hilbert space.

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complex $\longrightarrow$ Rotations on $\mathbb{R}^2$

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complex → Rotations on $\mathbb{R}^2$

∞-dim. → otherwise too easy (linear algebra)

separable

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Does every bounded linear operator on a (complex, ∞-dim., separable) Hilbert space have a nontrivial closed invariant subspace?

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- $\boxed{\text{complex}}$ → Rotations on $\mathbb{R}^2$
- $\boxed{\infty\text{-dim.}}$ → otherwise too easy (linear algebra)
- $\boxed{\text{separable}}$ → otherwise too easy (take the closure of span$\{T^n x_0\}$)

Recall: invariant means $T(M) \subset M$. We focus on closed subspaces M to avoid silly examples “erasing” the boundary of the full Hilbert space.

Every known Hilbert space satisfying the hypotheses has an invariant subspace. Should we bet that invariant subspaces always exist?

- **Yes?** Lomonosov (1973). True for any operator commuting with a compact operator.
- **No?** Enflo (1976-1987), Read (1984). There are counterexamples in Banach spaces. In fact there exists an operator on ℓ^1 without invariant subspaces (Read, 1985).

Gossiping about the invariant subspace problem:

- Lomonosov's proof of his theorem is astonishingly simple and elegant. His paper is only two pages long.
- It made obsolete the previous partial results on the problem. Even during some years it was unclear if Lomonosov's proof had fully solved it because any known operator satisfied the hypotheses.
- Enflo's paper is 101 pages long, took 7 years between submission and publication and 10 years since the initial announcement.
- Read's first paper is 64 pages long and it was published while Enflo's paper was in the refereeing process.

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The counterexamples are too complicated to give any insight. Let's ask for a third opinion

- **Maybe (not)?** Bishop (ca. 1950) suggested that the operator

$$f(t) \mapsto t\, f\big(\{t+\alpha\}\big)$$

on $L^2[0,1)$ with $\alpha \in \mathbb{R} - \mathbb{Q}$ and $\{\cdot\}$ = fractional part, could be a counterexample.

Bishop operators

Let us define the **Bishop operators** with a certain scaling

$$Tf(t) = et\, f\big(\{t+\alpha\}\big) \qquad e = 2.718182\dots$$

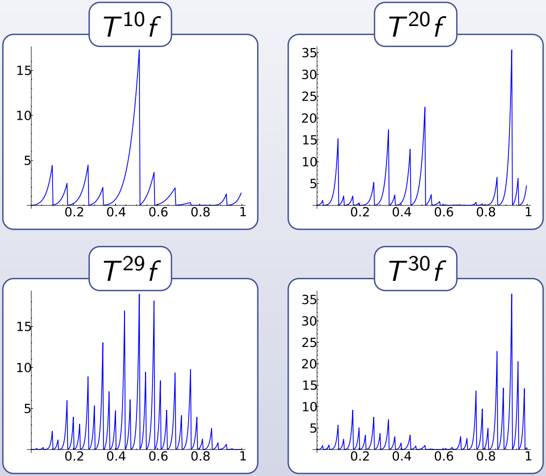
It is the composition of two well-known operators: a shift operator (with a continuous spectrum) and a multiplication operator (not invertible and no eigenfunctions). If $\alpha \in \mathbb{Q}$ the subspace of functions with small support around $k\alpha$ is invariant.

Possible rationale for it:

The equidistribution of $\{k\alpha\}$ for $\alpha \notin \mathbb{Q}$ mixes the support and the multiplication avoid eigenfunctions. If this implies that $\{T^k f\}$ gives everything we have solved the problem in the negative.

$$Tf(t) = et\, f\big(\{t+\alpha\}\big) \qquad e = 2.718182\ldots$$

$$f = \sin^2(\pi x), \qquad \alpha = \sqrt{2}$$



Davie theorem

More than 20 years later, A.M. Davie dealt Bishop's intuition a heavy blow.

Theorem (Davie, 1974). If α is not a Liouville number then T has nontrivial invariant subspaces.

Nevertheless many authors have continued the research on T and its variants. It seems that it is a natural operator to consider.

The folklore conjecture is the opposite of the original one: T has nontrivial invariant subspaces for every α.

Gossiping about Davie's paper:

- Halmos (1974) was rather tough in Mathematical reviews. "Several canons of both the pedantic grammarian and the mathematical expositor are violated. There is no hint to what motivates the proof [. . .] the techniques make use of nontrivial facts about subjects as diverse as diophantine approximations, Banach algebras, and quasi-analytic classes. The proofs appear, however, to be correct [. . .]"
- Halmos (1987) praises Davie and the result in his photo book.
- Flattot (2008) says that the proof is "very elliptic".

The proof is based on a method introduced by Wermer and extended by Atzmon. When it is applied, Davie had to confront some basic questions about Diophantine approximation.

Atzmon theorem (weak version)

Theorem (Atzmon, 1984). $L : H \longrightarrow H$, $f, g \in H - \{0\}$ with

$$\|L^{\pm n} f\|, \|(L^*)^{\pm n} g\| \leq W(n), \qquad W(n) = \exp\left(\frac{n}{(\log n)^{1+\epsilon}}\right)$$

$+$ technical condition (easy for Bishop's operators)

$\Rightarrow$ L has a nontrivial invariant subspace.

Sketch

Take $\phi(z) = \sum a_k z^k$, $\psi(z) = \sum b_k z^k$ with $\psi \cdot \phi = 0$ on S^1 and $a_k W(|k|), b_k W(|k|)$ small (possible by Denjoy-Carleman Theorem).

Define $u = \phi(L)f$, $v = \psi(L^*)g$ $\quad (\neq 0$ by the technical condition).
$M = \text{span}\{L^n u\}$ is invariant and it is nontrivial because $v \in M^\perp$:

$$\langle L^n u, v\rangle = \langle \phi(L)L^n f, \psi(L^*)g\rangle = \langle (\psi\phi)(L)L^n f, g\rangle = 0. \quad \square$$

Folklore conjecture. The Bishop operators $Tf(t) = et\, f(\{t+\alpha\})$ have nontrivial invariant subspaces for every $\alpha \notin \mathbb{Q}$.

Davie (1974). True for $\left|\alpha - \frac{a}{q}\right| > q^{-N}$, $\quad$ (N arbitrary).

.... (MacDonald, other authors) ....

Flattot (2008). True for $\left|\alpha - \frac{a}{q}\right| > \exp\big(-q^{1/3}\big)$, $\quad$ $\big(\frac{1}{3} \leftrightarrow \frac{1}{2} - \epsilon\big)$.

Our contribution (joint with Gallardo, Monsalve, Ubis 2018).

- Short proof of Flattot result.
- True for $\left|\alpha - \frac{a}{q}\right| > \exp\big(-q^{1-\epsilon}\big)$.
- For $\left|\alpha - \frac{a}{q}\right| \not> \exp\big(-\frac{Cq}{\log q}\big)$ $\nexists f, g$ to which Atzmon theorem can be applied (no possibility of improvement with the known methods).

Diophantine approximation

$$Tf(t) = et\, f\big(\{t+\alpha\}\big) \qquad e = 2.718182\ldots$$

A direct substitution proves

$$T^n f(t) = e^{L_n(t)} f\big(\{t+n\alpha\}\big), \quad T^{-n} f\big(\{t+n\alpha\}\big) = e^{-L_n(t)} f(t)$$

and similar formulas for T^* with

$$L_n(t) = \sum_{k=0}^{n-1} \big(1+\log\{t+k\alpha\}\big) \qquad \text{note} \int_0^1 \log = -1.$$

Idea (Davie). Choose the support of f in such a way that t is never close (mod. 1) to $m\alpha$, $m \in \mathbb{Z}$.

Bad guys. If α is very close to a rational, L_n is amplified.

Matching Flattot's result

Take $f = \mathbf{1}_{\mathcal{D}}$ with $\mathcal{D} = \left\{t : \langle t \pm n\alpha\rangle > \frac{1}{2019n^2},\ n \in \mathbb{Z}^+\right\}$ (Davie)

$a/q =$ convergent of α

$$L_n(t) = \sum_{k=0}^{n-1} \big(1 + \log\{t + k\alpha\}\big) \ll q + \frac{n+q}{q}\log(n+q).$$

Idea. Each q-block contributes $O(\log n + \log q)$.

Short proof of the result by Flattot

Atzmon theorem requires $L_n(t) \ll \frac{n}{(\log n)^{1+\epsilon}}$. It is achieved if $\frac{\log n}{q} \ll \frac{1}{(\log n)^{1+\epsilon}}$ for $n > q^{3/2}$.

$A/Q =$ next convergent, $q^{3/2} < n \le Q^{3/2} \rightsquigarrow \log Q \ll q^{1/2-\epsilon}$ and it is fulfilled for $\left|\alpha - \frac{a}{q}\right| > \exp(-q^{1/2-\epsilon})$. □

Going beyond

$t \in \mathcal{D} \Rightarrow \langle t \pm n\alpha \rangle > \epsilon_n \asymp n^{-2}$ and essentially

$$L_n(t) \ll q + nq^{-1} \log\left(\epsilon_n^{-1} + q\right) \quad \text{is best possible.}$$

Pessimistic view. Increase $\epsilon_n \Rightarrow |\mathcal{D}| = 0$ we ran out of points $\overset{(?)}{\Rightarrow}$ No chance of improvement.

Dreaming. With the nonsensical choice $\epsilon_n = 1$, Atzmon theorem imposes $\frac{n}{q} \log q \ll \frac{n}{(\log n)^{1+\epsilon}}$ and for $q^{3/2} < n \le Q^{3/2}$ gives $\log Q \ll q^{1-\epsilon}$ (Our result).

Hope. The dream only requires $\log(n+q) \to \log q$ to come true.

$$L_n(t) = \sum_{k=0}^{n-1} \left(1 + \log\{t + k\alpha\}\right) \qquad \frac{a}{q},\ \frac{A}{Q}\ \text{conv.}\ \alpha \notin \mathbb{Q}$$

$$\mathcal{D} = \left\{t \,:\, \langle t \pm n\alpha\rangle \gg \frac{1}{n^2},\ \forall n \in \mathbb{Z}^+\right\}$$

Dreamed bound. $L_n(t) \ll q + \frac{n}{q}\log q$ for $t \in \mathcal{D}$

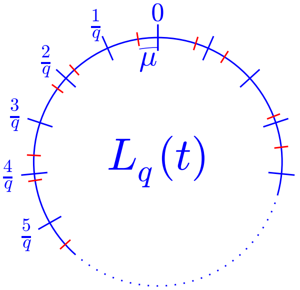
Proposition. For $Q \gg q^4$, $n \leq Q^{3/2}$, $t \in \mathcal{D}$

$$L_n(t) \ll q + \frac{n'}{q}\log q + \frac{n+Q}{Q}\log n$$

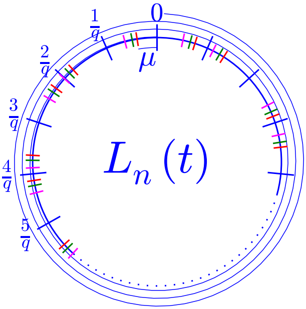
$n' =$ remainder when n is divided by Q.

Idea of the proof

$$L_n(t) = \sum_{k=0}^{n-1} \big(1 + \log\{t + k\alpha\}\big)$$



$$t \in \mathcal{D} \;\Rightarrow\; \mu \gg q^{-2} \;\Rightarrow\; \log q$$

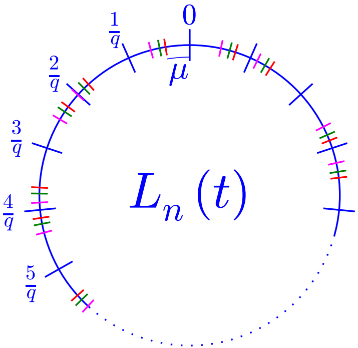


$$t \in \mathcal{D} \overset{q|n}{\Rightarrow} \mu \gg n^{-2} \Rightarrow \frac{n}{q} \log n$$

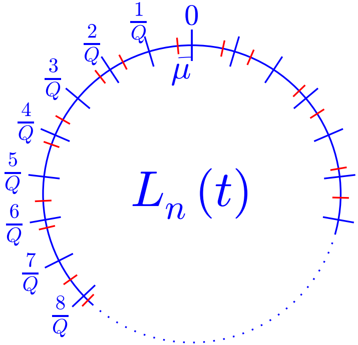
Target. Replace $\log n$ by $\log q$.

$\boxed{\text{Range to keep in mind } \log Q \ll q^{1-\epsilon}}$ $(q^{3/2} < n < Q^{3/2})$

Case $n \geq Q$ **[Use next convergent]**



$$\frac{n}{q}\log n$$



$$\frac{n}{Q}\log Q \ll \frac{n}{q}\log q \quad \checkmark$$

$\boxed{\text{Range to keep in mind } \log Q \ll q^{1-\epsilon}}$ $(q^{3/2} < n < Q^{3/2})$

Case $n \le Q$ **[Perturbation]** $\alpha = \frac{a}{q} + \frac{\delta}{qQ}, \quad n = qk + j$

$$\{t + n\alpha\} = \{t + j\alpha + \frac{k\delta}{Q}\}, \qquad t \in \mathcal{D} \Rightarrow \{t + j\alpha\} \gg \frac{1}{q^2}$$

If $n/qQ \ll q^{-2}$ then $k/Q \ll q^{-2} \to$ perturbed version of $L_q(t)$

$$L_n(t) \to \frac{n}{q}\widetilde{L}_q(t).$$

Then essentially

$$n \ll \frac{Q}{q} \quad \Rightarrow \quad L_n(t) \ll \frac{n}{q} \log q \quad \checkmark$$

It remains to consider $n \gg Q/q$

$\boxed{\text{Range to keep in mind } \log Q \ll q^{1-\epsilon}}$ $(q^{3/2} < n < Q^{3/2})$

Case $Q/q \ll n \le Q$ **[$j \neq 0 \to$ small scale]** $\alpha = \frac{a}{q} + \frac{\delta}{qQ}$, $n = qk + j$

$$L_n(t) \to \sum_{k=0}^{n/q-1} \sum_{j=0}^{q-1} \Big(1 + \log\big(\mu + \frac{j}{q} + \frac{k\delta}{Q}\big)\Big), \qquad \log \mu \ll \log n.$$

If $j \neq 0$, k/Q is in a smaller scale than $j/q \Rightarrow$ no interference. Essentially

$$L_n(t) - (\text{contrib. } j = 0) \ll \log n + \frac{n}{q} \log q \ll \frac{n}{q} \log q \quad \checkmark$$

Range to keep in mind $\log Q \ll q^{1-\epsilon}$ $(q^{3/2} < n < Q^{3/2})$

Case $Q/q \ll n \leq Q$ **[$j = 0 \to$ direct computation]** $\alpha = \frac{a}{q} + \frac{\delta}{qQ}$

The contribution of $j = 0$ is

$$\sum_{k=0}^{n/q-1} \Big(1 + \log\big(\mu + \frac{k\delta}{Q}\big)\Big)$$

with $\log \mu \ll \log n$. Then

$$\begin{aligned}
(\text{contrib. } j = 0) \quad &\ll \quad \frac{n}{q} + \log n + \sum_{k=1}^{n/q-1} \log \frac{qQ}{n} + \sum_{k=1}^{n/q-1} \log \frac{n}{qk} \\
&\ll \quad \frac{n}{q} + \log n + \quad (\text{trivial}) \quad + \quad (\text{Stirling}) \\
&\ll \quad \frac{n}{q} + \log n + \frac{n}{q}\log q + \frac{n}{q} \ll \frac{n}{q}\log q. \quad \checkmark
\end{aligned}$$

Sharpness

$$\boxed{\left|\alpha-\frac{a}{q}\right| \not\gg \exp\Big(-C\frac{q}{\log q}\Big) \ \Rightarrow\ \nexists f \text{ under Atzmon theorem.}}$$

The proof employs the result:

$$\frac{q^2\log q}{\epsilon^2} < n < \epsilon^2\frac{Q}{q} \ \Rightarrow\ L_n(t-n\alpha), -L_n(t) > \epsilon\frac{n}{q}\log q$$

for $t\in[0,1)$ except for a set S of measure $O(\epsilon)$.

$$\boxed{\|T^nf\|+\|T^{-n}f\| \gg \inf_{t\notin S}\Big(e^{L_n(t-n\alpha)}+e^{-L_n(t)}\Big)\|f\| \ \rightarrow\ \text{too large.}}$$

A copy of these slides is available in

https://www.uam.es/fernando.chamizo

Thank you for your attention!