Modular forms in a problem of spectral theory

Arithmétique en Plat Pays

(Journée printanière)

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Short abstract. The Weyl law is a remarkable formula in spectral theory that relates the asymptotic behavior of the frequencies of an oscillating manifold and its volume. A proof of a sharp version of the Weyl law for the Lie group $\mathrm{SO}(N)$ is presented using some properties of the classical modular forms and basic facts about some lattice point problems. No prerequisites are assumed about spectral theory or Lie groups.

Reference work. Chamizo, F.; Granados, J. A short proof of a sharp Weyl law for the special orthogonal group. *J. Spectr. Theory* 10 (2020), no. 1, 311–322.

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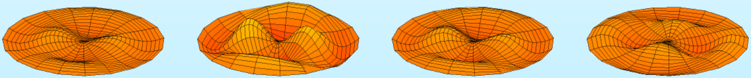
What is Weyl's law?

Physical formulation. When a membrane of area A oscillates, the frequencies $\nu_1, \nu_2, \ldots$ of the stationary waves satisfy

$$\frac{1}{N}\#\{n : \nu_n < N\} \sim \pi A \qquad \text{(for } N \text{ large).}$$



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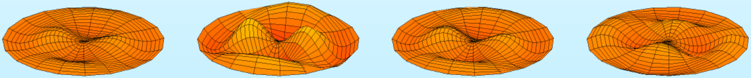

Old problem (M. Kac): So, we can "hear" the area of a drum. A refinement allows to hear the length of its boundary too. Can we hear its whole shape? Solved in the negative in 1992.

What is Weyl's law?

Mathematical formulation. $\Omega \subset \mathbb{R}^2$ smooth (except for corners), λ_n eigenvalues for $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$. Then

$$\mathcal{N}(\lambda) := \#\{n : \lambda_n < \lambda\} \sim \frac{|\Omega|}{4\pi}\lambda.$$

It generalizes to dimension d and manifolds as

$$\mathcal{N}(\lambda) \sim C_d \lambda^{d/2} \qquad \text{with} \quad C_d = \frac{2\,\mathrm{Vol}(M)}{d(4\pi)^{d/2}\Gamma(d/2)}.$$

It also extends to other elliptic operators.

Big problem: Prove good bounds for the error term.

Why is it true?

Recall linear algebra: If A is a (positive) self-adjoint on a subspace V

$$\lambda_n = \sup_{\vec{x} \in V - \{\vec{0}\}} \frac{\langle A\vec{x}, \vec{x} \rangle}{\|\vec{x}\|^2}$$

$V_- \subset V \subset V_+ \Rightarrow \lambda_n \downarrow$ in V_- and $\lambda_n \uparrow$ in V_+.

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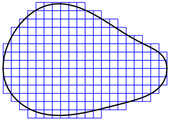
Assume $\Omega = \bigcup Q_j$, $Q_j =$ disjoint squares (ask Archimedes)

$$V = \{ u|_{\partial\Omega} = 0 \}$$



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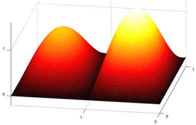

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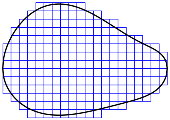
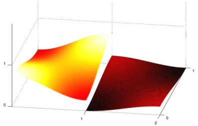


$V_+ = \{\frac{\partial u}{\partial n}|_{\partial Q_j} = 0\}$



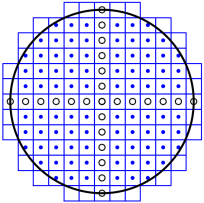
In $V_\pm$ the problem decouples. It is enough to study it on each tile.

Why is it true?

The whole problem reduces to prove Weyl's law for the unit square.

	$u = 0$ on ∂Q	$\partial u / \partial n = 0$ on ∂Q
eigenfunctions	$\sin(\pi m x) \sin(\pi n y)$	$\cos(\pi m x) \cos(\pi n y)$
eigenvalues	$\pi^2(m^2 + n^2)$ $\quad m, n > 0$	$\pi^2(m^2 + n^2)$ $\quad m, n \geq 0$

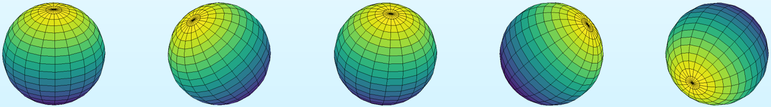
$$\mathcal{N}_Q(\lambda) \leftrightarrow \#\left\{(m, n) \in (\mathbb{Z}^+)^2 \text{ in the circle } x^2 + y^2 \leq \frac{\lambda}{\pi^2}\right\}$$



$$\mathcal{N}_Q(\lambda) \sim \frac{\lambda}{4\pi}.$$

Our result in words

We prove Weyl's law with error term for the manifold corresponding to the (Lie group of) rotations in $\mathbb{R}^N$.



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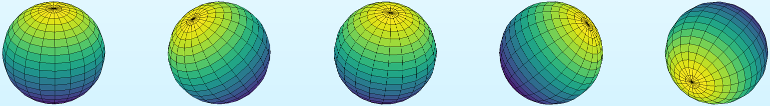

Two assets:

1. Except for $N = 4, 5, 6, 7$, the error term is sharp (read the fine print below).
2. Knowing the ingredients, the proof is very short.

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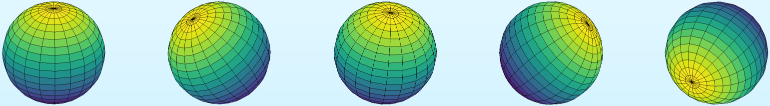

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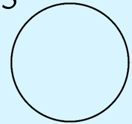
2. Knowing the ingredients, the proof is very short.

To say the whole truth, for $N = 8, 9$ the sharpness is in the exponent, a logarithmic factor could be improved. For $N > 9$ no improvement can be made.

Lie groups and eigenvalues

Essentially a Lie group is a manifold that is a group too i.e., we have a (differentiable) way of multiplying points.

S^1

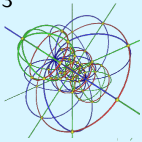


$p \mapsto z = e^{i\theta}$

S^2

(hairy ball theorem)

S^3



$p \mapsto$ quaternion $\|\mathbf{q}\| = 1$

The typical examples of compact Lie groups are the *classical groups*: $\mathrm{SO}(N)$ and allied groups.

$$\mathrm{SO}(N) \;\leftrightarrow\; \text{orthogonal } N \times N \text{ matrices} \;\leftrightarrow\; \text{rotations in } \mathbb{R}^N.$$

In fact, Lie groups are locally matrix groups.

Lie groups and eigenvalues

Good property of the Lie groups. Analytic and geometric problems are translated into linear algebra problems.

$$\text{Lie group} \longleftrightarrow \text{Lie algebra.}$$

Example: Lie group $SO(N) \to$ Lie algebra $\mathfrak{so}(N) =$ antisymmetric matrices. Straight line in $\mathfrak{so}(N) \to$ geodesic $\gamma(t) = e^{tA}$ in $SO(N)$.

Much more involved example: Eigenvalues of $-\Delta$ for $SO(N) \to$ properties related to the *root system*, eigenvalues of $X \mapsto AX - XA$ in $\mathfrak{so}(N)$.

Output: Explicit (but quite complicate) formulas for the eigenvalues of $-\Delta$ and their multiplicities for the classical groups.

Modular forms come on the scene

Recall: Modular form $f : \mathbb{H} \to \mathbb{C}$, $f(z) = \sum_{k=0}^{\infty} a_k e^{2\pi i k z}$ such that

$$f(\gamma(z)) = C_\gamma (cz+d)^r f(z) \qquad \text{for} \quad \gamma(z) = \frac{az+b}{cz+d}$$

with $\gamma \in \Gamma$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] < \infty$.

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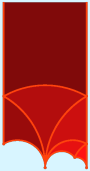
<u>Names</u>: r is called the *weight* and C_γ the *multiplier system*.

<u>Geometric interpretation for $r = 2$ and $C_\gamma = 1$</u>: $f(z)\, dz$ is a differential form on the Riemann surface $\Gamma \backslash \mathbb{H}$.

<u>Cusp forms</u>: They are the modular forms holomorphic at ∞.



If f is modular, $|\Im z|^{r/2} |f(z)|$ is invariant under Γ and it is uniformly bounded if f is a cusp form.

Modular forms come on the scene

Example. $\theta(z) = \sum_{k=-\infty}^{\infty} e^{2\pi i k^2 z}$ is a modular form of weight $1/2$,

$$\theta\Big(\frac{az+b}{cz+d}\Big) = C_\gamma (cz+d)^{1/2}\theta(z) \qquad \text{for} \quad 4 \mid c.$$

$C_\gamma \leftrightarrow$ Legendre symbol, (Poisson summation formula).
Raising to $d \to \theta^d(z) = \sum_{\vec{k}\in\mathbb{Z}^d} e^{2\pi i (k_1^2+\cdots+k_d^2)z}$ modular of weight $d/2$.

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Generalization. For $P = P(x_1, \ldots, x_d)$ homogeneous harmonic polynomial ($\sum \partial_j^2 P = 0$)

$$\Theta(z) = \sum_{\vec{k}\in\mathbb{Z}^d} P(\vec{k}) e^{2\pi i (k_1^2+\cdots+k_d^2)z}$$

is modular of weight $\frac{1}{2}d + \deg P$ and cusp form if $P \neq$ constant.

Summing up

1. Weyl's law is the asymptotic formula for the counting function of the eigenvalues and it involves the volume.
2. Our result is Weyl's law for $\mathrm{SO}(N)$ with optimal error term.
3. Lie groups are manifolds with a group structure. For $\mathrm{SO}(N)$ and the rest of the classical compact Lie groups it is possible to obtain an explicit formula for the eigenvalues.
4. Modular forms are holomorphic functions on the upper half plane given by Fourier series and satisfying a certain symmetry condition under a group of linear fractional transformations.
5. If we raise the Jacobi theta function to a power and we put a homogeneous harmonic polynomial as coefficient, we get a modular form.

Our result in formulas

Theorem. For the manifold $M = \mathrm{SO}(N)$

$$\mathcal{N}(\lambda) = C_d \lambda^{d/2} + \begin{cases} O(\lambda^{d/2-1}) & \text{if } n > 4 \\ O(\lambda^{d/2-1}\log\lambda) & \text{if } n = 4 \end{cases}$$

where $C_d =$ constant in Weyl's law, $d =$ dimension,
$n =$ *rank* as Lie group:

$$d = \frac{N(N-1)}{2} \qquad \text{and} \qquad n = \begin{cases} N/2 & \text{if } 2 \mid N \\ (N-1)/2 & \text{if } 2 \nmid N \end{cases}$$

Remark. The case $n = 1$ is trivial and $n = 2, 3$ are related to some old conjectures in number theory.

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Weyl's law as a lattice point problem

Formula for the eigenvalues and their multiplicities (N even):

$$\lambda_{\vec{b}} = \sum_{j=1}^{n} x_j^2 - K_n \qquad \text{and} \qquad m = \prod_{i=0}^{n-1} \prod_{j=i+1}^{n-1} \frac{(x_{n-j}^2 - x_{n-i}^2)^2}{(j^2 - i^2)^2}$$

where $n = \frac{1}{2}N$, $K_n = \frac{1}{6}n(n-1)(2n-1)$, $\vec{b} \in \mathbb{Z}^n$,

$$b_1 \geq b_2 \geq \cdots \geq |b_n| \qquad \text{and} \qquad x_j = b_j + n - j.$$

The formula is slightly more involved for N odd.

Weyl's law as a lattice point problem

Symmetry in b_j + elementary argument $\Rightarrow$

Lemma (relation to lattice point theory) For N even ($n = N/2$)

$$\mathcal{N}(\lambda) = \frac{1}{2^{n-1} n!} \sum_{\vec{x} \in \mathbb{Z}^n} m(\vec{x}) \chi_R(\vec{x})$$

with $\chi_R =$ charac. function of the ball of radius $R = \sqrt{\lambda + K_n}$.

And something similar when N is odd.

Regularization+Poisson summation (as in other LPP) $\rightarrow$ not so good idea
$\widehat{\chi}_R$ has large "wavefronts" when n grows (Morris & Taheri 2017)

The proof

Target: Estimate $S = \sum_{\vec{x}\in\mathbb{Z}^n} m(\vec{x})\chi_R(\vec{x})$, $\qquad m =$ hom. poly.

Fact from analysis: $Q =$ hom. poly., $\deg Q = g$ (even)

$$Q = \|\vec{x}\|^{2g} P_0 + \|\vec{x}\|^{2g-2} P_2 + \cdots + \|\vec{x}\|^2 P_{g-2} + P_g.$$

with $P_j =$ har. poly., $\deg P_j = j$.

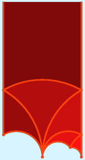
$$S = C \sum_{\substack{\vec{x}\in\mathbb{Z}^n \\ \|\vec{x}\|\le R}} \|\vec{x}\|^{2g} + O\Big(\sum_{\substack{\vec{x}\in\mathbb{Z}^n \\ \|\vec{x}\|\le R}} \|\vec{x}\|^{2g-2\ell} P_{2\ell}(\vec{x}) \Big) = C\mathcal{M} + O(\mathcal{E}).$$

Main: $\mathcal{M} = \sum_{k\le R^2} k^g r_n(k)$, $\qquad r_n = \#$repr. as sum of n squares
For $n > 4$ ($n = 4$), $r_n(k)$ very regular $\rightarrow$ sharp asymptotic formula.

The proof

Error: $\mathcal{E} = \sum_{k \le R^2} k^{g-\ell} a_k,$ $\qquad a_k = k$-th Fourier coeff. of Θ

$\Theta(z) = \sum_{\vec{k} \in \mathbb{Z}^d} P(\vec{k}) e^{2\pi i (k_1^2 + \cdots + k_d^2) z} = \sum_{k=1}^{\infty} a_k e^{2\pi i k z}$



Θ cusp form $\Rightarrow |\Im z|^{r/2} |\Theta(z)|$ unif. bounded

$|\Theta(x+iy)| \le C y^{-r/2}$ as $y \to 0^+$ $\qquad (r = d/2 + \deg P)$

$$\sum_{k=1}^{K} a_k = \int_0^1 \Theta(x+iy) \sum_{k=1}^{K} e^{-2\pi i k (x+iy)} dx$$ under control

Choosing $y = 1/K$, the error term $\mathcal{E}$ is proved to be negligible. $\square$

The low-dimensional cases

n	group	dim.	problem	bound for $\mathcal{E}$
1	SO(2), SO(3)	1, 3	trivial	$\lambda^{(d-1)/2}$ (sharp)
2	SO(4), SO(5)	6, 10	circle problem	$\lambda^{(d/2-1)+27/82}$
3	SO(6), SO(7)	15, 21	sphere problem	$\lambda^{(d/2-1)+1/4}\log\lambda$

Trivial case $N = 2$**:** $SO(2) \cong S^1$ (an angle α determines a plane rotation).
Eigenfunctions $\to e^{ik\alpha}$, , $\Delta = -\frac{d^2}{d\alpha^2}$, eigenvalues $\to k^2$

$$\mathcal{N}(\lambda) = \#\{k \in \mathbb{Z} \,:\, k^2 \le \lambda\} = 2\lambda^{1/2} + O(1).$$

These slides will be available in

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https://matematicas.uam.es/~fernando.chamizo

Thank you for your attention!

Je vous remercie de votre attention!

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