Fractal and multifractal Fourier series

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Hauteurs, Modularité et Transcendance

Credits

This talk is based mainly on the joint work with A. Ubis:

- Multifractal behavior of polynomial Fourier series. *Adv. Math.* 250 (2014), 1–34.

There is also a (unpublished) part coauthored with S. Ruiz-Cabello and included in his recent PhD thesis to be defended in two weeks:

- Prime generators, approximate identities and multifractal functions. PhD Thesis 2014.

Fractal and multifractal functions

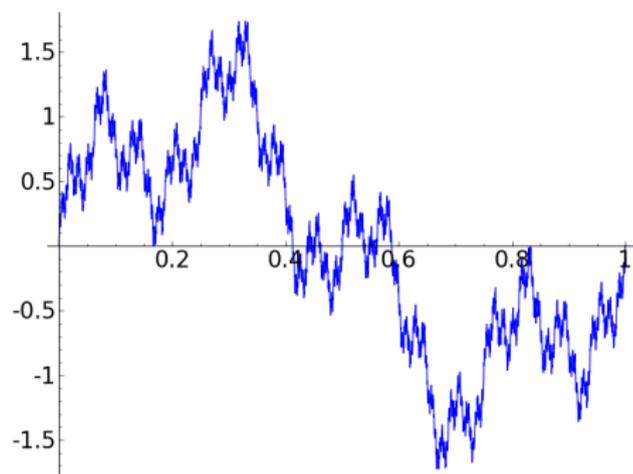A way of constructing fractals is to consider low regularity functions.
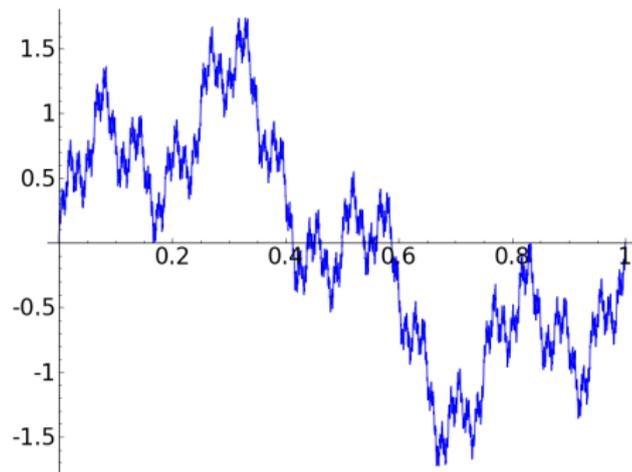
Weierstrass function

$$\sum_{n=0}^{\infty} \frac{\sin(2\pi 4^n x)}{2^n}$$

$$\beta_f(x) = \sup \left\{ \gamma \le 1 \ : \ |f(x+h) - f(x)| = O(|h|^\gamma) \right\}$$

[Definition to be modified at points of differentiability]

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$$\beta_f(x) = 1/2$$

A function with $\beta_f(x) = \beta$ (constant) gives a fractal graph of dimension $2 - \beta$.

For Weierstrass function one could suspect that $\beta_f(x) = 1/2$ because it has almost $1/2$ of derivative.

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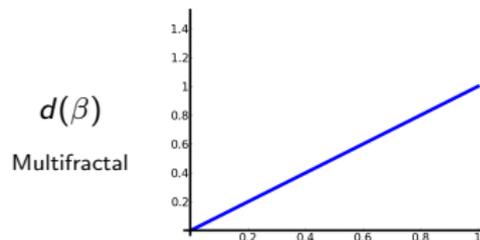
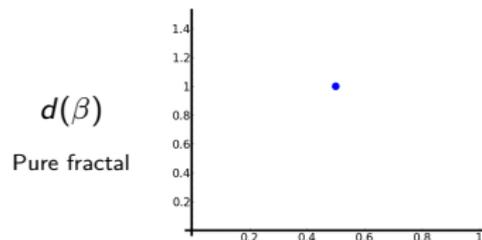### Spectrum of singularities

$$d(\beta) = \dim\{x \ : \ \beta_f(x) = \beta\}$$

where dim is the Hausdorff dimension ($\dim \emptyset = -\infty$ or undefined).

Multifractal function

It is a function with a non-discrete spectrum of singularities.

Why should I care about multifractals? Multifractal analysis is employed in many areas, ranging from fluid mechanics to medical imaging.

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In this talk the *simple function* is mainly a Fourier series with integral polynomial frequencies

$$F(x) = \sum_{n=1}^{\infty} \frac{e(P(n)x)}{n^{\alpha}}, \qquad P \in \mathbb{Z}[x] \quad \text{where } e(x) = e^{2\pi i x}.$$

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The question: What is the spectrum of singularities of F?

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The question: What is the spectrum of singularities of F?

Short answer: I don't know.

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The question: What is the spectrum of singularities of F?

Short answer: I don't know.

Long answer: The case $\deg P = 2$ is settled, there is a general lower bound, and also an upper bound proving that F is a multifractal function in certain ranges, there are some conjectural formulas and heuristics... [The rest of the talk].
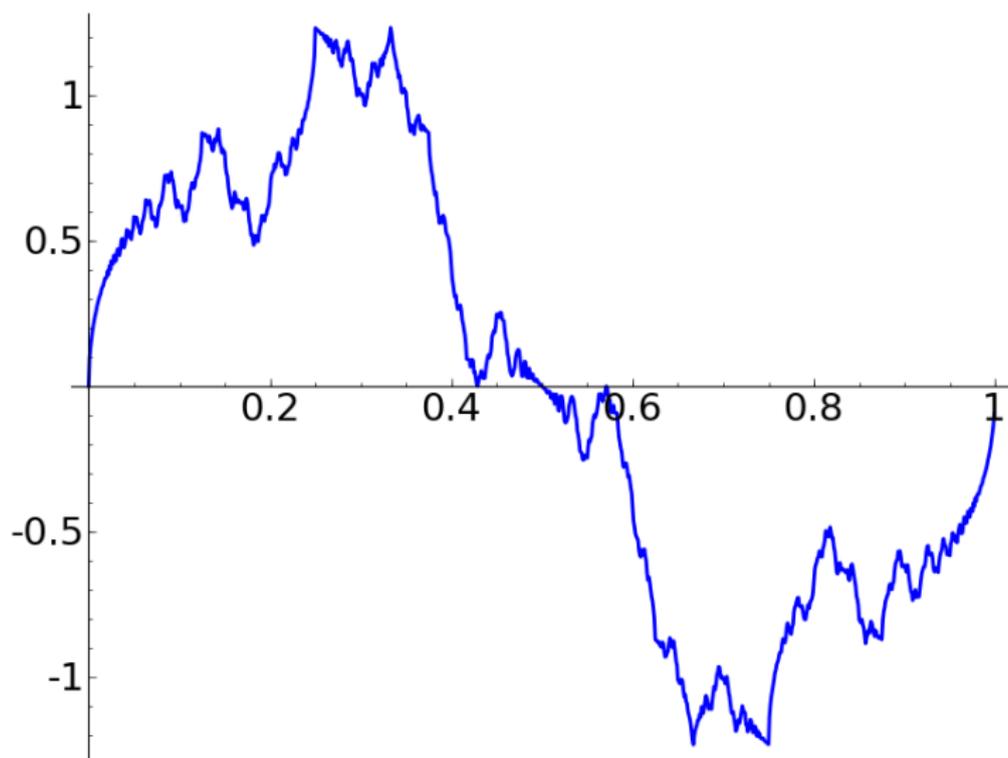
What is special in the quadratic case?

The simplest quadratic example is indeed old and famous.

Riemann's example

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2}$$

According to Weierstrass (1861), Riemann considered R to be an example of a continuous nowhere differentiable function.

Graph of the Riemann example

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2}$$

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Wrong proof, wrong answer

$R'(x) \stackrel{?}{=} 2\pi \sum \cos(2\pi n^2 x)$ does not converge for any $x \in \mathbb{R}$. For x irrational, $\cos(2\pi n^2 x)$ is dense in $[-1, 1]$. For $x \in \mathbb{Q}$ it oscillates.

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$$\theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z), \qquad \Im z > 0$$

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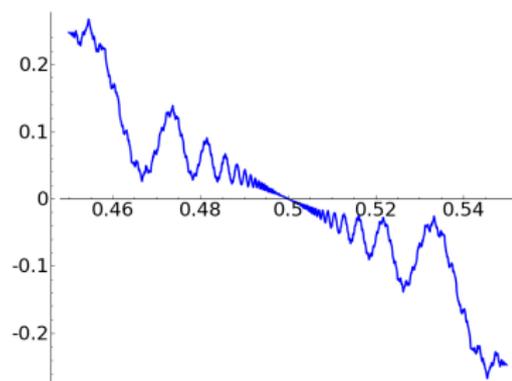
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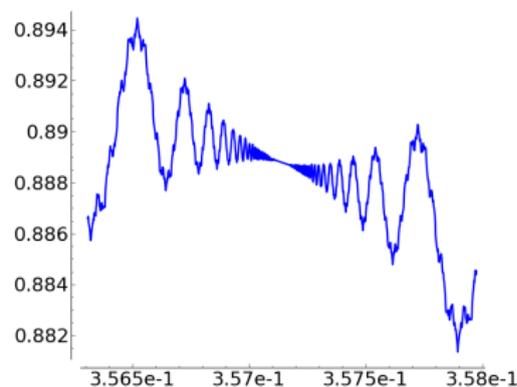
$z = x + iy$

$R'(x) = \left(\lim_{y \to 0^+} \Im \sum \frac{e(n^2 z)}{n^2} \right)'$

$R'(x) \overset{?}{=} \pi \lim_{y \to 0^+} \Re(\theta(z) - 1)$

$\exists \lim_{y \to 0^+} \Re\theta(z) \Leftrightarrow x = \frac{a}{2b}, \ 2 \nmid b$.

close to x = 1/2

close to x = 5/14

Some harmonic analysts dub as *chirps* this kind of oscillations resembling to the function $x^\alpha \sin x^{-\beta}$.
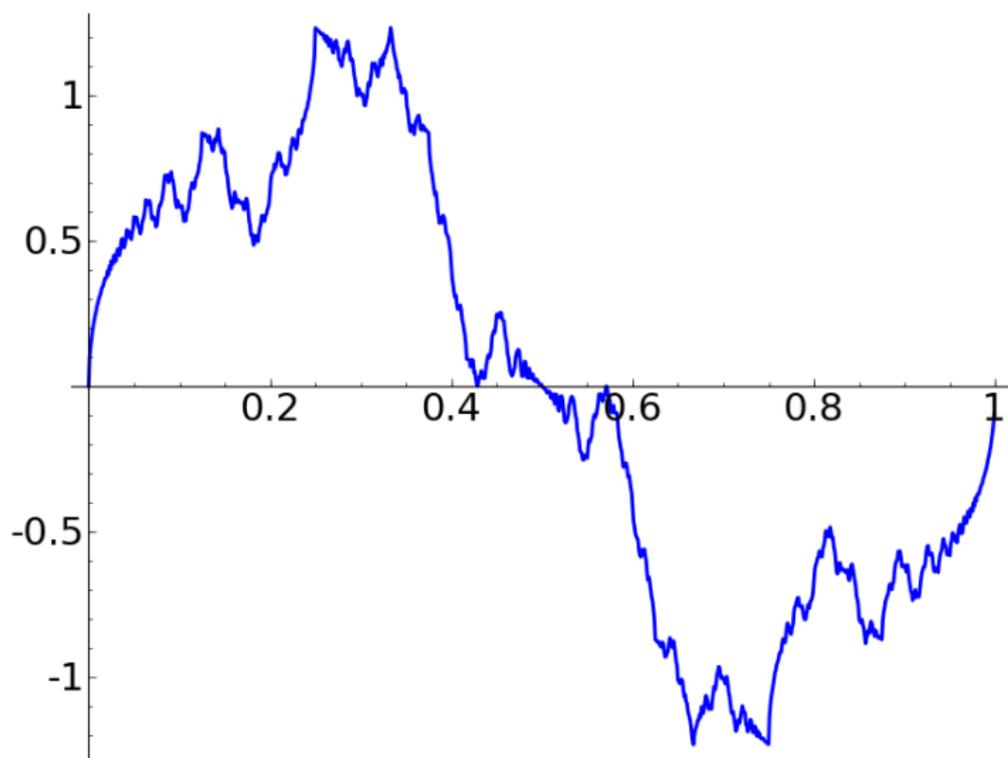
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Poisson's summation $\qquad \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}$

$\qquad\qquad\qquad\quad \updownarrow$

Modular relation $\qquad \theta(z) = \sqrt{\dfrac{i}{2z}}\, \theta\big(\dfrac{-1}{4z}\big)$

Analytic tools $\quad \rightarrow \quad R(h) - R(0) = Ch^{1/2} + O(h).$

Graph of the Riemann example

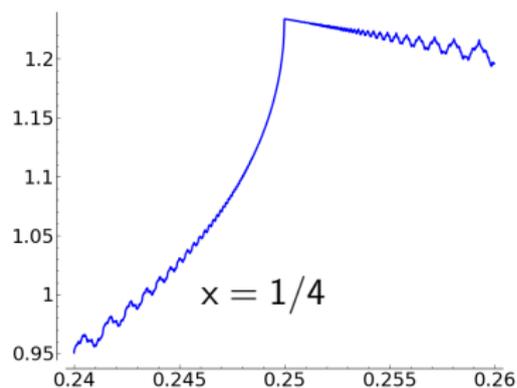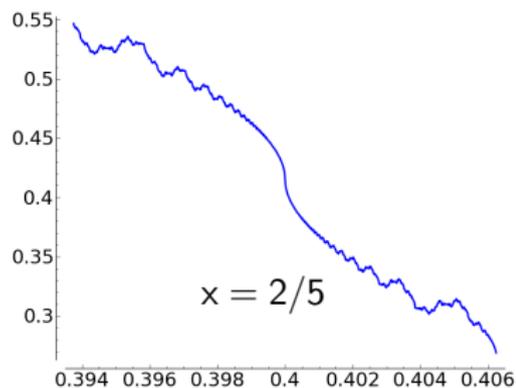$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n^2 x)}{n^2}$$

Expansion at the cusps (Poisson's summation in arithmetic progressions), for $h > 0$

$$R\left(\frac{a}{q} + h\right) - R\left(\frac{a}{q}\right) = q^{-1} G h^{1/2} + O\left(h q^{1/2}\right)$$

with G essentially a Gauss sum.

Diophantine approximation $\rightarrow$ characterization of the Hölder exponent in terms of the continued fraction.
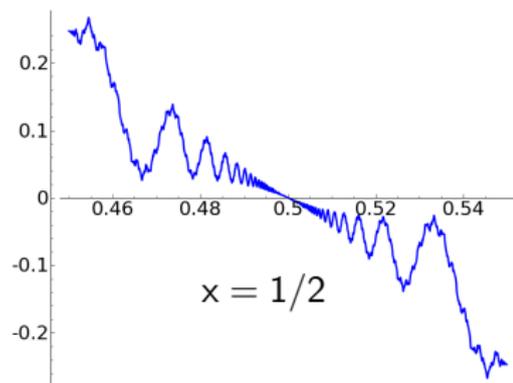
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In the rest of the cases, we have something like $q^{-1/2}h^{1/2}$. The typical approximation by rationals has $h \approx q^{-2}$. Then for the most of the points the expected Hölder exponent is $3/4$ and $d(3/4) = 1$.

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Quicker approximations by rationals, say $h \approx q^{-r}$, give $h^{1/2+1/2r}$. If r is large, this suggests $d(1/2) = 0$ and a continuous variation with r.

Jaffard (1996) $\rightarrow$ The function $R(x)$ is multifractal.

$$R_\alpha(x) = \sum_{n=1}^\infty \frac{\sin(2\pi n^2 x)}{n^\alpha}, \qquad \alpha > 1.$$
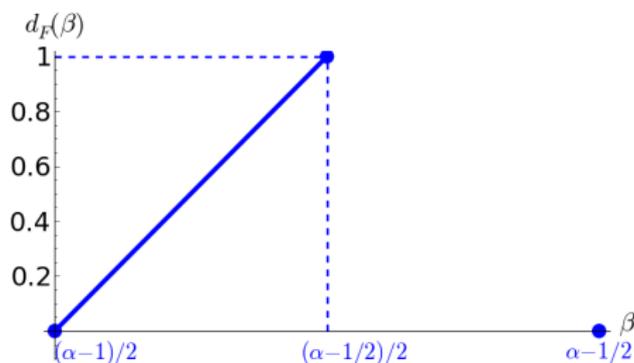
The general quadratic case is treated with other θ-functions.

The automorphic setting

J.-P. Serre, H.M. Stark (1977): θ-functions and modular forms of weight $1/2$ are the same thing.

quadratic case $\leftrightarrow$ weight $1/2$

Natural question: What happen with other weights?

$$f(z) = \sum a_n e(nz) \quad \text{automorphic}$$

$$\text{fractional integral} \quad \rightarrow \quad f_\alpha(x) = \sum \frac{a_n}{n^\alpha} e(nx).$$

Some results

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Petrykiewicz (2013) If f is a classical Eisenstein series then the Hölder exponent can be determined at every point in some ranges.

Ch. & Ruiz-Cabello (2014) If f is not a cusp form, f_α is a multifractal under strict conditions on α and the weight.

In the last result, we only treat the case with $\beta_f \leq 1$ and it forces the restrictions on the ranges. We plan to extend the result in a near future.
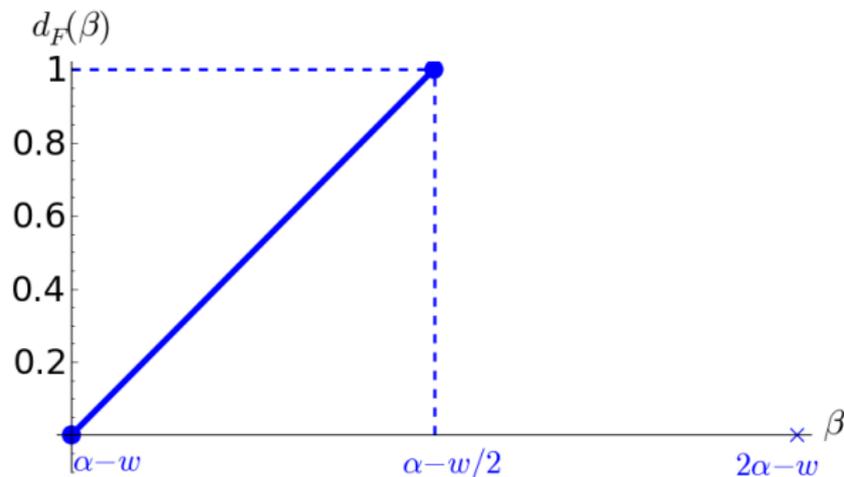
The spectrum of singularities of f_α

$w =$ weight of the modular form.

Why the case of higher degree is completely different?

In the case of degree k, Poisson summation allows to control the (asymptotics of the) oscillation of the function on each interval

$$\left(\frac{a}{q} - h, \frac{a}{q} + h\right), \qquad h < \frac{1}{q^k}.$$

If $k = 2$, they define a covering of $\mathbb{R}$. In general, the union of these intervals has Hausdorff dimension $2/k$. Very thin for $k > 2$.

A more important barrier to treat the case $k > 2$ is that the error term after using Poisson's summation, is better than trivial only when $h < q^{-k/2}$ while for the Diophantine approximation process we have to deal with h almost like q^{-1}.

What have we actually proved?

$$F(x) = \sum_{n=1}^{\infty} \frac{e(P(n)x)}{n^{\alpha}}.$$

Define $\nu_0 = \max(\nu_F, 2)$ with ν_F the maximal multiplicity of a zero of P'.
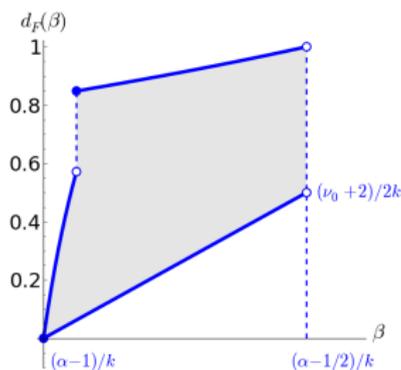
Theorem

For $1 + k/2 < \alpha < k$, $0 \le \beta < 1/2k$,

$$d\big(\beta + \frac{\alpha - 1}{k}\big) \ge (\nu_0 + 2)\beta.$$

$$F(x) = \sum_{n=1}^{\infty} \frac{e\big(P(n)x\big)}{n^{\alpha}}.$$

We have also an upper bound. We think that in the "typical case" the lower bound gives the true dimension.

Proved
F is multifractal
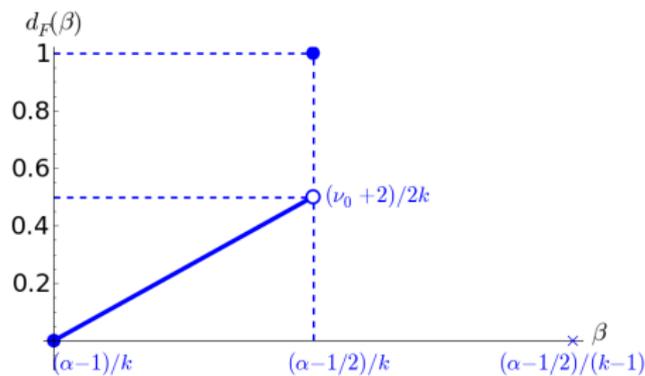
Conjectured in the "typical case"

What are the ideas under the proof?

The proof of the theorem is long and technical and involves several different ingredients (Poisson summation, variants of large sieve, Turán's method, Weyl's inequality, generalized Cantor sets. . .)

Instead of reviewing the steps and the role of these ingredients in the proof we try to motivate the result.

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The subsequent slides do <u>not</u> reflect the scheme of the actual proof but they give some hints about the underlying ideas. The point is

How could one imagine the result?

Assume $h > 0$ and write $k = \deg P$

$$F\left(\frac{a}{q} + h\right) - F\left(\frac{a}{q}\right) = \sum e\left(\frac{aP(n)}{q}\right) w_h(n), \quad w_h(n) = \frac{e\left(hP(n)\right) - 1}{n^\alpha}$$

w_h oscillates for $n > h^{-1/k}$, and it is small for n small. The bulk of the contribution comes from $n \asymp h^{-1/k}$ where $w_h(n) \asymp h^{\alpha/k}$.

Model

If $h = q^{-r}$, $r > k$, $F(a/q + h) - F(a/q)$ should behave like

$$h^{\alpha/k} \sum_{n \asymp q^{r/k}} e\left(\frac{aP(n)}{q}\right).$$

The analytic theory of exponential sums appeals very often to the heuristics of *square root cancellation*.

[The (unproved, out of reach) $(\epsilon, 1/2 + \epsilon)$ exponent pair]

It would solve the Gauss circle problem, the Lindelöf hypothesis for ζ, etc.

Rough idea in many contexts

A really oscillatory exponential sum S with N terms should verify $|S| \leq C_\epsilon N^{1/2+\epsilon}$, and typically $|S| \geq C'_\epsilon N^{1/2-\epsilon}$.

$$\text{Quadratic Gauss sum} \quad \sum_{n=1}^{q} e\left(\frac{n^2}{q}\right) = \frac{1 + (-i)^q}{1 - i}\sqrt{q}.$$

Model

If $h = q^{-r}$, $r > k$, $F(a/q + h) - F(a/q)$ should behave like

$$h^{\alpha/k} \sum_{n \asymp q^{r/k}} e\left(\frac{aP(n)}{q}\right).$$

Note that $e\left(aP(n)/q\right)$ is q-periodic. The square root cancellation heuristics applied to each q-block, leads to

$$h^{\alpha/k} q^{r/k-1} q^{1/2} = h^{(\alpha-1)/k+1/2r}$$

Jarník-Besicovitch theorem

$$\dim \left\{ x \; : \; \left| x - \frac{a}{q} \right| < q^{-r} \text{ infinitely often} \right\} = \frac{2}{r}$$

If $h = q^{-r}$, $r > k$, $F(a/q + h) - F(a/q)$ should behave like

$$h^{\alpha/k} \sum_{n \asymp q^{r/k}} e\left(\frac{aP(n)}{q}\right) \approx h^{(\alpha-1)/k+1/2r}.$$

Hölder exponent $\dfrac{\alpha - 1}{k} + \dfrac{1}{2r}$ in a set of $\dim = \dfrac{2}{r}$

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Hölder exponent $\dfrac{\alpha - 1}{k} + \dfrac{1}{2r}$ in a set of $\dim = \dfrac{2}{r}$

Writing $\beta = 1/2r$ we have the lower bound

$$d\left(\beta + \frac{\alpha - 1}{k}\right) \geq 4\beta$$

that is part of the theorem.

This bound can be improved if P has multiple zeros because in this case the square root cancellation is violated.

Model example: $q = p^k$, $p \nmid a$, $p \nmid k$

$$\sum_{n=1}^{q} e\left(\frac{an^k}{q}\right) = q^{1-1/k}$$

In this case

$$h^{\alpha/k} \sum_{n \asymp q^{r/k}} e\left(\frac{aP(n)}{q}\right) \approx h^{(\alpha-1)/k+1/kr}$$

In general, if $\nu > 1$ the maximal multiplicity of a zero of P', and $q = p^{\nu+1}$, the model suggests that if $h = q^{-r}$, $r > k$, $F(a/q + h) - F(a/q)$ should behave like

$$h^{\alpha/k} \sum_{n \asymp q^{r/k}} e\left(\frac{aP(n)}{q}\right) \gg h^{(\alpha-1)/k+1/(\nu+1)r}.$$

(Modified) Jarník-Besicovitch theorem

$$\dim\left\{x \, : \, \left|x - \frac{a}{q}\right| < q^{-r}, \; q = p^{\nu+1} \text{ inf. often}\right\} = \frac{1}{r} + \frac{1}{(\nu+1)r}.$$

Hölder exponent $\dfrac{\alpha-1}{k}+\dfrac{1}{(\nu+1)r}$ in a set of $\dim = \dfrac{1}{r}+\dfrac{1}{(\nu+1)r}$

Hölder exponent $\dfrac{\alpha - 1}{k} + \dfrac{1}{(\nu + 1)r}$ in a set of $\dim = \dfrac{1}{r} + \dfrac{1}{(\nu + 1)r}$

Writing $\beta = 1/(\nu + 1)r$ we have the lower bound

$$d\left(\beta + \frac{\alpha - 1}{k}\right) \geq (\nu + 2)\beta$$

that is the remaining part of the theorem.

What are the main steps in the actual proof?

1. Local analysis

<u>Tools</u>: Poisson's summation, Weyl's inequality.
<u>Output</u>: Approximation around rational values.

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<u>Output</u>: Some control of the oscillation averaging over thin sets.

3. Lower bounds for exponential sums

<u>Tools</u>: Turán's method, polynomials over finite fields.
<u>Output</u>: General lower bound for families of exponential sums.

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4. Some Cantor-like sets

<u>Tools</u>: Elementary measure theory.
<u>Output</u>: Dimension of general limit set of sequences of nested intervals.

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5. Construction of the fractal set

<u>Tools</u>: Jarník-Besicovitch, Diophantine approximation.

<u>Output</u>: A fractal set in which the Hölder exponent is constant.

What is the theorem that we have not proved?

- Some artificial examples with unbalanced multiplicities produce a serious obstruction to the square root cancellation philosophy and its modification. We have not a clear conjecture for the general case but it seems to be exact for most polynomials (including those with successive derivatives having simple zeros).

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- Our approach only can deal with irrationals very well approximated by rationals. We expect maximal cancellation in the rest of the cases.

- The behavior in the rationals depends on the non-vanishing of a finite exponential sum.

- Near to $\mathbb{Q}$: Cancellation in terms of multiplicities.
- Far away from $\mathbb{Q}$: Maximal cancellation.
- In $\mathbb{Q}$: Arithmetical properties.
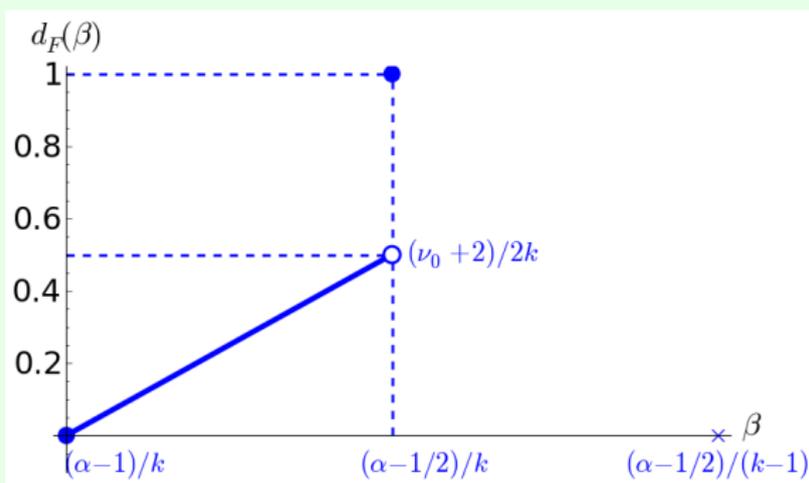
Conjecture for the "typical case"

Some related papers

- F. Chamizo and S. Ruiz-Cabello. Modular forms and multifractal Fourier series. In preparation.
- F. Chamizo and A. Ubis. Multifractal behavior of polynomial Fourier series. *Adv. Math.*, 250:1–34, 2014.
- F. Chamizo and A. Ubis. Some Fourier series with gaps. *J. Anal. Math.*, 101:179–197, 2007.
- F. Chamizo. Automorphic forms and differentiability properties. *Trans. Amer. Math. Soc.*, 356(5):1909–1935, 2004.
- F. Chamizo and A. Córdoba. Differentiability and dimension of some fractal Fourier series. *Adv. Math.*, 142(2):335–354, 1999.

Presentation available in

`http://www.uam.es/fernando.chamizo`