Ramanujan, Kronecker and a classical series evaluation

Fernando Chamizo (UAM-ICMAT)

Universidad Autónoma de Madrid

Universidad Carlos III de Madrid

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Abstract. We introduce a method to evaluate a series giving a special value of a theta function related to one of the most emblematic formulas due to Ramanujan and to the so-called Kronecker limit formula. The series evaluation is classical. The novelty of our approach is that the requirements barely exceed basic complex analysis, in particular no background about elliptic functions is needed.

F. Chamizo. A simple evaluation of a theta value, the Kronecker limit formula and a formula of Ramanujan. Ramanujan J., 59(3):947–954, 2022.

One of the most emblematic Ramanujan formulas

$$\left(\sum_{n=-\infty}^{\infty}\frac{\cos(\pi nx)}{\cosh(\pi n)}\right)^{-2}+\left(\sum_{n=-\infty}^{\infty}\frac{\cosh(\pi nx)}{\cosh(\pi n)}\right)^{-2}=K$$

where

$$K=\frac{\pi^3}{2}\left(\int_{-\infty}^{\infty}e^{-t^4}\,dt\right)^{-4}=1.435540\dots$$

It is symmetric and unexpected (Ramanujan's trademark)

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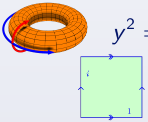
Proof? $\begin{cases}\text{Ramanujan's} \to \text{unknown}\\ \text{Natural} \to \text{elliptic functions}\\ \qquad\qquad\quad\ \text{complex multiplication}\end{cases}$



$y^2=x^3+x$

B. C. Berndt in its edition of the *Notebooks*: "One wonders how Ramanujan ever discovered this most unusual and beautiful formula".

Pos. def. $Q(x,y) = ax^2 + bxy + cy^2$, $D = 4ac - b^2$

$$\zeta(s,Q) = \sum_{\mathbf{n}\in\mathbb{Z}^2-\{\mathbf{0}\}} \frac{1}{Q(\mathbf{n})^s}, \qquad z_Q = \frac{-b + i\sqrt{D}}{2a}$$

Kronecker limit formula (non standard form)

$$\lim_{s\to 1^+} \Big(\frac{\sqrt{D}}{4\pi}\zeta(s,Q) - \zeta(2s-1)\Big) = \log \frac{\sqrt{a/D}}{\big|\eta(z_Q)\big|^2}.$$

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Here

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{and} \qquad \eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} \big(1 - e^{2\pi i n z}\big).$$

Meaning: $\zeta(s) \sim \frac{1}{s-1}$, $\quad \zeta(2s-1) \sim \frac{1}{2(s-1)}$

The formula gives a_{-1} and a_0 in $\zeta(s,Q) = \dfrac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \dots$

Our aim is

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2} = \frac{\Gamma(1/4)}{\pi^{3/4}\sqrt{2}}$$

with $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt$ the Gamma function.

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This is a special value of $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$.

$$\theta(i) = \frac{\Gamma(1/4)}{\pi^{3/4}\sqrt{2}} = \frac{1}{\sqrt[4]{2\pi}}\sqrt{\frac{\Gamma(1/4)}{\Gamma(3/4)}} = \frac{\sqrt{2}}{\pi^{3/4}}\int_{-\infty}^{\infty} e^{-t^4}\,dt$$

$$\theta(i) = 1.08643\dots$$

What is the relation between these three things?

Ramanujan formula $x = 0$

$$\Rightarrow \quad S := \sum_{n=-\infty}^{\infty} \frac{1}{\cosh(\pi n)} = \sqrt{\frac{2}{K}}.$$

$$\frac{1}{\cosh(\pi n)} = \frac{2e^{-\pi n}}{1 + e^{-2\pi n}} = 2\sum_{k=0}^{\infty}(-1)^k e^{-\pi n(2k+1)} \qquad n > 0.$$

Input: (classic, "elementary")
$\#\{(x,y) \in \mathbb{Z}^2 \,:\, x^2 + y^2 = m\} = 4\sum_{2k+1|m}(-1)^k.$

$$S = 1 + 4\sum_{m=1}^{\infty}\sum_{2k+1|m}(-1)^k e^{-\pi m} = \Big(\sum_{n=-\infty}^{\infty} e^{-\pi n^2}\Big)^2 = \theta(i)^2.$$

What is the relation between these three things?

Kronecker formula
$Q(x,y) = x^2 + y^2$

$$\Rightarrow \quad z_Q = \frac{-0 + i\sqrt{4}}{2 \cdot 1} = i.$$

Input: (classic, ““elementary””) $|\eta(i)| = \theta(i)/\sqrt{2}$.

$$\lim_{s\to 1^+} \Big(\frac{1}{2\pi} \sum_{n_1^2+n_2^2\neq 0} \frac{1}{(n_1^2+n_2^2)^s} - \zeta(2s-1) \Big) = -2\log\theta(i).$$

The sum is $\sum_{m=1}^{\infty} m^{-s} \cdot 4 \sum_{2k+1|m} (-1)^k$. Hence $-2\log\theta(i)$ equals

$$\lim_{s\to 1^+} \Big(\frac{2}{\pi}\zeta(s)L(s) - \zeta(2s-1) \Big) \quad \text{with} \quad L(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}.$$

Summing up. . .

Ramanujan formula $\Rightarrow$ evaluation of $\theta(i)$

Kronecker formula $\Rightarrow$ $\theta(i)$ as a limit

$$\theta(i)=\sum_{n=-\infty}^{\infty} e^{-\pi n^2}, \quad \zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \quad L(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^s}$$

$$\theta(i)=\lim_{s\to 1^+}\exp\Big(\frac{1}{2}\zeta(2s-1)-\frac{1}{\pi}\zeta(s)L(s)\Big)$$

The plan

Here

Simple proof of
Kronecker formula
$+$
Simple computation of
the limit

$\Rightarrow$

Evaluation of $\theta(i)$
(no ell. funct. theory)

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In the supporting paper

(RF = Ramanujan formula)

Simple proof of
LHS in RF is constant
$+$
evaluation of $\theta(i)$

$\Rightarrow$

Simple proof of RF

The limit as a derivative

$$\ell = \lim_{s\to 1^+} \Big(\frac{1}{2}\zeta(2s-1) - \frac{1}{\pi}\zeta(s)L(s)\Big)$$

Input: $\zeta(s) = \frac{1}{s-1} + \gamma + \dots$ (Hint: $\zeta(s) = s\int_1^\infty \frac{\lfloor x\rfloor}{x^{s+1}}\,dx$), $\gamma = -\Gamma'(1)$

$$\frac{\zeta(s)}{2\zeta(2s-1)} = \frac{\frac{1}{s-1} + \gamma + \dots}{\frac{2}{2s-2} + \gamma + \dots} = 1 - \gamma(s-1) + \dots = \Gamma(s) + \dots$$

$$\ell = \lim_{s\to 1^+} \frac{\frac{1}{4} - \frac{1}{\pi}\Gamma(s)L(s)}{s-1}$$

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$$\ell = \lim_{s\to 1^+}\frac{\frac{1}{4}-\frac{1}{\pi}\Gamma(s)L(s)}{s-1} \underset{\text{L'H}}{=} -\frac{1}{\pi}\frac{d}{ds}\Big|_{s=1}\big(\Gamma(s)L(s)\big)$$

The limit as an integral

$$\Gamma(s)L(s) \underset{\text{def. } L}{=} \frac{\Gamma(s)}{1^s} - \frac{\Gamma(s)}{3^s} + \frac{\Gamma(s)}{5^s} - \frac{\Gamma(s)}{7^s} + \dots$$

$$\underset{\text{def. } \Gamma}{=} \int_0^\infty t^{s-1}\left(e^{-t} - e^{-3t} + e^{-5t} - e^{-7t} + \dots\right) dt$$

$$= \int_0^\infty \frac{t^{s-1}}{2\cosh t}\, dt$$

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$$\ell = -\frac{1}{\pi}\frac{d}{ds}\Big|_{s=1}\left(\Gamma(s)L(s)\right) = -\frac{1}{2\pi}\int_0^\infty \frac{\log t}{\cosh t}\, dt$$

Computation of the integral

$$\int_0^\infty \frac{\log t}{\cosh t}\,dt \underset{t\mapsto 2\pi t}{=} \int_{-\infty}^\infty \frac{\pi \log|t|}{\cosh(2\pi t)}\,dt + \frac{1}{2\pi}\int_0^\infty \frac{\log(2\pi)}{\cosh t}\,dt$$

Last integral: standard $u = e^t$.

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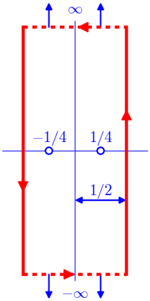

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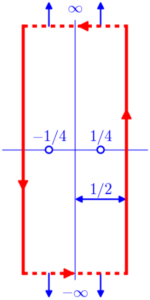

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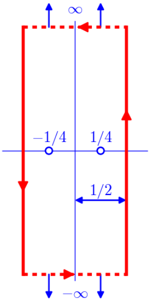

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Residue theorem

$$\log\frac{\Gamma(3/4)}{\Gamma(1/4)} = \int_{\mathbb{R}} \frac{\log\Gamma(1+it) - \log\Gamma(it)}{\cosh(2\pi t)}\,dt$$

$$= \int_{\mathbb{R}} \frac{\log|t|}{\cosh(2\pi t)}\,dt$$

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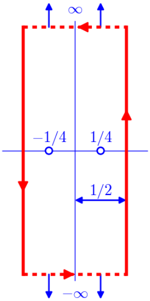

$$f(z) = i\,\frac{\log\Gamma(1/2+z)}{\cos(2\pi z)}$$

$$2\pi i\,\mathrm{Res}(f,\pm\tfrac14) = \pm\log\Gamma\big(\tfrac12 \pm \tfrac14\big)$$

Residue theorem

$$\log\frac{\Gamma(3/4)}{\Gamma(1/4)} = \int_{\mathbb{R}} \frac{\log\Gamma(1+it)-\log\Gamma(it)}{\cosh(2\pi t)}\,dt$$

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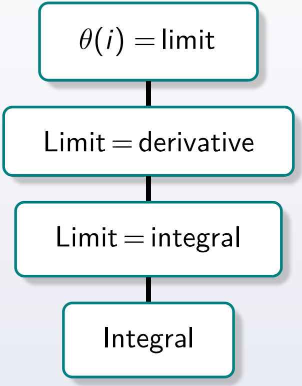
$\Rightarrow \theta(i)$

Summary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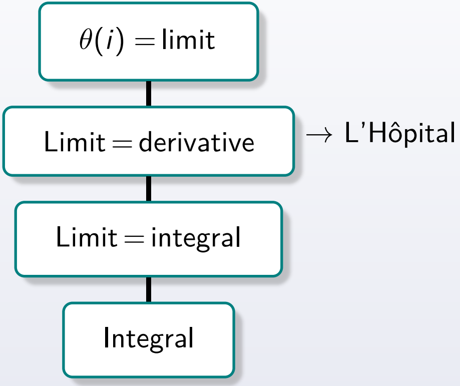
Limit = derivative
Limit = integral
Integral

Summary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Limit $=$ derivative $\rightarrow$ L'Hôpital

Limit $=$ integral $\rightarrow$ definition of Γ

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Integral $\rightarrow$ Residue theorem

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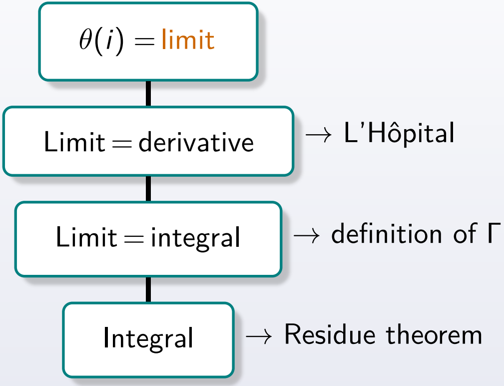

Limit formula ↔ Residue theorem

About a simple proof of the Kronecker formula

Relation $\zeta(s,Q) \leftrightarrow \eta(z_Q)$ $Q = x^2+y^2$, $z_Q = i$, $g_s(x) = \frac{2}{(x^2+1)^s}$

$$\sum_{n^2+m^2\neq 0} \frac{1}{(n^2+m^2)^s} - 2\zeta(2s) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \sum_{m\in\mathbb{Z}} g_s\Big(\frac{m}{n}\Big)$$

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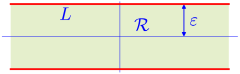

$$2\pi i n \,\mathrm{Res}\big(\cot(\pi n z), m/n\big) = 2i$$

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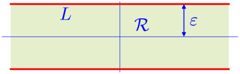

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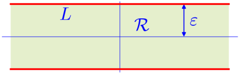

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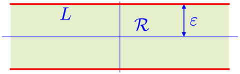

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$1 \underset{s\to 1^+}{\to} \zeta(2s-1)$ singularity

$i\cot(\pi n z) - 1 \underset{s\to 1^+}{\to} \int_{\mathbb{R}} g_1(x)e^{2\pi i n x}\,dx$ factors of $|\eta(i)|$ (*undergraduate*)

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There is a copy of this presentation in

https://matematicas.uam.es/~fernando.chamizo/

Thank you for your attention!

Something to take home . . .

Ramanujan's formula is also related to the Γ-free evaluation:

$$\prod_{m \text{ odd}} \tanh\left(\frac{\pi m}{2}\right) = \frac{1}{\sqrt[8]{2}}.$$

Equivalently,

$$\frac{e^{\pi/2} - e^{-\pi/2}}{e^{\pi/2} + e^{-\pi/2}} \cdot \frac{e^{3\pi/2} - e^{-3\pi/2}}{e^{3\pi/2} + e^{-3\pi/2}} \cdots = \frac{1}{\sqrt[8]{2}}.$$

Is it possible to get a direct elementary proof of it?