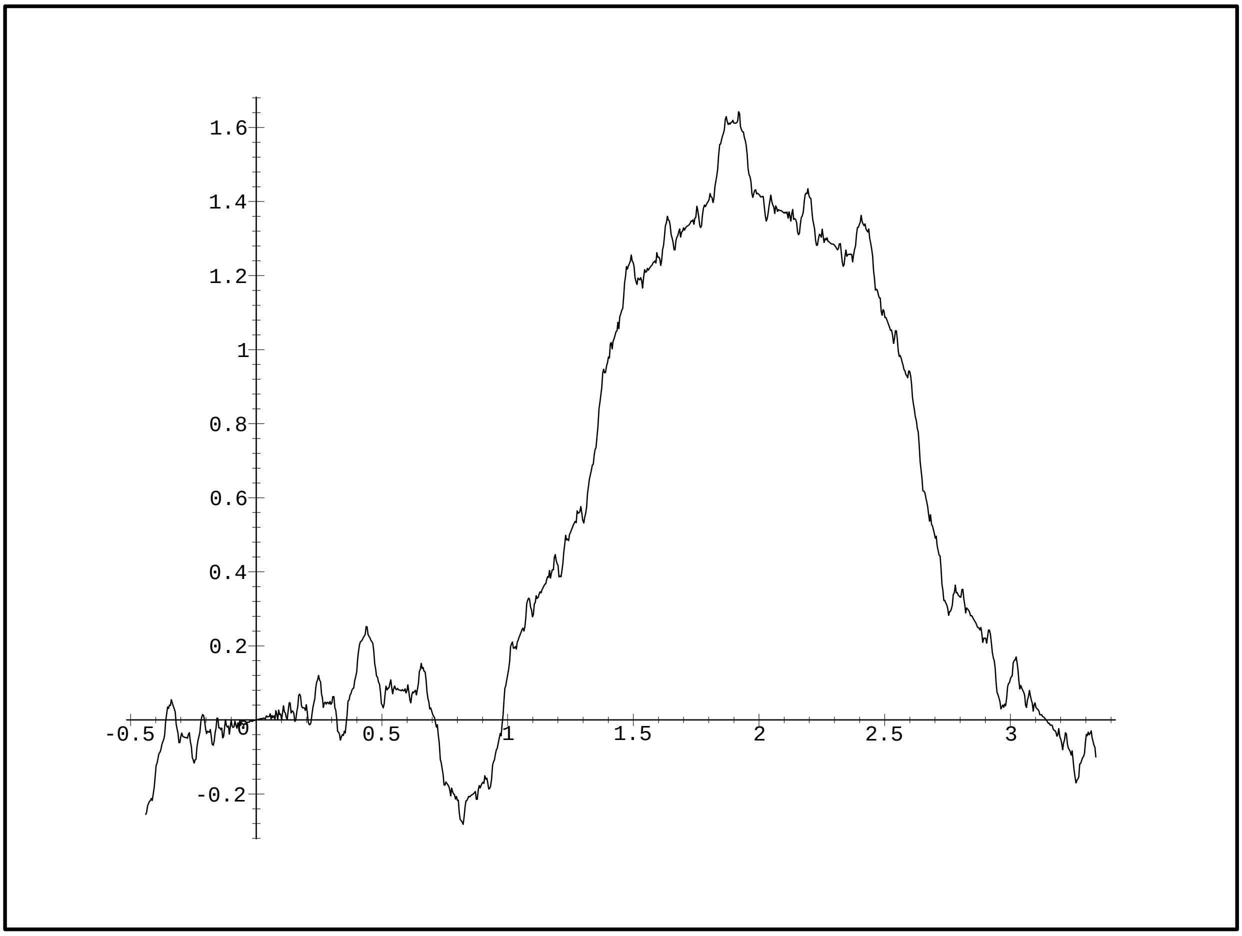
The "simplest" elliptic curve

$$E: \; y^2 + y = x^3 - x^2 - 10\,x - 20$$

Hasse-Weil L-function $\longrightarrow$ $L(E,s) = \sum \frac{a_n}{n^s}$

Graph of $\sum \frac{a_n}{n^{7/4}} \sin(nx)$ in $[0, \pi]$

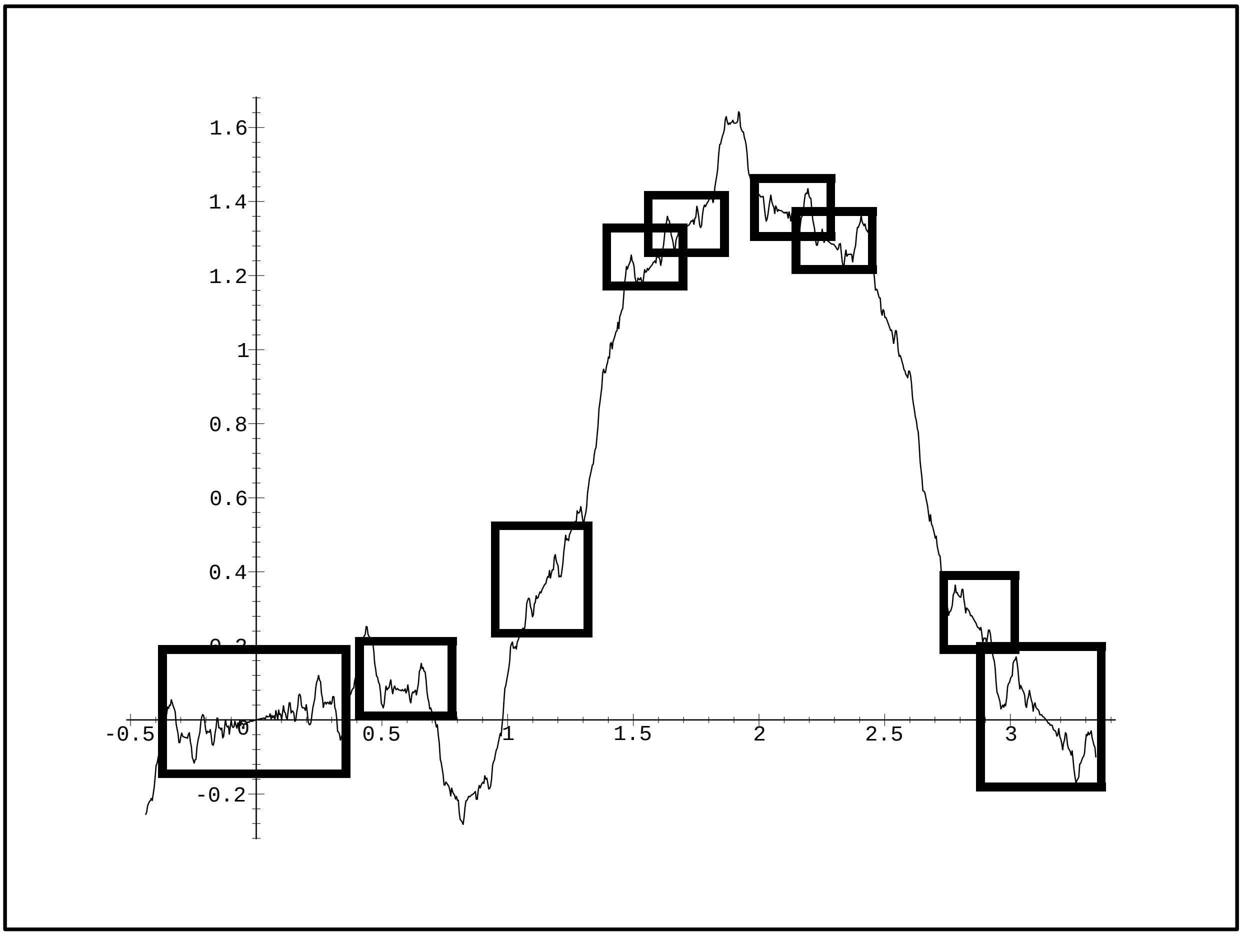


The "simplest" elliptic curve

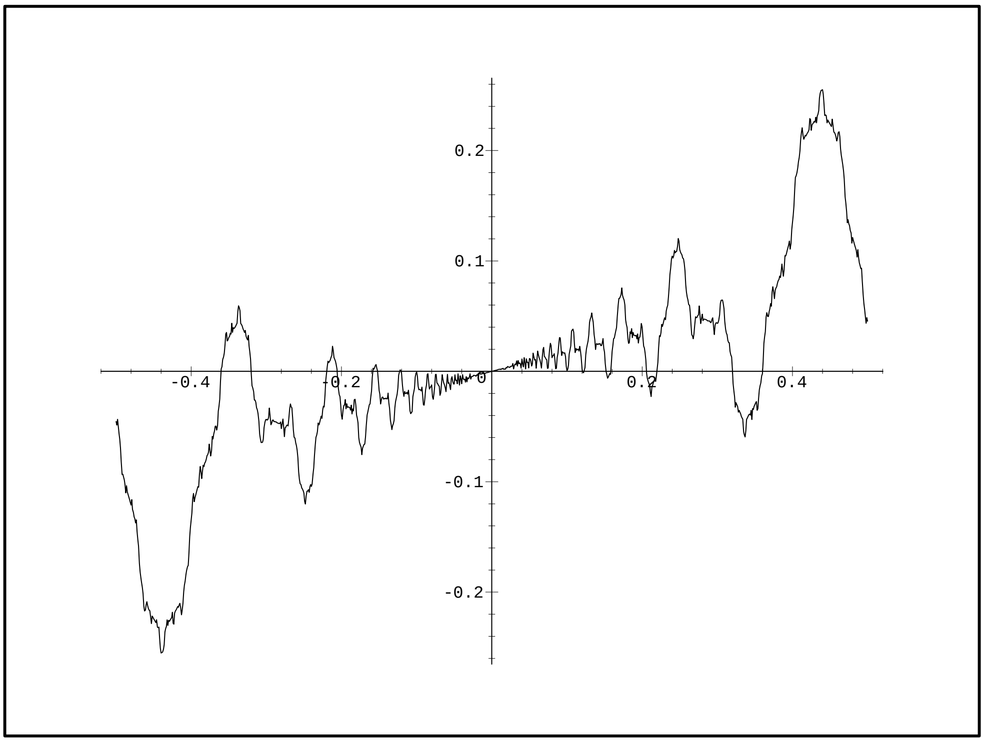
$$E:\ y^2 + y = x^3 - x^2 - 10\,x - 20$$

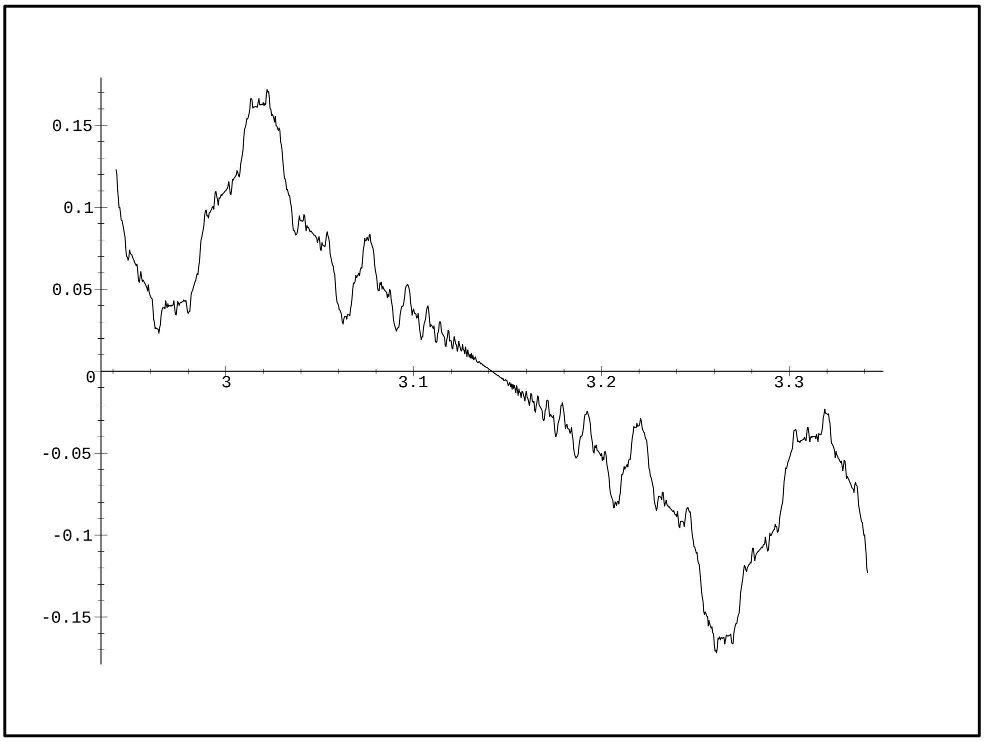
Hasse-Weil L-function $\longrightarrow$ $L(E,s) = \sum \frac{a_n}{n^s}$

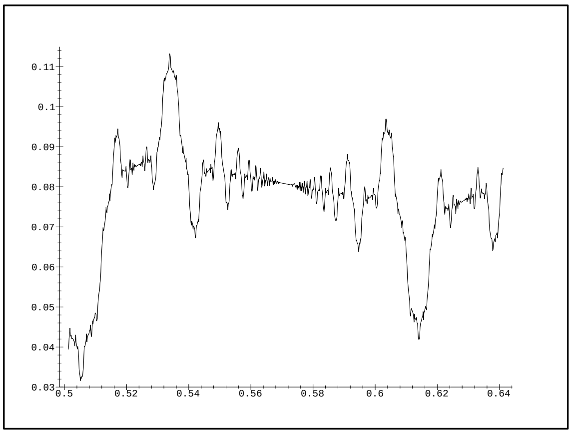
Graph of $\sum \frac{a_n}{n^{7/4}} \sin(nx)$ in $[0,\pi]$

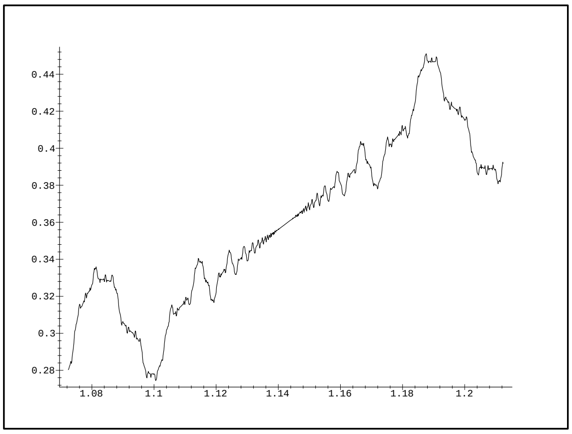


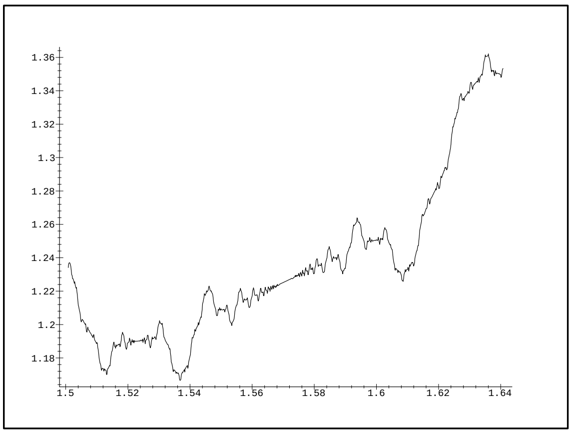
A closer look around zero and π

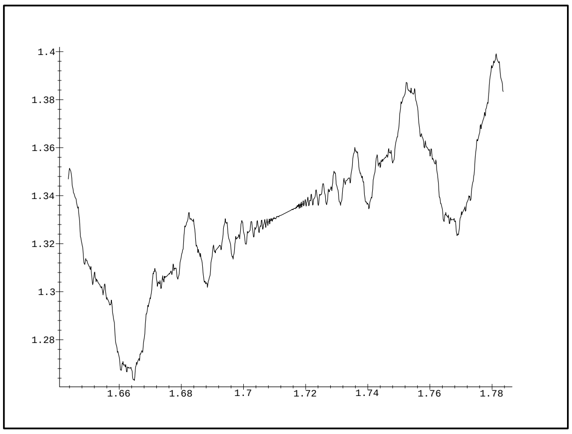


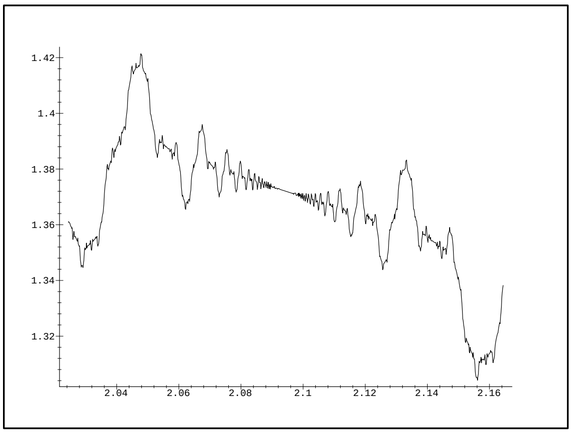


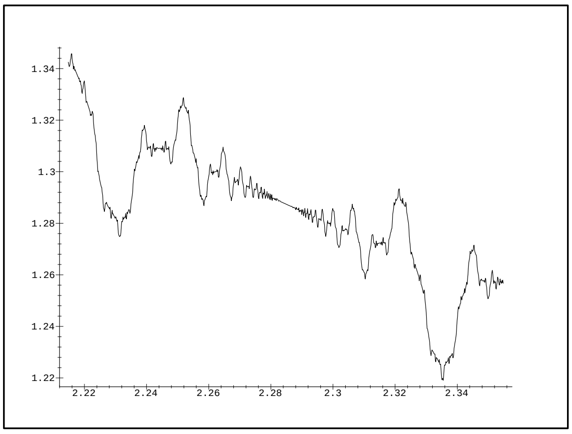

0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.5
0.52
0.54
0.56
0.58
0.6
0.62
0.64

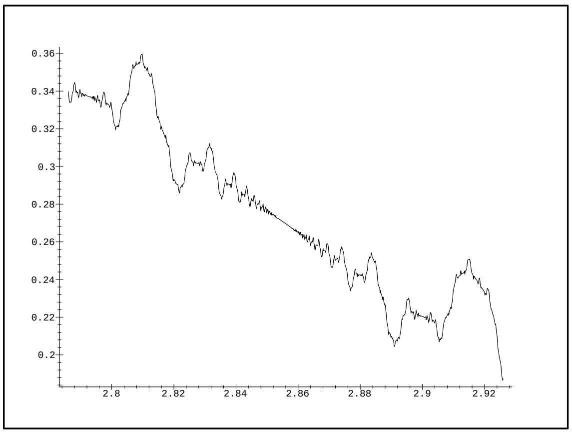

0.44
0.42
0.4
0.38
0.36
0.34
0.32
0.3
0.28
1.08
1.1
1.12
1.14
1.16
1.18
1.2

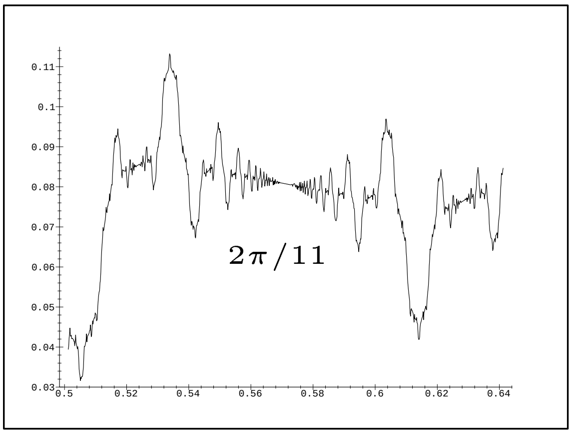

1.36
1.34
1.32
1.3
1.28
1.26
1.24
1.22
1.2
1.18
1.5
1.52
1.54
1.56
1.58
1.6
1.62
1.64

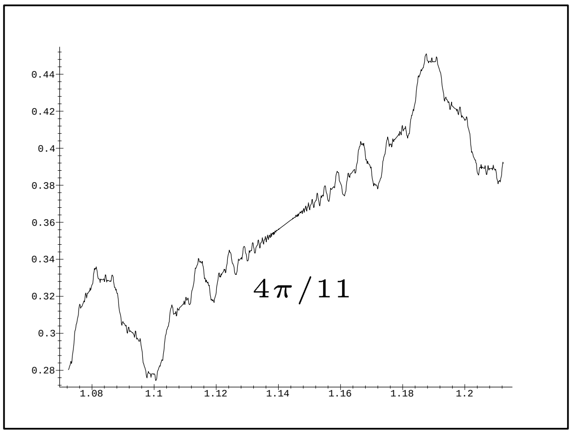

1.4
1.38
1.36
1.34
1.32
1.3
1.28
1.66
1.68
1.7
1.72
1.74
1.76
1.78

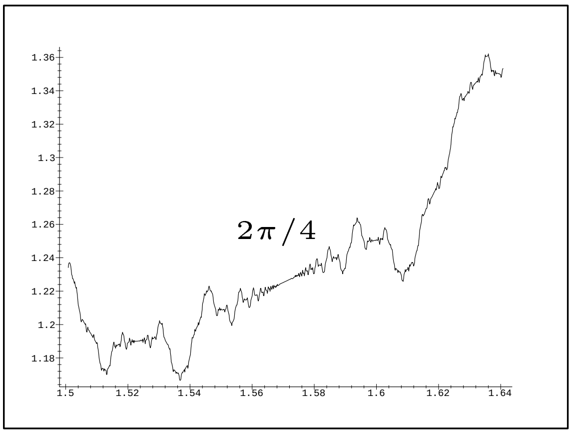

1.42
1.4
1.38
1.36
1.34
1.32
2.04
2.06
2.08
2.1
2.12
2.14
2.16

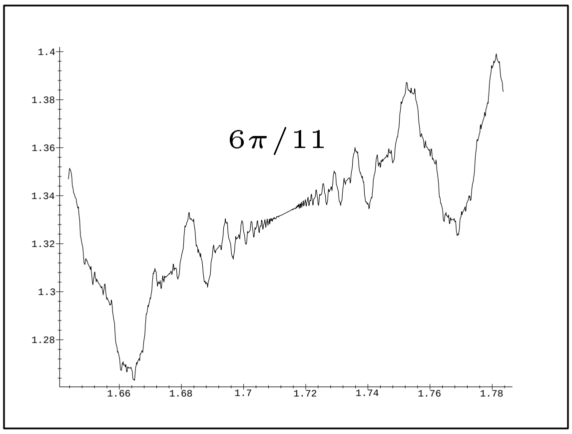

1.34
1.32
1.3
1.28
1.26
1.24
1.22
2.22
2.24
2.26
2.28
2.3
2.32
2.34

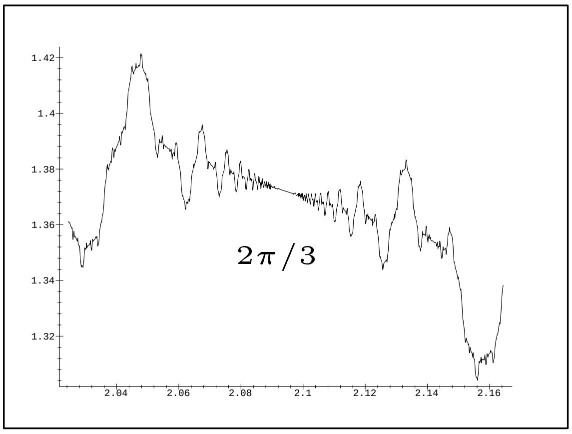

0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
2.8
2.82
2.84
2.86
2.88
2.9
2.92

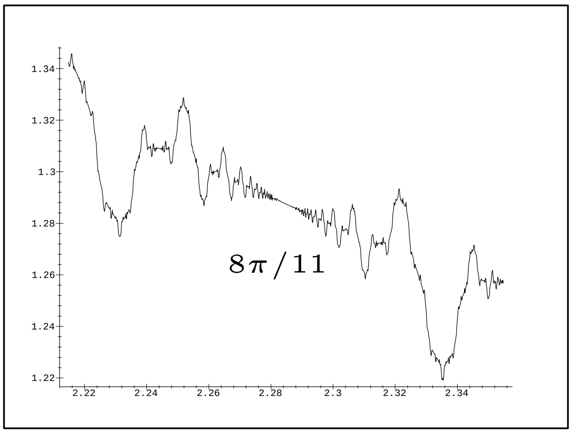


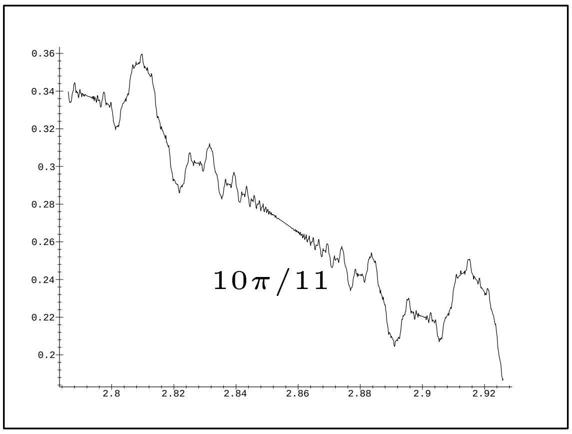












Why can we observe
oscillation only
at these points
to the naked eye?

Can we extract arithmetic information from this kind of graphics?

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} \sin(nx) \overset{(?)}{\Rightarrow} f'(x) \overset{(?)}{=} \sum_{n=1}^{\infty} \frac{a_n}{n} \cos(nx)$$

(This latter does not even make sense in L^2)

$$f'(0) \overset{(?)}{=} \sum_{n=1}^{\infty} \frac{a_n}{n} \overset{(?)}{=} L(E,1)$$

f is flat at $x = 0$ $\Rightarrow$ E contains infinitely many rational points

Seeing (weak) B–S-D conjecture with our own eyes

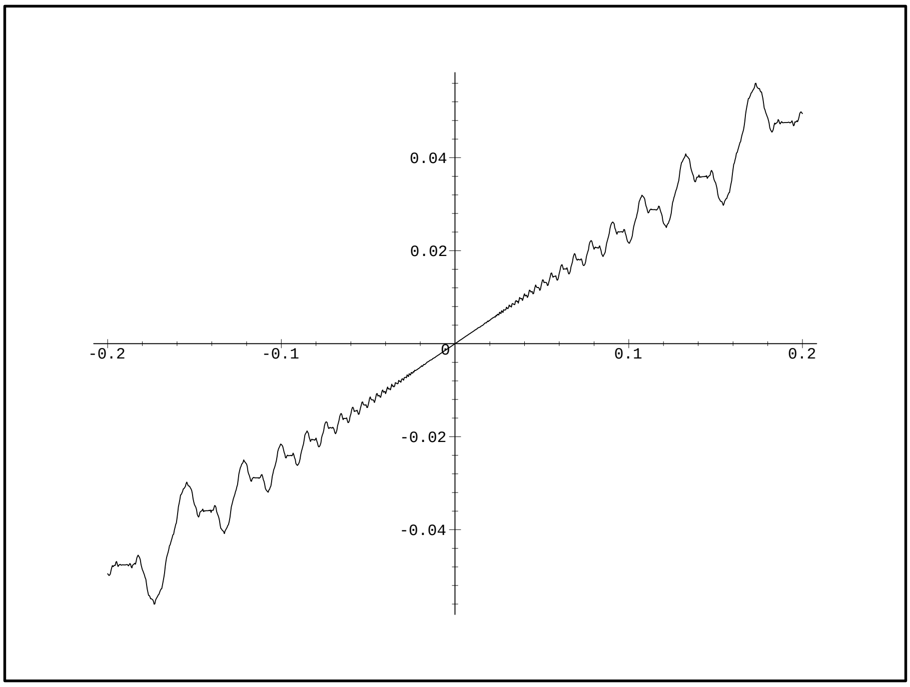
$$E:\ y^2 + y = x^3 - x^2 - 10\,x - 20$$

$$\sum \frac{a_n}{n^2}\sin(nx)$$

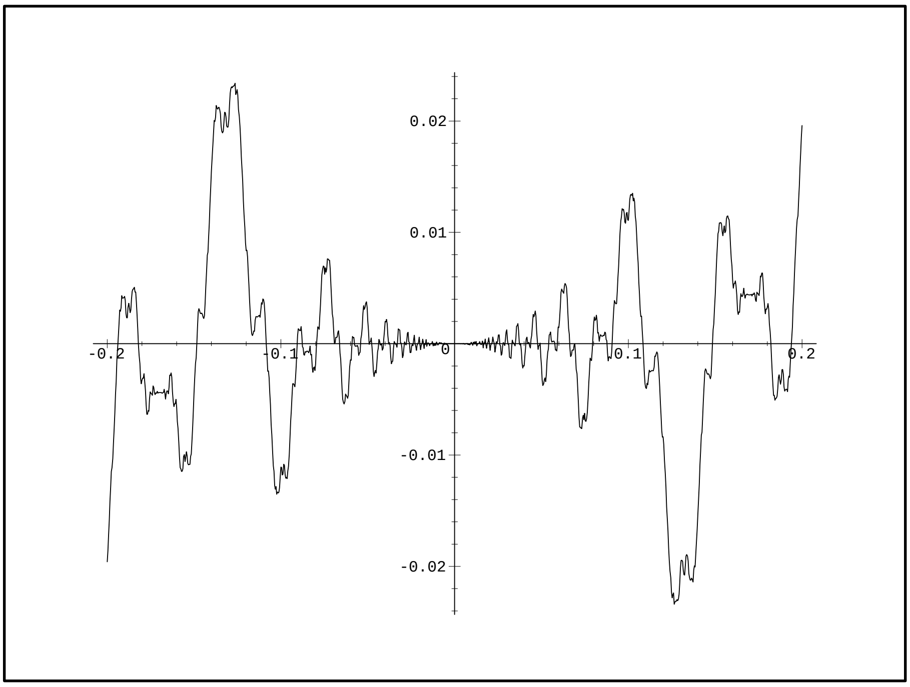
$$f'(0) \neq 0 \ \Rightarrow$$

$$L(E,1) \neq 0$$

$$\mathrm{rk}(E) = 0$$



$$E:\ y^2 + y = x^3 - x$$



$$\sum \frac{a_n}{n^2}\sin(nx)$$

$$f'(0) \neq 0 \ \Rightarrow$$

$$L(E,1) = 0$$

$$\mathrm{rk}(E) = 1$$

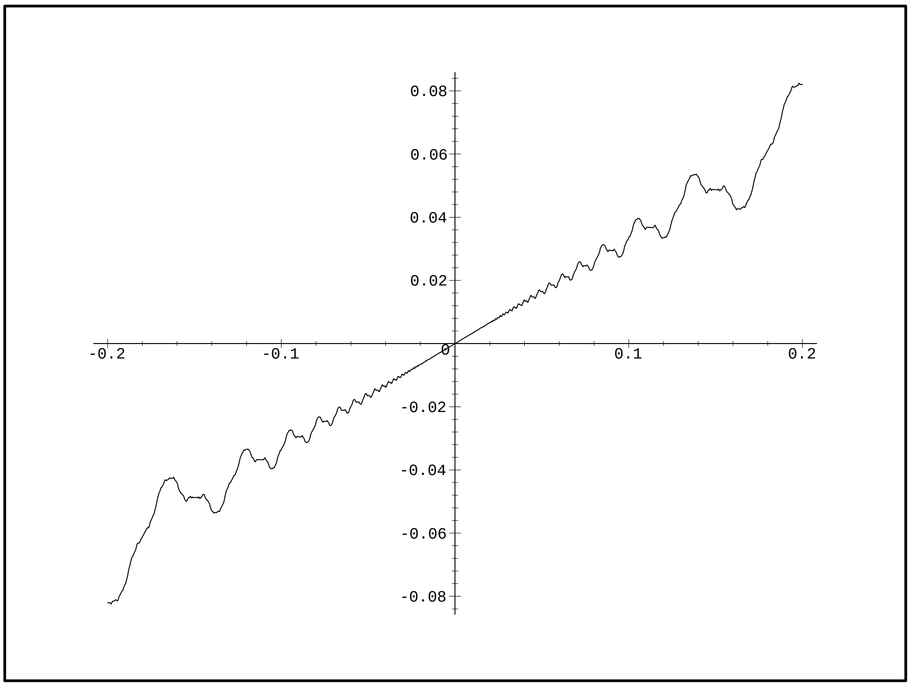
$$E:\ y^2 + y = x^3 + 4\,x - 6$$

$$\sum \frac{a_n}{n^2}\sin(nx)$$

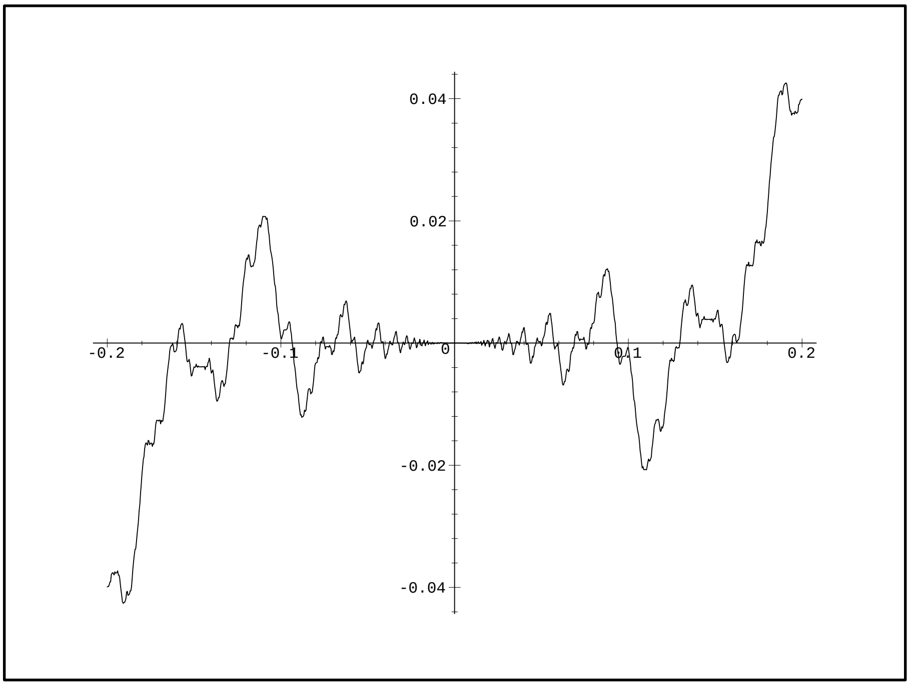
$$f'(0) \neq 0 \ \Rightarrow$$

$$L(E,1) \neq 0$$

$$\mathrm{rk}(E) = 0$$



$$E:\ y^2 + y = x^3 + x^2$$



$$\sum \frac{a_n}{n^2}\sin(nx)$$

$$f'(0) \neq 0 \ \Rightarrow$$

$$L(E,1) = 0$$

$$\mathrm{rk}(E) = 1$$

What is the general theorem?

Th. *Let* $E/\mathbb{Q}$ *be an elliptic curve,* $\sum a_n n^{-s}$ *its Hasse-Weil L-function and*

$$A_\alpha(x)=\sum \frac{a_n}{n^\alpha}\cos(2\pi n x), \quad B_\alpha(x)=\sum \frac{a_n}{n^\alpha}\sin(2\pi n x).$$

Then

a) For $3/2 < \alpha < 2$ *these functiones are differentiable in* $\mathbb{Q}$ *and non-differentiable in* $\mathbb{R}\setminus\mathbb{Q}$ *.*

b) For $1 < \alpha < 2$ *the graphs of this functions are fractal sets of dimension* $3-\alpha$*.*

c) $B_2'(0) = 0 \;\Leftrightarrow\; L(E,1) = 0$ $(\Rightarrow\; rk(E) > 0)$*.*

---

b) $\Rightarrow$ Generic close views are highly oscillatory.

a) $\Rightarrow$ The oscillation is tamed around rational points.

Remarks: dimension = Minkowski dimension.
$\alpha < 1 \;\Rightarrow A_\alpha, B_\alpha \notin L^2$ (they are not even continuous).
$\alpha > 2 \;\Rightarrow A'_\alpha, B'_\alpha \in L^2$ (they are a.e. differentiable).

What is the even more general theorem?

Th. *Let* $f(z) = \sum a_n e(nz)$ *be an automorphic form of weight* $r > 0$ *(non necessarily integral) with respect to* Γ *(*$[SL_2(\mathbb{Z}) : \Gamma] < \infty$*, unit. trans.*$\in \Gamma$*) and let*

$$f_\alpha(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^\alpha} e(nx), \quad A_\alpha = \mathrm{Re}\, f_\alpha, \quad B_\alpha = \mathrm{Im}\, f_\alpha$$

a) It is possible caracterize in some ranges the functional spaces (Sobolev spaces, local and global Lispchitz spaces) to which f_α *belongs.*

b) For $\frac{r+1}{2} < \alpha < \frac{r}{2} + 1$

A_α *is differentiable at* $x_0 \Leftrightarrow f$ *is cuspidal at* x_0

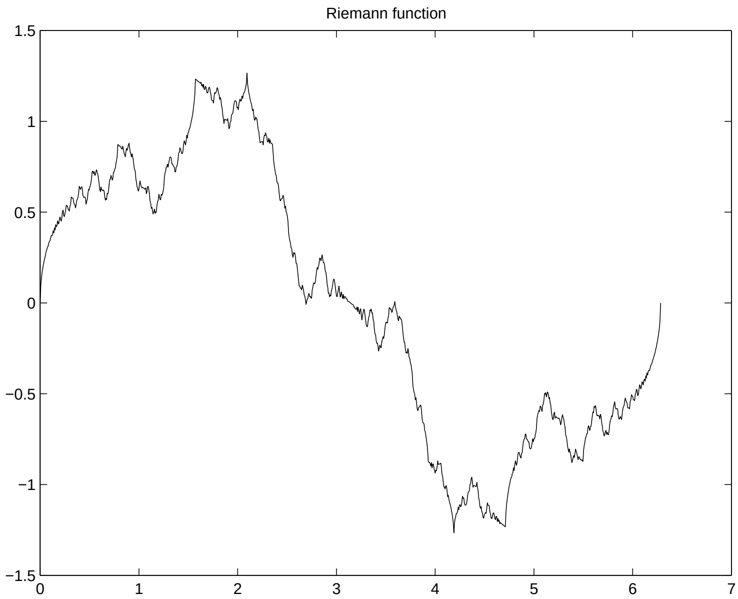
B_α *is differentiable at* $x_0 \Leftrightarrow f$ *is cuspidal at* x_0

c) The graphs of A_α *and* B_α *are commonly fractal sets. For instance, if* f *is a cusp form*

$$\dim = \max\left(1, 2 - \alpha + \frac{r}{2}\right).$$

A historical example

Riemann (non-zeta) function : $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$



According to Weierstrass (1872), Riemann claimed in 1861 that f is nowhere differentiable.

Wrong proof, wrong answer

$f'(x) = \sum \cos(n^2 x)$ doesn't converge for any $x \in \mathbb{R} \overset{(?)}{\Rightarrow}$ it is nowhere differentiable

Riemann was right (?)

Wrong proof, right answer

$f'(x) =$derivative of

$$\lim_{y\to 0} \Big(\sum e^{-n^2 y} \frac{\sin(n^2 x)}{n^2} \Big)$$

$$= \lim_{y\to 0} \frac{\theta(\frac{x}{2\pi} + iy) - 1}{2} = -\frac{1}{2}$$

for $x = a\pi/b$, $2\nmid a, b$

Hardy (1916) $\to$ It is not differentiable at any irrational.

Gerver (1970) $\to$ f is differentiable at $x = \dfrac{a}{b}\pi$, $2 \nmid a, b$.

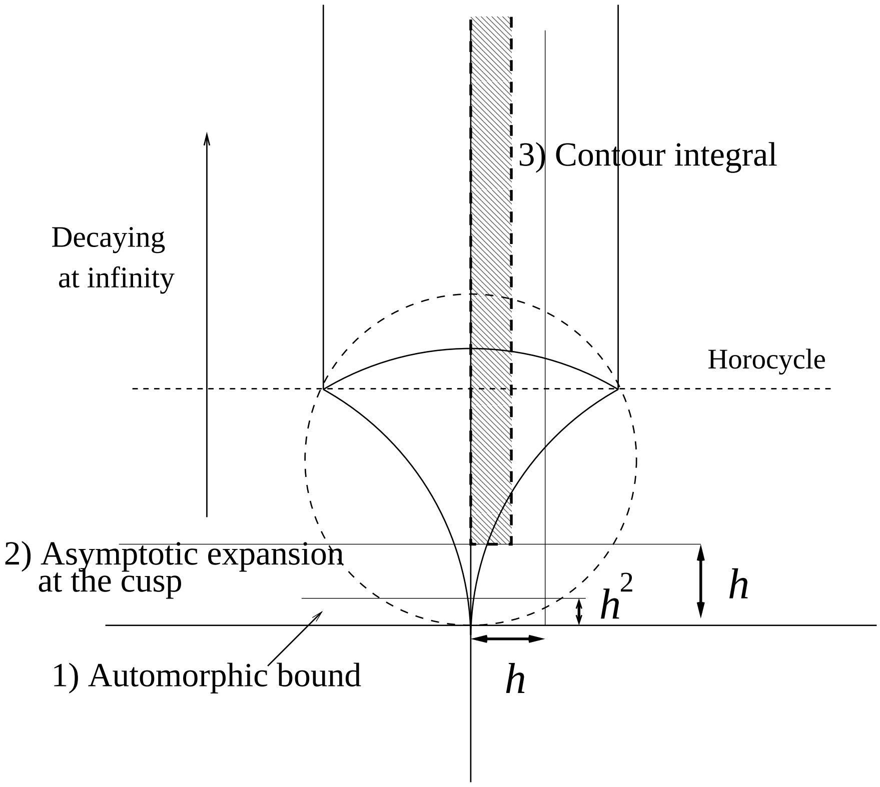
(30 pages long elementary proof. But it is just Poisson summation!)

More recent results:

Ch, Córdoba (1993) $\to$ dim(graph of f) $= \dfrac{5}{4}$.

Conrey, Farmer, Soundararajan (2000) $\to$ f appears in the asymptotics of $\displaystyle\sum_{2 \nmid n} \sum_{2 \nmid m} \left(\tfrac{n}{m}\right)$.

Differentiability properties



$$f_\alpha(h) - f_\alpha(0) = \frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \big(f(h+it) - f(it) \big) dt$$

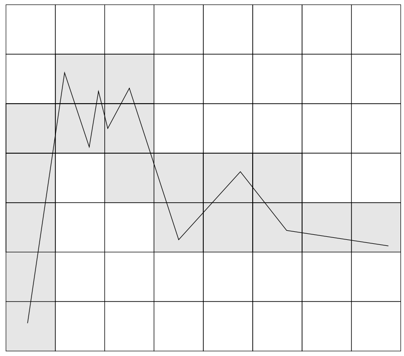
1) f cusp form $\Rightarrow$ $|\mathrm{Im}\, z|^{r/2}|f(z)|$ bounded in $\Gamma\backslash\mathbb{H}$. In general $|f(h+it)| \ll h^r t^{-r} + h^{-r}$.

2) $f(z) \sim C_1 z^{-r} e(-C_2/z)$

3) $f(h+it) - f(it) = hf'(\xi + it)$ and move the line of integration.

$$f_\alpha(h) - f_\alpha(0) = h\frac{(2\pi)^\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} f'(it)dt + O(h^{2\alpha-r} + h^2)$$

Fractal dimension



$g = \mathrm{Re}\, f_\alpha,\ \mathrm{Im}\, f_\alpha$

$\dim = \frac{\log \mathcal{N}}{\log \Delta x} = 1 + \frac{\log \sum |\Delta g|}{\log h^{-1}}$

<u>Upper bound</u>

Control of Fourier coeff. $\overset{\text{Large sieve}}{\Longrightarrow}$ Control of $\sum |\Delta g|$

<u>Lower bound</u>

1. *Cuspidal case*:

$$\sum |\Delta g| \geq h^{-1} \|g(\cdot + h) - g(\cdot)\|_1 \geq \frac{\|\cdot\|_2^2}{\|\cdot\|_\infty}$$

upper bound for $\sum a_n e(n\alpha)$+asymptotics of $\sum |a_n|^2$.

2. *Non cuspidal case*:

Lower bound for $f(\cdot + ih)$ at $h-$spaced points

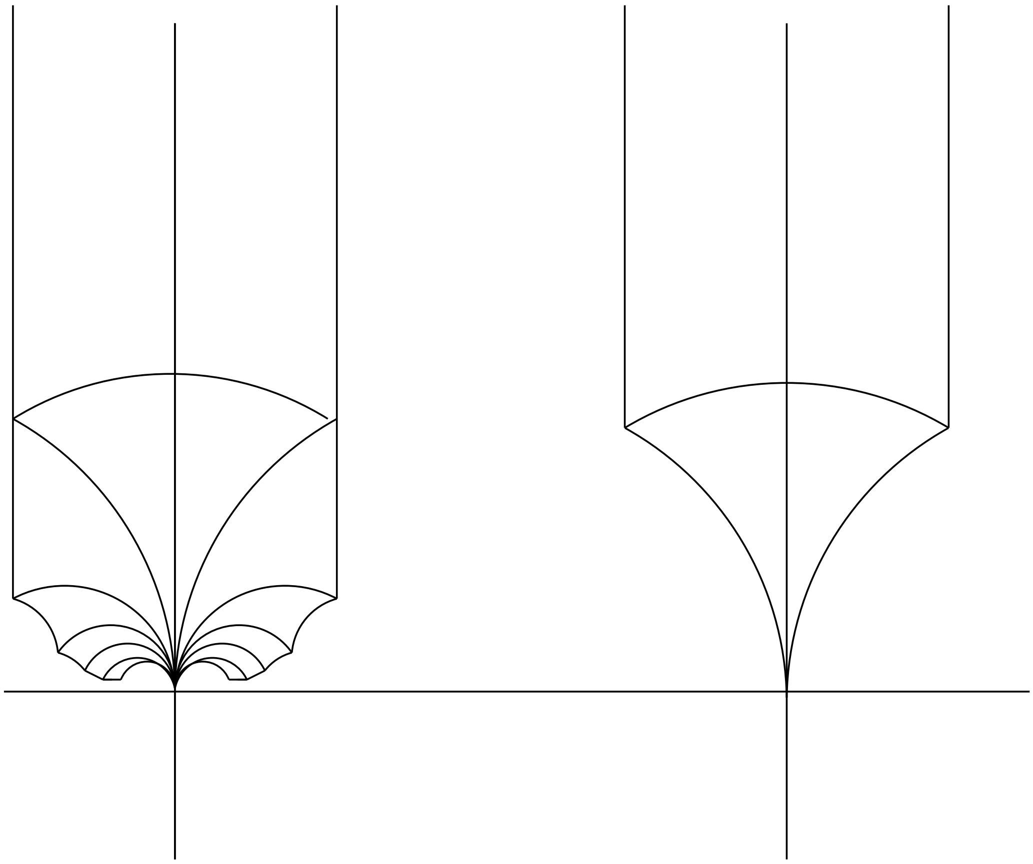
$\Rightarrow$ lower bound for $\sum |\Delta g|$.

(Poisson integral representation + covering lemma).

$f(x+ih)$ is large for $x = \dfrac{a}{b} \notin \{\text{cusps}\}$ with b small.

Once again: Why could we observe oscillation to the naked eye, in our example, only at rationals with very small denominator or 11?

$$\Gamma = \Gamma_0(11) \qquad [SL_2(\mathbb{Z}) : \Gamma] = 12$$



Cusps equiv. to 0

width = 11

Decaying as $e^{-t/11}$

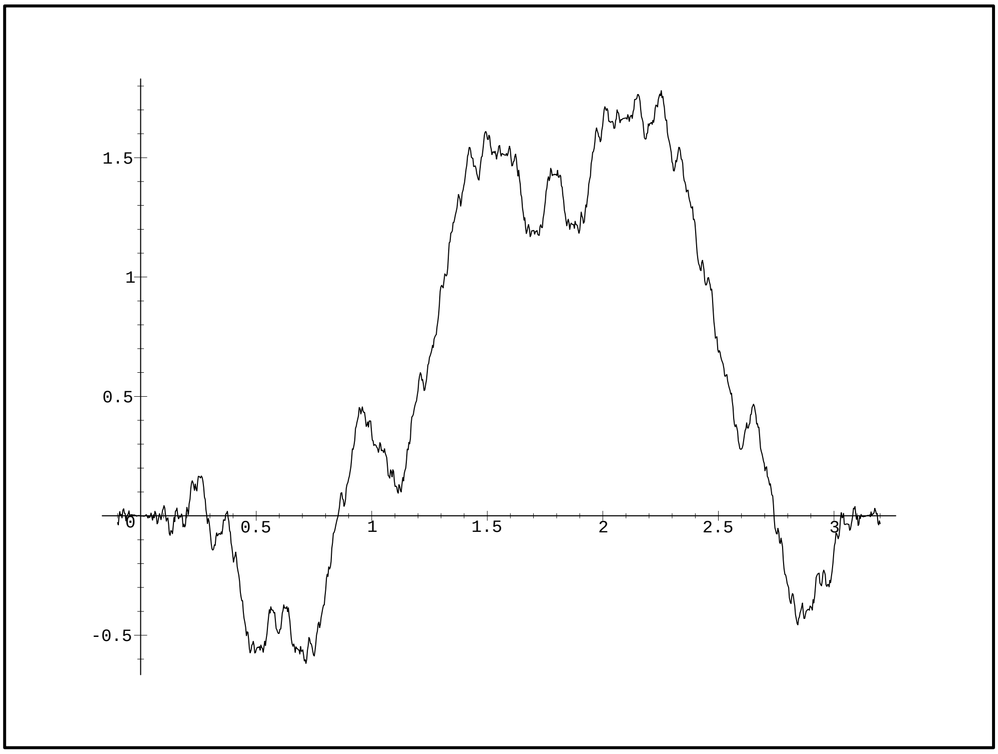
Cusps equiv. to 1/11

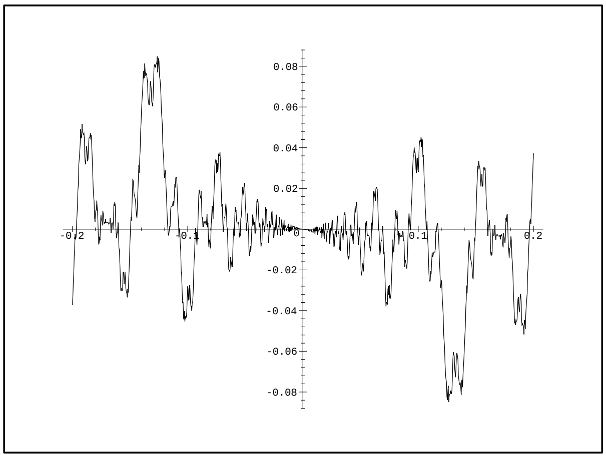
width = 1

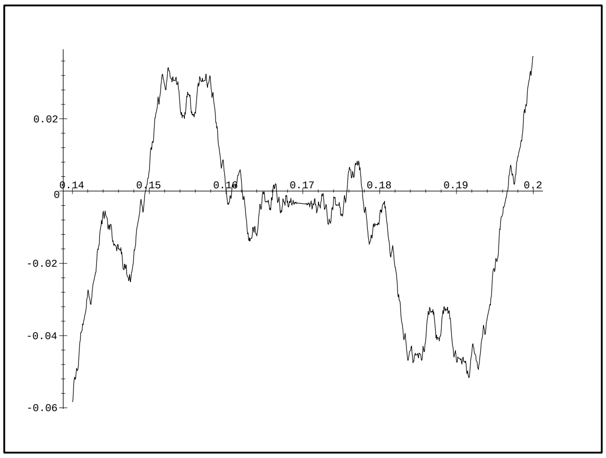
Decaying as e^{-t}

We can see "eleven times" further at a/b when $11|b$.

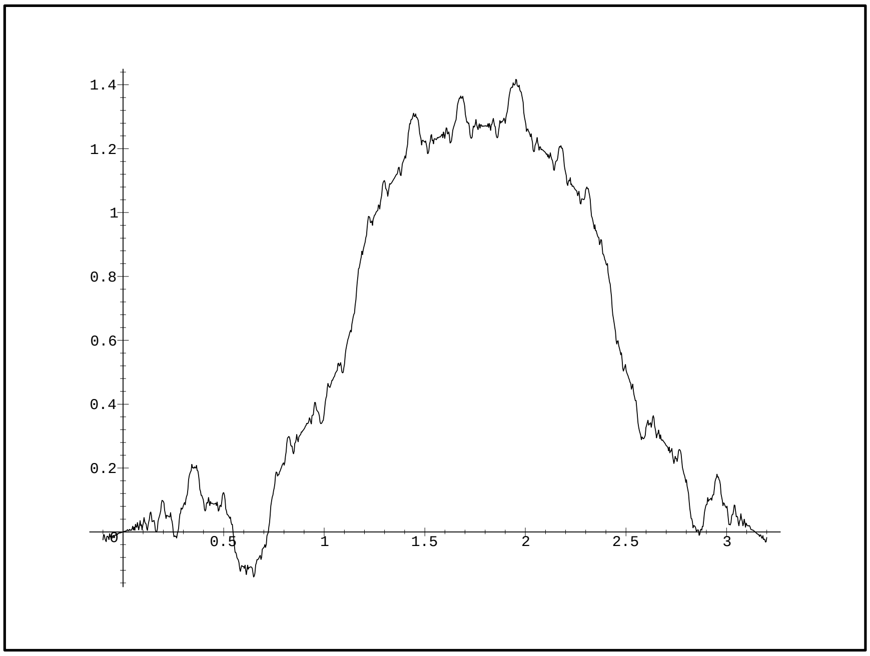
$$E : y^2 + y = x^3 - x \qquad \text{Conductor } = 37$$

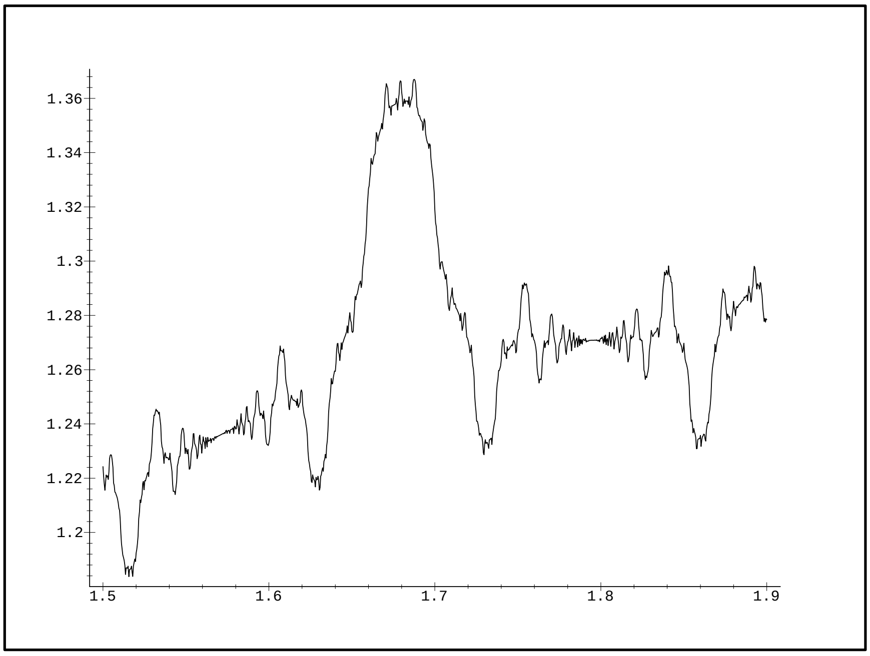






$$\frac{2\pi}{37} = 0.169815819\ldots$$





$$\frac{\pi}{2} = 1.570796327\ldots \qquad \frac{4\pi}{7} = 1.795195802\ldots$$

$$E : y^2 + y = x^3 + 4\,x - 6 \qquad \text{Conductor } = 14 = 2 \cdot 7$$