Fourier series with gaps and arithmetic

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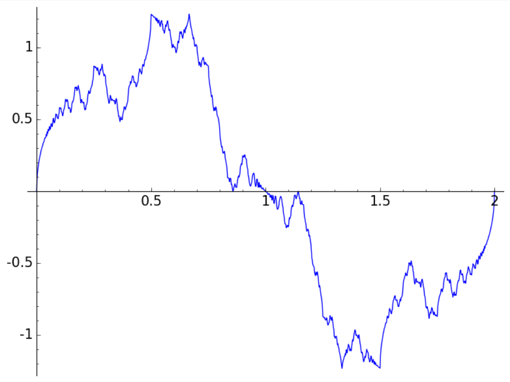
21 mars 2019

To be or not to be
. . . differentiable

According to Weierstrass, Riemann claimed that f is nowhere differentiable

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2}$$



Quick conclusion: Riemann and Weierstrass did not have a computer.

Two fake proofs

Wrong proof, wrong answer	Wrong proof, right answer
$f'(x) = \pi \sum \cos(\pi n^2 x)$ does not converge for any real number $x \stackrel{(?)}{\Rightarrow}$ f is nowhere differentiable and Riemann's alleged claim is right.	$f'(x)$ is the derivative of $\lim_{y\to 0} \sum e^{-\pi n^2 y} \frac{\sin(\pi n^2 x)}{n^2} = \pi \lim_{y\to 0} \frac{1}{2}\big(\vartheta(x+iy)-1\big)$ and it exists and takes the value $-\pi/2$ exactly at the *cusps* $x = a/b$, $2 \nmid a, b$.

$\vartheta(z) = \sum_{n\in\mathbb{Z}} e(\frac{1}{2}n^2 z)$ (Jacobi)

Notation: $e(x) = e^{2\pi i x}$

A couple of unexpected connections

① Riemann's example appears in the asymptotics of the sum of the Jacobi symbol $\displaystyle\sum_{2\nmid n<X}\sum_{2\nmid m<Y}\left(\frac{n}{m}\right)$.

Conrey, J. B.; Farmer, D. W.; Soundararajan, K. Transition mean values of real characters. J. Number Theory 82 (2000), no. 1, 109–120.

② Riemann's example appears in the evolution of a polygonal vortex under a fluid dynamic model.

de la Hoz, F.; Vega, L. Vortex filament equation for a regular polygon. Nonlinearity 27 (2014), no. 12, 3031–3057.

Square frequencies are linked to Schrödinger equation and it opens the opportunity for several applications, for instance the quantum Talbot effect.

Berry, M. V.; Klein, S. Integer, fractional and fractal Talbot effects. J. Modern Opt. 43 (1996), no. 10, 2139–2164, (Vega, L.; Eceizabbarrena, D. Work in progress)

Gaps: Good or bad?

A couple of unexpected connections

Riemann's example is only differentiable in a meager set and Weierstrass constructed his fractal-like functions with lacunary Fourier series. Do gaps imply chaos?

Some theorems suggests that lacunary frequencies have a smoothing effect. Namely for $f \sim \sum e(n_j x)$ with $n_{j+1}/n_j > c > 1$ it is known:

If f is bounded then its Fourier series is absolutely convergent.

If f is integrable then it belongs to L^2. In particular $\sum |c_n|^2 < \infty$.

These results are blatantly false if lacunarity is dropped.

Conjecture (Hardy, Littlewood, Rudin, Córdoba): The "kernel" $\sum e(n^2x)$ is a multiplier $L^2 \longrightarrow L^p$ for $p < 4$ i.e., L^2 Fourier series with squares frequencies are automatically in L^p for $p < 4$.

It implies Rudin's conjecture: An arithmetic progression of length N may contain at most $O(N^{1/2+\epsilon})$ squares.

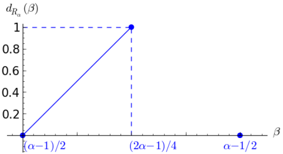
Uncertainty principle: To examine details at level ϵ we need a frequency range of at least ϵ^{-1}. $[x, p] = i\hbar, \quad 4\pi\|xf\|\|\xi\hat{f}\| \geq \|f\|^2$

The gaps give us an opportunity in our everlasting fight against uncertainty principle. On the other hand the possible resonances are sometimes related to deep arithmetical problems.

Intricate theory of exponential sums estimation in number theory.

How chaotic is your function?

Theorem (Jaffard, 1996). Riemann's example and its fractional derivatives are multifractals.



Theorem (Ch. Ubis, 2014). The functions $\sum n^{-\alpha}e(P(n))$ with $P \in \mathbb{Z}[x]$, $\deg P > 2$ are also multifractal in some ranges.

The higher gaps for $\deg P > 2$ do not payback because one loses self-similarity.

Visiting a Riemann's relative

We consider a relative of Riemann's example at the edge of convergence

$$F(x) = \sum_{n=1}^{\infty} \frac{e(n^2 x)}{n}.$$

Remarks

1. It clearly diverges in a dense set of rationals.
2. It nevertheless defines a BMO function.

Our basic naive question

How much does it differ from being a bounded function?

BMO, an interlude

(when mean oscillation is not mean)

The space of **B**ounded **M**ean **O**scillation was introduced by John and Nirenberg in 1961 and readily applied by Moser to get the regularity of some elliptic PDEs. Its relevance was boosted by C. Fefferman duality theorem $(H^1)^* = \text{BMO}$.

$$f \in \text{BMO} \quad \text{means} \quad \|f\|_I = \frac{1}{|I|}\int_I |f - f_I| < C$$

uniformly on intervals I, where f_I is the average of f.

Comment. $f : [a,b] \longrightarrow \mathbb{R}$ bounded $\Rightarrow f \in \text{BMO}$.

(Counter)example. $\log x$ as a function $(0,1] \longrightarrow \mathbb{R}$ belongs to BMO.

- From the point of view of harmonic analysis is a substitute for L^∞ well-behaved under interpolation and duality.
- By John-Nirenberg inequality, roughly speaking, BMO is like L^∞ allowing at most logarithmic singularities.

Theorem. (Ch., Córdoba, Ubis 2019) If $\nu_{n+1}/\nu_n \ge 1 + \delta \max(|a_n|, |a_{n+1}|)$

$$\limsup_{|I|\to 0} \Big\| \sum_{n=1}^{\infty} a_n e(\nu_n x) \Big\|_I \le \frac{3}{\delta}\sqrt[3]{12\pi}.$$

$$\sum_{n=1}^{\infty} \frac{e(n^k x)}{n} \in \mathrm{BMO} \quad \text{with some control on its “BMO norm”.}$$

Paradox: Bigger gaps $\to$ closer to vanishing mean oscillation.

Converging to something

Our results (joint with Córdoba and Ubis 2019)

$$F(x) = \sum_{n=1}^{\infty} \frac{e(n^2 x)}{n}$$

Rough formulation

Theorem 1. Full characterization of the convergence set of the series.

Seuret, Ubis 2017 (Ann. Inst. Fourier 67, no. 5, 2237–2264) proved the convergence and the divergence for classes of irrational values.

Theorem 2. Precise estimation of the oscillation in small intervals.

John, Nirenberg 1961 (Comm. Pure Appl. Math. 14 415–426) proved that $\frac{1}{|I|}\big|\{x \in I : |f(x) - f_I| > \lambda\}\big|$ is $O\big(e^{-C\lambda}\big)$ for BMO functions.

Our results (joint with Córdoba and Ubis 2019)

Exact formulation

Theorem 1. For x irrational, the series converges if and only if

$$\frac{1}{2}\sum_{j=1}^{\infty}\frac{\theta_{p_j/q_j}}{\sqrt{q_j}}\log\frac{q_{j+1}}{q_j} \quad \text{does.}$$

In fact the difference between F and this sum is bounded in the convergence set. Here p_j/q_j are the convergents of the continued fraction of x and θ_{p_j/q_j} are the normalized Gauss sums.

- $\theta_{p_j/q_j} \in \{0, \pm 1, \pm i, \pm 1 \pm i\}$. Then $\frac{\theta_{p_j/q_j}}{\sqrt{q_j}}$ has an exponential decay.
- Corollary (Seuret, Ubis). If $q_{j+1} \neq o\big(q_j \exp\big(q_j^{1/2}\big)\big)$ for $2 \nmid q_j$, it diverges.

Our results (joint with Córdoba and Ubis 2019)

Exact formulation

Theorem 2. There are constants $c_1, c_2, C > 0$ such that

$$C^{-1}e^{-c_2\lambda\sqrt{q}} \le \frac{1}{|I|}\big|\{x \in I : |F(x) - F_I| > \lambda\}\big| \le Ce^{-c_1\lambda\sqrt{q}}$$

for I the interval of real numbers with continued fraction extending that of p/q. In fact $c_1 = c_2$ if $4 \mid q$.

- $|I|$ is comparable to and interval centered at p/q of length q^{-2}.
- The upper bound still holds for any subinterval of I.
- The oscillation depends on the Diophantine approximation, being exponentially small on the denominator in the Farey dissection.

A modular forms as Poisson summation in disguise

> *Poisson summation for number theory is what a car is for people in modern communities –it transports things to other places and it takes you back home when applied next time– one cannot live without it.*

H. Iwaniec, E. Kowalski

Analytic Number Theory

Some avatars of the invariance of Gaussians under the Fourier transform:

$$\sum_{n\in\mathbb{Z}} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}$$

$\theta(z)$ is a half weight modular form under the theta group

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$$\sum_{n\le N}{}' e\Big(\frac{1}{2}n^2x\Big) = \frac{e(1/4)}{\sqrt{x}} \sum_{n\le Nx}{}' e\Big(-\frac{n^2}{2x}\Big) + O\Big(\frac{1}{\sqrt{x}}\Big)$$

The last one establishes the connection with continued fractions.

Fiedler, Jurkat, Körner. Acta Arith. 32 (1977), 129–146.

Hardy, Littlewood. Acta Math. 37:155–191, 193–239, 1914.

Poisson summation in arithmetic progressions proves

$$\sum_{q\le n<Q} e(n^2x) = \frac{\theta_{p/q}}{\sqrt{q}} \int_q^Q e(ht^2)\,dt + O(q^{1/2})$$

with p/q, P/Q are consecutive convergents of x and $h = x - p/q$.

Replacing Q by $N < Q$, applying partial summation

$$\sum_{q\le n<Q} \frac{e(n^2x)}{n^2} = \frac{\theta_{p/q}}{\sqrt{q}} \int_q^Q \frac{e(ht^2)}{t}\,dt + O(q^{-1/2}).$$

Outcome:

Characterization of the convergence.

Just take $q = q_j$, $Q = q_{j+1}$, estimate the integral and sum in j.

The average F_I gives a blur version of F at scale $q^{-2} = q_{j_0}^{-2}$, the size of I.

The uncertainty principle suggests that $\sum_{n^2<q^2} e(n^2x)/n^2$ contains alike information.

If this is so, the oscillation is represented by $\sum_{n>q} e(n^2x)/n^2$.

Outcome:

$$F(x) - F_I = \frac{1}{2}\sum_{j\geq j_0} \frac{\theta_{p_j/q_j}}{\sqrt{q_j}} \log\frac{q_{j+1}}{q_j} + O\Big(\frac{1}{\sqrt{q_{j_0}}}\Big) \qquad I = I_{p_{j_0}/q_{j_0}}.$$

Continued and dis-continued fractions

Two metric results after Khinchin

Although the partial quotients in the continued fraction are not exactly independent, they behave like that.

$I = I_{p/q}$, $\frac{p}{q} = \frac{p_{j_0}}{q_{j_0}}$

So we can control the measure when we specify a partial quotient...

Lemma. The measure of the $x \in I$ with a partial quotient $a_j = k$ for some fixed $j > j_0$ is comparable to $k^{-2}|I|$

Or even a tail sequence...

Lemma.
If $A_n \ge 1$ and $S = \sum A_n^{-1} < \infty$ then

$$e^{c_1 S} \le \frac{1}{|I|}\big|\{x \in I \ : \ a_{j_0+n}(x) \le A_n \text{ for } n \in \mathbb{Z}^+\}\big| \le e^{c_2 S}.$$

Measuring vanishing sets

Recall

$$F(x) - F_I = \frac{1}{2}\sum_{j\ge j_0} \frac{\theta_{p_j/q_j}}{\sqrt{q_j}} \log \frac{q_{j+1}}{q_j} + O\Big(\frac{1}{\sqrt{q}}\Big) \qquad I = I_{p_{j_0}/q_{j_0}}.$$

We have $q_{j+1}/q_j = a_{j+1} + O(1)$ and according to the metric theory, the most of the $x \in I$ have not outrageously large partial quotients $\{a_{j_0+n}\}_{n\ge 2}$.

Note that $1/\sqrt{q_j}$ shows, at least, a geometric decay

Commonly

$$F(x) - F_I \approx \frac{1}{2}\frac{\theta_{p_{j_0}/q_{j_0}}}{\sqrt{q}} \log a_{j_0+1}$$

and the approximation improves when a_{j_0+1} grows because it becomes more dominant.

Outcome:

Large values of $|F(x) - F_I|$ require exponentially large partial quotients and this happens only in an exponentially small proportion of the interval.

Technical point: If $\theta_{p_{j_0}/q_{j_0}}$ vanishes, the expected dominant term is shifted.

Je vous remercie de votre attention!