Antonio Córdoba and Number Theory

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The Poetry of Analysis

Conference in honour of Antonio Córdoba
on the occasion of his 60th birthday

Colegio mayor Juan Luis Vives
June 26, 2009

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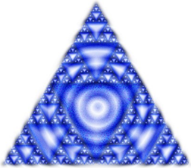
Harmonic analysis



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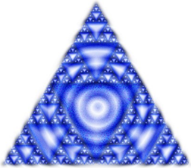

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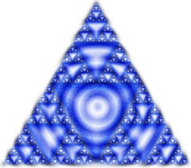


Strange series

Atoms

Irrational

Intertwining Number Theory and Harmonic Analysis

Kernels

Analysis	Number Theory
$1/x$	$\sum e^{2\pi i n^2 x}$
$\Omega(\vec{x})/\|\vec{x}\|^n$	$\sum e^{2\pi i p x}$, p prime

Kernels in analysis are simpler in nature with isolated singularities. The payback is that they act on huge class of (possibly singular) functions.

In analytic number theory the situation is commonly the opposite.

For instance, in some sense Hardy-Littlewood circle method is an approach to deal with *ultra-singular integrals*

$$\int_{C_r} \frac{1}{|1-z|} f(z)\, dz \qquad r_k(N) = \frac{1}{2\pi i}\int_{C_r} \Big(\sum_{p \text{ prime}} z^p\Big)^k z^{-N-1}\, dz$$

singular $\quad \xrightarrow{r\to 1^-} \quad$ ultra-singular

Kloosterman variant of the circle method gets rid of some theoretical analytic limitations in some special cases.

Minor and major arcs in circle method (arguably) have some resemblances with bad and good sets in Calderón-Zygmund decomposition.

Rudin's conjecture

Rudin 1960 (still open)

$$T\Big(\sum a_n e^{2\pi i n x}\Big) = \sum a_{n^2} e^{2\pi i n^2 x} \quad \text{sends } L^2 \text{ into } L^{4-\epsilon} \ (?)$$

Córdoba 1987

T sends $L^2 \cap \{f : a_n \downarrow 0\}$ into weak L^4.

$$T : L^2 \longrightarrow L^4 \Rightarrow T : f = N^{-1/4}\sum_{k=1}^{N} e^{2\pi i(a+qk)x} \in L^{4/3} \mapsto L^2$$

$$T(f) \in L^2 \Leftrightarrow \#\{a + qk = \square,\ 1 \le k \le N\} < CN^{1/2}$$

Bombieri, Granville, Pintz 1992, Bombieri, Zannier 2002

Arithmetic geometry + sieve $\rightarrow N^{3/5+\epsilon}$.

Restriction theorems (in $\mathbb{R}^2$)

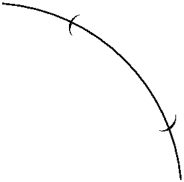
How bad/good can Fourier transform be when we restrict it to a curve?

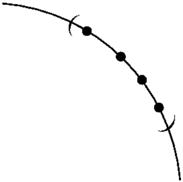
Example: $f \in L^{6/5}(\mathbb{R}^2) \Rightarrow \widehat{f}\big|_{S^1} \in L^2(S^1)$.

In the Fourier series analog a concentration of lattice points in arcs would impose a limit.

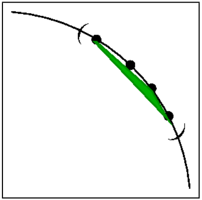
Weak forms of this problem appears in the Cantor-Lebesgue theorem in $\mathbb{T}^2$ (Zygmund).

Continuous versus discrete





Lattice points on the circle



$$\#\text{points} > m$$

$$\Downarrow$$

$$\text{length} > R^{1/2-1/(4[m/2]+2)}$$

Córdoba, Cilleruelo 1992

On the circle $\|\vec{x}\| = R$ an arc of length less than $\sqrt{2}R^{1/2-1/(4[m/2]+2)}$ contains at most m lattice points.

The same holds for (rational) ellipses.

Córdoba, Cilleruelo 1992

For $1/2 < \alpha < 1$

$$\Big\| \sum_{N \le n \le N+N^\alpha} e^{2\pi i n^2 x} \Big\|_4 = 2^{1/4} N^{\alpha/2} + O\big(N^{(3\alpha-1)/4+\epsilon}\big).$$

Conjecture:

$$\Big\| \sum_{N \le n \le N+N^\alpha} a_n e^{2\pi i n^2 x} \Big\|_4 \overset{?}{\le} C \Big\| \sum_{N \le n \le N+N^\alpha} a_n e^{2\pi i n^2 x} \Big\|_2.$$

Geometric approach to Fourier analysis problems:

- A. Córdoba. Geometric Fourier analysis. Ann. Inst. Fourier (Grenoble) 32 (1982), no. 3, vii, 215–226.
- A. Córdoba. A note on Bochner-Riesz operators. Duke Math. J. 46 (1979), no. 3, 505–511.
- A. Córdoba. The Kakeya maximal function and the spherical summation multipliers. Amer. J. Math. 99 (1977), no. 1, 1–22.
- A. Córdoba. The multiplier problem for the polygon. Ann. of Math. (2) 105 (1977), no. 3, 581–588.

$\dim = 2$

Functions $\longrightarrow$ union of rectangles

Fourier transforms $\longrightarrow$ union of dual rectangles

Snapshot from *The Kakeya maximal function...*

Remark 1. Part (d) of the proof of Proposition 1.2 admits the following description: Suppose that we have a square room Q of side 1, and we want to illuminate the side AB with beams of light placed on the opposite side CD.

Suppose that our beams have width N^{-1} (i.e., each one illuminates only an interval of length N^{-1} on AB) but we can place them arbitrarily on CD and also we have freedom to choose the direction of the light for each beam (Fig. 2). Then, if the whole wall AB is illuminated and if P is the portion of room illuminated, we have the estimate:

$$|P| \geqslant \frac{1}{\operatorname{Log} N}$$

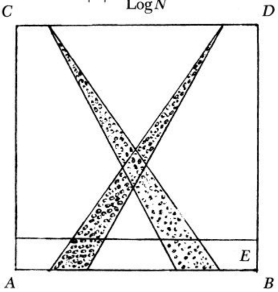
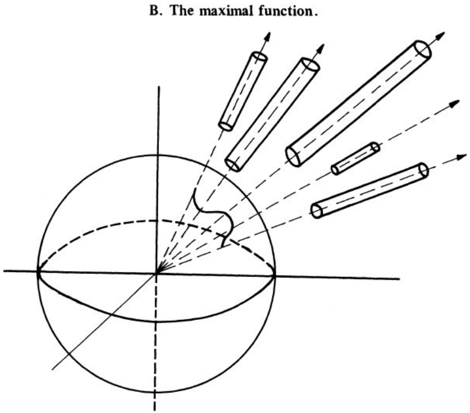


Fig. 2

Snapshot from *Geometric Fourier analysis...*



Snapshot from *The multiplier problem for the polygon*

Proof of the claim. Given a bad rectangle $R \in F_\alpha$ with eccentricity $\geq N$, let $\tilde{R}$ be the rectangle containing R, such that: i) the direction of R = the direction of $\tilde{R}$; ii) the length of the longer side of $\tilde{R}$ = twice the length of the longer side of R; iii) eccentricity of $(\tilde{R}) = N$ (see Figure 1).

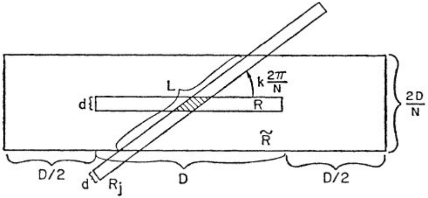
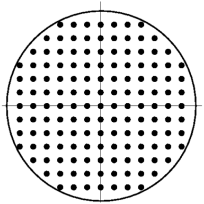
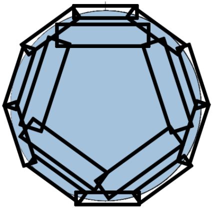


FIGURE 1

These geometric ideas lead readily to the *trivial exponent* in lattice point problems





Geometry of dual rect. $\longrightarrow \#\{\vec{n} \in \mathbb{Z}^2 \ : \ \|\vec{n}\| < R\} = \pi R^2 + O(R^{2/3})$

In Bochner-Riesz problem (*spherical* summation of Fourier series) the optimal result requires to study the geometry of intersections of sums of rectangles.

F. Chamizo and A. Córdoba.
Lattice points. In *Margarita mathematica*, pages 59–76. Univ. La Rioja, Logroño, 2001.

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$B_2[\infty]$-sequences of square numbers. *Acta Arith.*, 61(3):265–270, 1992.

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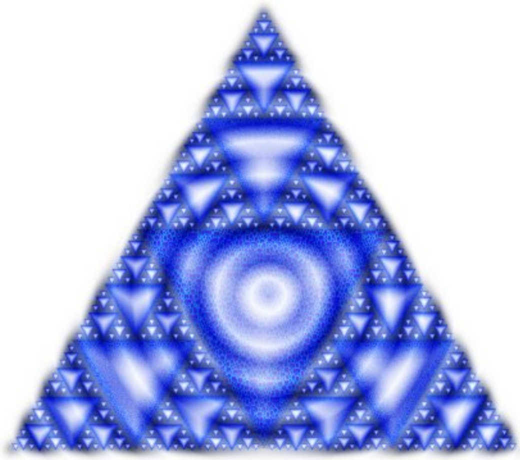
Trigonometric polynomials and lattice points. *Proc. Amer. Math. Soc.*, 115(4):899–905, 1992.

J. Cilleruelo and A. Córdoba.
Lattice points on ellipses. *Duke Math. J.*, 76(3):741–750, 1994.

A. Córdoba.
Lattice points. In *Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996)*, volume 3, pages 859–870, 1997.

Strange Fourier series



What is the right way of summing Fourier series?

- Sharp cut of the series (Carleson's theorem)
- Smoothing the summation (Fejér's theorem)

From a physical point of view we can consider Fourier coefficients as measurements affected by uncertainty in such a way that zero and close to zero coefficients are indistinguishable

Physical summations?

$$\lim_{\lambda\to 0^+} \sum_{|a_n|>\lambda} a_n e^{2\pi i n x}, \qquad \lim_{\lambda\to 0^+} \sum_{\substack{|a_n|>\lambda^\delta \\ |n|<\lambda^{-1}}} a_n e^{2\pi i n x}$$

Körner 1996

There exists $f \in L^2$ such that $\lim_{\lambda\to 0^+} \sum_{|a_n|>\lambda} a_n e^{2\pi i n x} = \infty$ a.e.

Córdoba, Fernández-Gallardo 1996

- Explicit example $f \in L^p$, $p < 4/3$ using Gauss sums.
- Explicit example $\|\mathcal{M}f\|_p = \infty$ (maximal function), for $1 \le p < 2$ using sums of primes.

Córdoba, Fernández-Gallardo 1996

$$f(x) = \sum_{k=0}^{\infty} 2^{-k/2+\epsilon k} \sum_{n=2^{2k}}^{2^{2k+2}-1} a_n \cos(2\pi nx)$$

where

$$a_n = 1 + \frac{1}{n} \quad \text{if } n = \square, \qquad a_n = 1 + \frac{1}{2^{2k}} \quad \text{if } n \neq \square$$

satisfies

$$\lim_{\lambda \to 0^+} \sum_{|\hat{f}(n)|>\lambda} \hat{f}(n) e^{2\pi i n x} = \infty \quad \text{a.e.}$$

and $f \in L^p$, $p < 4/3$ depending on ϵ.

Fourier series

General:

$$\sum a_n e^{2\pi inx}, \quad n \in \mathbb{Z}$$

- **Lacunary:**

$$\sum a_n e^{2\pi i f_n x}, \quad f_{n+1}/f_n > c > 1$$

  Good behaviour ($L^2 \Rightarrow L^p$)

- **Sublacunary:**

$$f_{n+1} - f_n \to \infty, \quad f_{n+1}/f_n \to 1$$

  e.g., f_n polinomial. Not well understood. Related to arithmetical problems: distribution of squares in arithmetic progressions, etc.

Chamizo, Córdoba 1999

Polynomial frequencies $\Rightarrow$ Global caotic behaviour

$$F(x) = \sum_{n=1}^{\infty} a_n e^{2\pi i n^k x} \qquad 0 < \limsup,\ \liminf\ n^{\alpha} a_n < \infty$$

- The graphs of $\Im F$ are $\Re F$ are fractal sets of dimension

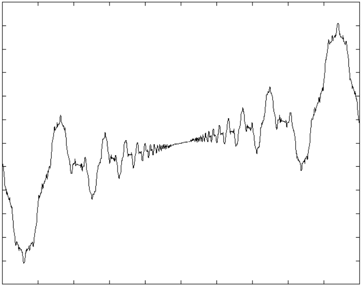
$$\dim = \max\left(1, 2 + \frac{1-2\alpha}{2k}\right) \qquad \text{for } \alpha \geq \frac{k+2}{4}$$

- Under certain arithmetical conjectures (sharp Hua's inequality) the formula is valid for every α (> 1).

$$F(x) = \sum_{n=1}^{\infty} a_n e^{2\pi i n^k x} \qquad n^\alpha a_n \to l \neq 0, \infty \quad \text{monotonic}$$

Determination of the local behaviour in $\mathbb{Q}$

F is differentiable at a/q (irreducible fraction) $\Leftrightarrow \alpha > k - 1/2$ and for some prime power $p^\gamma || q$ we have $\gcd(k, p-1) = 1$ and $k | \gamma - 1$.



F. Chamizo and A. Córdoba.
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Riemann fractals: numbers and figures. *Gac. R. Soc. Mat. Esp.*, 1(1):37–47, 1998.

F. Chamizo and A. Córdoba.
Differentiability and dimension of some fractal Fourier series. *Adv. Math.*, 142(2):335–354, 1999.

A. Córdoba and P. Fernández.
Convergence and divergence of decreasing rearranged Fourier series. *SIAM J. Math. Anal.*, 29(5):1129–1139, 1998.

Atomic Number Theory!

Atomic pictures

Classical

Quantum

Hamiltonian and ground state energy (electrons as bosons)

$$\mathcal{H} = -\sum_{i=1}^{Z}\left(\Delta_{x_i} + \frac{Z}{\|x_i\|}\right) + \frac{1}{2}\sum_{i\neq j}\frac{Z}{\|x_i - x_j\|}$$

Z = atomic number

$$E(Z) = \inf_{\|\Psi\|=1}\langle \mathcal{H}\Psi, \Psi\rangle.$$

Córdoba, Fefferman, Seco 1995

$$E(Z) = C_{\mathrm{TF}} Z^{7/3} + C_{\mathrm{Sc}} Z^2 + C_{\mathrm{SD}} Z^{5/3} + \phi(Z) + \dots$$

with $\phi(Z) =$ sum of fractional parts.

Rough analogy:

Potential well $\longrightarrow -\Psi'' - U_0\Psi = E\Psi, \quad \Psi(0) = \Psi(\pi) = 0,\ E > 0$

$\Psi = A\sin(x\sqrt{E + U_0}), \qquad E_0 = 2\sqrt{U_0} + 1 + f^2 - 2f(\sqrt{U_0} + 1),$

$f =$ fractional part of $\sqrt{U_0}$.

A method in Analytic Number Theory

$$\sum_{n=1}^{N} d(n) \quad \xrightarrow{n=k\cdot l} \quad \sum_{k=1}^{N} [N/k]$$

Fourier expansion of $[x] - x$ + van der Corput method $\longrightarrow$ bound for the error term.

Córdoba, Fefferman, Seco 1995

Extra regularity allows to get a sharp result:

- $|\phi(Z)| = O\big(Z^{3/2}\big)$
- $|\phi(Z)| = \Omega\big(Z^{3/2}\big) \longrightarrow$ No new main term.

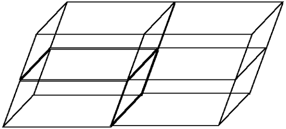
Dirac Combs — Córdoba 1989

Every "crystalline" Poisson summation formula in $\mathbb{R}^n$ is the usual one up to linear transformations.

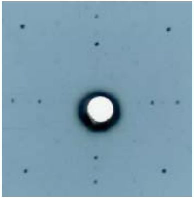
Molecules with a finite number of types of atoms — Bragg's peaks

$$\mu = \sum_{j=1}^{N} a_j \sum_{x\in\Lambda_j} \delta_x \qquad \text{and} \qquad \widehat{\mu} = \sum_k b_k \delta_{y_k}$$

$\Longrightarrow$ each Λ_j is a finite union of lattices.



$\Longrightarrow$



$\underset{?}{\Longleftarrow}$

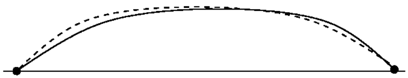
Periodic Dirac deltas

Dirac deltas

Another motivation from Physics

Usual average results in lattice point problems (dilations, translations) are actually averages on the lattice.

Is it possible to perform a real average on the curve?.



Jarník proved that the trivial exponent is best possible for particular convex arcs. Is it so in *general*?

Chamizo, Córdoba 2002

Keeping in mind Feynmann's path integral formulation of Quantum Mechanics the error term for *random arcs* is optimal.

F. Chamizo and A. Córdoba.
A path integral approach to lattice point problems. *J. Math. Pures Appl. (9)*, 81(10):957–966, 2002.

A. Córdoba, C. Fefferman, and L. Seco.
A trigonometric sum relevant to the nonrelativistic theory of atoms. *Proc. Nat. Acad. Sci. U.S.A.*, 91(13):5776–5778, 1994.

A. Córdoba, C. L. Fefferman, and L. A. Seco.
A number-theoretic estimate for the Thomas-Fermi density. *Comm. Partial Differential Equations*, 21(7-8):1087–1102, 1996.

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Weyl sums and atomic energy oscillations. *Rev. Mat. Iberoamericana*, 11(1):165–226, 1995.

Irrational and rational thoughts

A proof that A. Córdoba did not miss

R. Apéry proved in 1978 that $\zeta(3) \notin \mathbb{Q}$.

F. Beukers got a simple proof in 1979, a proof that was also obtained independently by A. Córdoba.

For $P, Q \in R[x, y]$ there exist $A, B \in R$

$$\int_0^1 \int_0^1 \frac{P(x,y)}{1-xy} dxdy = A\zeta(2) + B$$

$$\int_0^1 \int_0^1 \frac{Q(x,y)}{1-xy} \log(xy)\, dxdy = A\zeta(3) + B$$

Proof: expand $(1-r)^{-1} = 1 + r + r^2 + \dots$ and integrate.

$$I_Q := \int_0^1 \int_0^1 \frac{Q(x,y)}{1-xy} \log(xy)\, dxdy = A\zeta(3) + B$$

$$P \in Z[x,y] \Longrightarrow A, B \in \mathbb{Z}$$

$$\text{Wise choice of } Q \in \{Q_n\}_{n=1}^{\infty} \Longrightarrow I_Q \to 0 \text{ and } I_Q \neq 0$$

$$\Longrightarrow \zeta(3) \notin \mathbb{Q}$$

A sequence of integers converging to zero is eventually constant

Easy proof of $\zeta(2) = \pi^2/6$

This kind of manipulations have remained along the years...

Córdoba 2001

$$\zeta(2) = \frac{1}{3}\int_{-1}^{1}\int_{-1}^{1}\frac{dxdy}{1-x^2y^2} \qquad \text{(expand } (1-r)^{-1} = 1 + r + r^2 + \dots\text{)}$$

Change of variables $x = \tanh\frac{s+t}{2}$, $y = \tanh\frac{s-t}{2}$

$$\zeta(2) = \frac{1}{6}\int_{-\infty}^{\infty}\frac{ds}{\cosh s}\cdot\int_{-\infty}^{\infty}\frac{dt}{\cosh t} = \frac{1}{6}\Big(\int_0^{\infty}\frac{2\,du}{1+u^2}\Big)^2 = \frac{\pi^2}{6}.$$

J. Cilleruelo and A. Córdoba.
La Teoría de los Números. Mondadori, Madrid, 1992.

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This talk closes the conference

The Poetry of Analysis

Thank you, Antonio!