Post-Newtonian approximations

Fernando Chamizo

Master Theoretical Physics
Universidad Autónoma de Madrid
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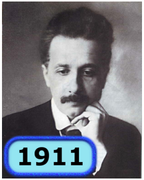
Einstein's double prediction



A ray of light grazing the Sun suffers a deflection of about

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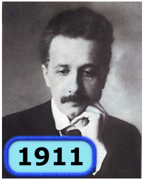
(Annalen der Physik 35: 898–908, 1911)

Luckily the experiment was not done (pre-war crisis, weather) because. . .

Einstein's double prediction



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A ray of light grazing the Sun suffers a deflection of about

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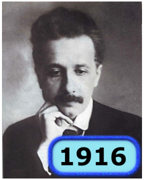
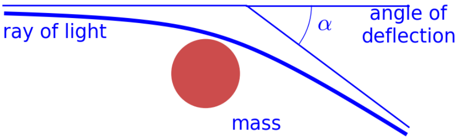
(Annalen der Physik 46: 769–822, 1916)



Actually the first prediction was done by J.G. von Soldner (more than a century before Einstein!) just assuming that light is affected by Newtonian gravitation as any particle.



von Soldner 1804	$\rightarrow \alpha \sim 2GM/R_0$
Einstein 1911	$\rightarrow \alpha \sim 2GM/R_0$
Einstein 1916	$\rightarrow \alpha \sim 4GM/R_0$

Newtonian vs Post-Newtonian

Looking into retrospective, we can interpret today this double prediction saying that in the first prediction the metric of space-time was approximated by

$$(1-2\mathcal{U})dt^2 - (dx^2+dy^2+dz^2) \qquad \text{with} \quad \mathcal{U} = \frac{GM}{r}$$

and in the second by

$$(1-2\mathcal{U}+2\mathcal{U}^2)dt^2 - (1+2\mathcal{U})(dx^2+dy^2+dz^2).$$

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Both are approximations to the actual (Schwarzschild's) metric but the first one does not go beyond the Newtonian (von Solder's) results and the second, more precise, is **post-Newtonian**. It doubles the prediction although $\mathcal{U}$ is small in non-relativistic units.

$$\boxed{\text{General relativity}} \qquad\qquad \boxed{\text{Newtonian theory}}$$

$$\begin{tikzcd}
\dfrac{Du}{d\tau} = 0 \arrow[r, "v/c\to 0"] \arrow[d, "G \to 0"] & \dfrac{d\vec{v}}{dt} = -\nabla\Phi \arrow[d, "G \to 0"] \\
\dfrac{du}{d\tau} = 0 \arrow[r, "v/c\to 0"] & \dfrac{d\vec{v}}{dt} = 0
\end{tikzcd}$$

(relativistic kinematics) (Newtonian kinematics)

$$
\begin{tikzcd}[column sep=huge]
\boxed{\text{General relativity}} & & \boxed{\text{Newtonian theory}} \\
R_{\alpha\beta} = 0 & & \Delta\Phi = 0 \arrow[ll, "\text{geod. deviat.}"'] \\
\dfrac{Du}{d\tau} = 0 \arrow[rr, "v/c\to 0"] \arrow[d, "G \to 0"] & & \dfrac{d\vec{v}}{dt} = -\nabla\Phi \arrow[d, "G \to 0"] \\
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\text{(relativistic kinematics)} & & \text{(Newtonian kinematics)}
\end{tikzcd}
$$

Before doing any approximation there are two questions to answer:

1. How far should we approximate the metric?
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2. What kind of quantities should we use in the approximations?

We work under the assumption of weak field and slow motion of the gravitational sources. We want to employ a nearly global Minkowskian coordinate system (t, x, y, z).
The usual special relativistic action for a particle is

$$S = -m \int \sqrt{g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}}.$$

In terms of $v^i = dx^i/dt$

$$S = -m\int L\,dt \qquad \text{with} \quad L = \sqrt{g_{00} + 2g_{0j}v^j + g_{jk}v^j v^k}.$$

The Newtonian limit is $g_{00} = 1 - 2\mathcal{U}$, $g_{0j} = 0$, $g_{ij} = -\delta_{ij}$, hence

$$L = \sqrt{1 - 2\mathcal{U} - v^2}.$$

Therefore the Newtonian limit corresponds to terms $O[2]$ in L, where $O[n]$ means $O(v^n)$. Terms $O[3]$ do not appear in realistic models (loss of energy) and we skip to $O[4]$ that corresponds to

$$g_{00} = 1 - 2\mathcal{U} + O[4], \qquad g_{0j} = O[3], \qquad g_{ij} = -\delta_{ij} + O[2].$$

The approximations are done in terms of the *matter variables* ρ, Π (internal energy), v and p (pressure), with $p/\rho = \Pi = O[2]$.

The PPN formalism

Each metric theory has its own post-Newtonian approximations and the way of finding them can be long and difficult and completely different in each case.

Although these potentially big differences, the final expressions keep the same flavor.

The **parametrized post-Newtonian formalism** (PPN) is a comprehensive family of post-Newtonian approximations depending on parameters that can be adjusted to represent, in practice, every meaningful metric theory of gravitation.

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2. Standardize the way of presenting the experimental results.
3. (?) Compute general relativity corrections in astrophysics.

The first point is the main one. Theories sharing the same PPN parameters could only be distinguished with *post-post-Newtonian* experiments that are commonly out of reach with current methods because the PPN formalism mimics relativistic theories with high accuracy.

The modern formalism, uses ten parameters γ, β, ξ, α_1, α_2, α_3, ζ_1, ζ_2, ζ_3 and ζ_4 having physical meaning, and the metric

$$\begin{cases} g_{00} = 1 - 2\mathcal{U} + \lambda_1\mathcal{U}^2 + \lambda_2\Phi_W + \lambda_3\Phi_1 + \lambda_4\Phi_2 \\ \qquad +\lambda_5\Phi_3 + \lambda_6\Phi_4 + \lambda_7\mathcal{A} \\ g_{0j} = \lambda_8 V_j + \lambda_9 W_j \\ g_{ij} = -\delta_{ij} + \lambda_{10}\mathcal{U}\delta_{ij} \end{cases}$$

where

$$\begin{cases} \lambda_1=2\beta \\ \lambda_2=2\xi \\ \lambda_3=-2\gamma-2-\alpha_3-\zeta_1+2\xi \\ \lambda_4=-2(3\gamma-2\beta+1+\zeta_2+\xi) \\ \lambda_5=-2(1+\zeta_3) \end{cases} \qquad \begin{cases} \lambda_6=-2(3\gamma+3\zeta_4-2\xi) \\ \lambda_7=\zeta_1-2\xi \\ \lambda_8=\frac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1-2\xi) \\ \lambda_9=\frac{1}{2}(1+\alpha_2-\zeta_1+2\xi) \\ \lambda_{10}=-2\gamma \end{cases}$$

$$\begin{cases} g_{00} = 1 - 2\mathcal{U} + \lambda_1 \mathcal{U}^2 + \lambda_2 \Phi_W + \lambda_3 \Phi_1 + \lambda_4 \Phi_2 \\ \qquad + \lambda_5 \Phi_3 + \lambda_6 \Phi_4 + \lambda_7 \mathcal{A} \\ g_{0j} = \lambda_8 V_j + \lambda_9 W_j \\ g_{ij} = -\delta_{ij} + \lambda_{10} \mathcal{U} \delta_{ij} \end{cases}$$

The functions of accompanying the coefficients are the *potentials* of the formalism. The simplest one is the Newtonian potential

$$\mathcal{U} = \int \frac{\rho(\vec{y}, t)}{|\vec{x} - \vec{y}|} \, d^3\vec{y}$$

and the most complicate (the variable t is not displayed)

$$\Phi_W = \int \rho(\vec{y})\rho(\vec{z}) \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \cdot \left(\frac{\vec{y} - \vec{z}}{|\vec{x} - \vec{z}|} - \frac{\vec{x} - \vec{z}}{|\vec{y} - \vec{z}|} \right) d^3\vec{y} \, d^3\vec{z}.$$

The parameter ξ that multiplies this latter potential was introduced historically to cover a gravitational theory proposed by A.N. Whitehead. The parameters α_1, α_2 and α_3 measure the existence of a *universal rest frame*. On the other hand ζ_1, ζ_2, ζ_3 and ζ_4 (and in part α_3) are related to the violation of global conservation laws.

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When all of these parameters are set to zero (as in the case of general relativity) the PPN formalism reduces considerably. If also $p = \Pi = 0$ (as happen with point masses) and the gravitational sources are static, only the terms with $\mathcal{U}$ survive. In this case, we have

$$\big(1 - 2\mathcal{U} + 2\beta\mathcal{U}^2\big)dt^2 - \big(1 + 2\gamma\mathcal{U}\big)\big(dx^2 + dy^2 + dz^2\big).$$

If we do not assume static sources, we have to add to g_{00} the term

$$-2(\gamma+1)\int \frac{\rho(\vec{y})v^2(\vec{y})}{|\vec{x}-\vec{y}|}d^3\vec{y}$$

and cross terms appear

$$g_{0j}=\frac{1}{2}(4\gamma+3)V_j+\frac{1}{2}W_j$$

where

$$V_j=\int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}v_j(\vec{y})\ d^3\vec{y}$$

and

$$W_j=\int \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|}\frac{x_j-y_j}{|\vec{x}-\vec{y}|}\frac{\vec{x}-\vec{y}}{|\vec{x}-\vec{y}|}\cdot\vec{v}(\vec{y})\ d^3\vec{y}.$$

For general relativity

$$\gamma = \beta = 1, \qquad \text{rest of the parameters} = 0$$

$$\big(1 - 2\mathcal{U} + 2\beta\mathcal{U}^2\big)\, dt^2 - \big(1 + 2\gamma\mathcal{U}\big)\big(dx^2 + dy^2 + dz^2\big)$$

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Example: Schwarzschild's isotropic metric, $r \mapsto r(1 + GM/2r)^2$,

$$\Big(\frac{1 - GM/2r}{1 + GM/2r}\Big)^2 dt^2 - \Big(1 + \frac{GM}{2r}\Big)^4 \big(dx^2 + dy^2 + dz^2\big)$$

and use

$$\Big(\frac{1 - x/2}{1 + x/2}\Big)^2 = 1 - 2x + 2x^2 + O(x^3), \quad \big(1 + \frac{x}{2}\big)^4 = 1 + 2x + O(x^2)$$

with $x = \mathcal{U} = GM/r$.

The classical tests

Usually the following facts are considered the classical tests of general relativity:

(i) **The gravitational red shift**
(ii) **The deflection of light**
(iii) **The perihelion shift of Mercury**

The classical tests

Usually the following facts are considered the classical tests of general relativity:

(i) **The gravitational red shift**
(ii) **The deflection of light**
(iii) **The perihelion shift of Mercury**

Actually (i) is not a (good) test of GR because it is a consequence of the equivalence principle. It holds for any reasonable (metric) theory.

(ii) with the right deflection and (iii) are Post-Newtonian but. . .

Deflection of light

L. Schiff:
The deflection of light is a consequence of the equivalence principle and special relativity.

(American Journal of Physics 28: 340–343, 1960)

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And 8 years latter...

W. Rindler:
The deflection of light cannot be derived from the equivalence principle and special relativity.

(American Journal of Physics 36: 540–544, 1968)

In the PPN formalism the equations of motion for light are

$$\begin{cases} 1 - 2\mathcal{U} - \left|\frac{d\vec{x}}{dt}\right|^2 (1 + 2\gamma\mathcal{U}) = 0 \\ \frac{d^2x^j}{dt^2} = \frac{\partial\mathcal{U}}{\partial x^j}\left(1 + \gamma\left|\frac{d\vec{x}}{dt}\right|^2\right) - 2\frac{dx^j}{dt}\left(\frac{d\vec{x}}{dt}\cdot\nabla\mathcal{U}\right)(1+\gamma) \end{cases}$$

First approximation $\rightarrow$ Newtonian equation $\frac{d^2\vec{x}}{dt^2} = \frac{\partial\mathcal{U}}{\partial x^j}$

It only depends on γ then the deflection of light is a good method to measure this parameter.

Tracking of Cassini space probe $\rightarrow \gamma = 1 \pm 2.3 \cdot 10^{-5}$

Perihelion shift

The PPN formalism with $\alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = 0$ or under the assumption of a negligible planet mass, gives

$$\alpha \sim \pi GM\Big(\frac{1}{r_a} + \frac{1}{r_p}\Big)\Big(2 + 2\gamma - \beta\Big) + \frac{3\pi}{4} J_2 R^2 \Big(\frac{1}{r_a} + \frac{1}{r_p}\Big)^2$$

where J_2 = quadrupole moment, R = mean radius of the Sun.

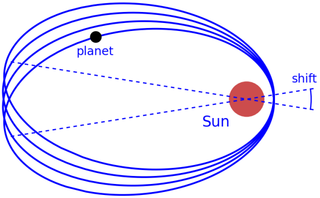
A possible discrepancy between GR and the experimental solar oblateness found by R.H. Dicke raised a long controversy during the 60's and early 70's.

Mercury observations $\rightarrow |2\gamma - \beta - 1| < 3 \cdot 10^{-3}$.

The PN approximation of GR gives a "rotation of the ellipse" of angle

$$\alpha \sim 3\pi GM\left(\frac{1}{r_a}+\frac{1}{r_p}\right)$$

per each orbital period, where r_a and r_p are the (distances to) the aphelion and perihelion.



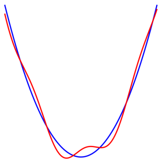
General relativity predictions

Mercury	$\rightarrow$	42.98" /century
Venus	$\rightarrow$	8.63" /century
Earth	$\rightarrow$	3.84" /century
Mars	$\rightarrow$	1.35" /century

In many books of general relativity it is assumed a nearly circular orbit in the perihelion shift. This is not very convincing:

1. The main interest is to apply it to Mercury and its orbit is clearly eccentric.
2. In a nearly circular orbit the perihelion is too sensitive to approximations.

In the perihelion a minimum is attached. Minimum attaching points are not well localized when using flat function approximations:



It is better to use that for every cubic polynomial

$$P(x) = \epsilon x^3 - bx^2 + cx - d \quad \text{with } \epsilon, b > 0$$

and three real roots $0 < x_1 < x_2 < x_3$ we have the approximation

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{P(x)}} \sim \frac{\pi}{\sqrt{b}} + \frac{3\pi}{4b^{3/2}}(x_1 + x_2)\epsilon$$

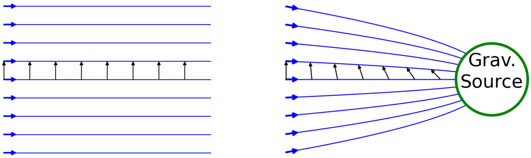
which is valid whenever $\epsilon x_2/b$ is small.

In the application, $\epsilon = 2GM$ for GR and $\epsilon = 2AGM$ (with A essentially the quadrupole moment over the square of the angular momentum of the planet) for the Newtonian theory.

Appendix: From Newton to Einstein

Parametrize the trajectories of dust of Newtonian test particles $\alpha(t,s) = \big(x(t,s), y(t,s), z(t,s)\big)$ and consider the vector fields that points to the "adjacent" particle

$$\alpha(t,s) = \big(x(t,s), y(t,s), z(t,s)\big) \qquad \vec{V} = \frac{\partial \alpha}{\partial t}(t,0).$$



Appendix: From Newton to Einstein

The, so to speak, relative acceleration is

$$\frac{d^2V^i}{dt^2} = -\delta^{ij}\frac{\partial}{\partial s}\Big|_{s=0}\partial_j\Phi\big(\alpha(t,s)\big) = -\delta^{ij}\big(\partial_k\partial_j\Phi\big)V^k.$$

Defining $A^i_k = \delta^{ij}\partial_k\partial_j\Phi$ we can re-write this as

$$\frac{d^2V^i}{dt^2} + A^i_kV^k = 0 \qquad \text{with} \quad A^i_i = 4\pi G\rho.$$

Comparing with the geodesic deviation formula

$$\frac{D^2V^\alpha}{dt^2} + \mathcal{A}^\alpha_\gamma V^\gamma \qquad \text{where} \quad \mathcal{A}^\alpha_\gamma = R^\alpha_{\beta\gamma\delta}\dot{x}^\beta\dot{x}^\delta$$

one can infer (imagine) Einstein's field equations.