Electron anomalous magnetic moment: history and current status

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Prehistory

Schrödinger equation (1925–1926)

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

Hydrogen atom

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r}$$

$g = 0$?

normal Zeeman effect is OK but . . . there is an anomalous Zeeman effect.

Last line in his famous paper: *in what way the electron spin has to be taken into account in the present theory is yet unknown*.

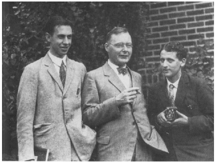
The spin enters into play (1925)

The electron behaves as a magnet. The introduction of the spin of the electron was motivated by spectroscopy (Zeeman effect) not by the Stern-Gerlach experiment.

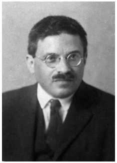
Who discovered/invented the spin of the electron?
R. Kronig was first but was criticized by W. Pauli, shortly after S. Goudsmit and G. Uhlenbeck arrived to similar ideas and they were supported by P. Ehrenfest.

R. Kronig

W. Pauli



G. Uhlenbeck, H. K., S. Goudsmit

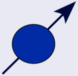


P. Ehrenfest

The right equation

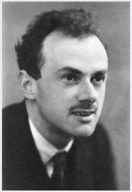
Something strange: Does the electron really spin?

$$\vec{\mu} = g\frac{e}{2m}\vec{S}$$



Classic electrodynamics suggests $g = 1$ but it is not true!

Dirac equation (1928)



It is the "square root" of the Klein-Gordon equation

$$(i\hbar\partial\!\!\!/ - m)\psi = 0, \qquad \partial\!\!\!/ = \gamma^\mu \partial_\mu$$

relativistic and first order.

Complications: it is spinorial, ψ has 4 components.

You should read this paper!

The Quantum Theory of the Electron.

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received January 2, 1928.)

The new quantum mechanics, when applied to the problem of the structure of the atom with point-charge electrons, does not give results in agreement with experiment. The discrepancies consist of " duplexity " phenomena, the observed number of stationary states for an electron in an atom being twice the number given by the theory. To meet the difficulty, Goudsmit and Uhlenbeck have introduced the idea of an electron with a spin angular momentum of half a quantum and a magnetic moment of one Bohr magneton. This model for the electron has been fitted into the new mechanics by Pauli,* and Darwin,† working with an equivalent theory, has shown that it gives results in agreement with experiment for hydrogen-like spectra to the first order of accuracy.

The right (approximate) g factor

Stepping back: Pauli equation (1927)

The spin suggests two coordinates for the wave functions (spin up and down). Pauli equation is a kind of variant of Schrödinger equation in this way with

$$H\psi = \frac{(\vec{\sigma}\cdot\hat{\mathbf{p}})^2}{2m}\psi$$

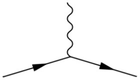
When $\hat{\mathbf{p}}$ is replaced by $\hat{\mathbf{p}} + e\mathbf{A}$ then, using the properties of Pauli matrices, one gets

$$\frac{(\hat{\mathbf{p}} + e\mathbf{A})^2}{2m} + \mathbf{2}\frac{e}{2m}\vec{S}\cdot\mathbf{B} \qquad \text{with} \quad \vec{S} = \frac{1}{2}\vec{\sigma}.$$

Hence, comparing to $\vec{\mu}\cdot\mathbf{B}$ we have $g = 2$.

The modern interpretation

1. The Pauli equation is the non-relativistic limit of the Dirac equation
   ψ_R and ψ_L collapse to give only two coordinates.
2. $\hat{\mathbf{p}} \longmapsto \hat{\mathbf{p}} + e\mathbf{A}$
   is the application of the gauge principle for U(1).
3. QED Lagrangian = Dirac + gauge principle + electromagnetic
   $\bar{\psi}(i\hbar \not{D} - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, $D_\mu = \partial_\mu + eA_\mu$.
4. The non-anomalous value $g = 2$ comes from tree level

The basic Feynman diagram in QED 

QED at work. The 1-loop correction

J. Schwinger

The 1-loop correction is probably the most celebrated result by J. Schwinger, one of the best physicists of the 20th century. It is engraved on his tombstone.

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} + O(\alpha^2)$$

Numerically

$$\frac{\alpha}{2\pi} \approx 1.1614 \cdot 10^{-3}.$$

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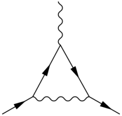
The paper (1949)

Schwinger's paper is rather impressive: 28 pages full of intricate formulas (the result about g is at some point in the middle) with very few and schematic physical considerations. It is difficult to compare to the modern treatment.

Schwinger was not advocated to Feynman diagrams but nowadays we understand his result writting an integral corresponding to just one Feynman diagram.



One recalls the (unfair) claim by R. Oppenheimer: others gave talks to show others how to do the calculation, while Schwinger gave talks to show that only he could do it.

Aspect of one othe pages of Schwinger's paper:

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However,

$$\frac{1}{k^2-2kp'-i\epsilon}\,\frac{1}{k^2-2kp''-i\epsilon}\,\frac{1}{k^2-i\epsilon}=\frac{1}{2k(p'-p'')}\left[\frac{1}{2kp'}\left(\frac{1}{k^2-2kp'-i\epsilon}-\frac{1}{k^2-i\epsilon}\right)-\frac{1}{2kp''}\left(\frac{1}{k^2-2kp''-i\epsilon}-\frac{1}{k^2-i\epsilon}\right)\right], \quad (1.80)$$

and, on extracting the imaginary part divided by π, we again encounter (1.74).

The second part of K_μ, (1.62), can also be readily expressed in Fourier integral form

$$K_\mu^{(2)}(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\left[\frac{1}{2\kappa}p_\lambda'\frac{\partial}{\partial p_\lambda'}\gamma_\nu(i\gamma(p'-k)-\kappa)\gamma_\nu\left(\frac{\delta((k-p')^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{(k-p')^2+\kappa^2}\right)\gamma_\mu+\gamma_\mu\frac{1}{2\kappa}p_\lambda''\frac{\delta}{\partial p_\lambda''}\gamma_\nu(i\gamma(p''-k)-\kappa)\gamma_\nu\left(\frac{\delta((k-p'')^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{(k-p'')^2+\kappa^2}\right)\right]. \quad (1.81)$$

To evaluate the derivatives with respect to p_λ' and p_λ'', we observe that

$$p_\lambda\frac{\partial}{\partial p_\lambda}(i\gamma(p-k)+\kappa)(i\gamma(p-k)-\kappa)f((p-k)^2+\kappa^2)=0, \quad (1.82)$$

where $f(x)$ is $\delta(x)$ or $1/x$. On differentiating and multiplying to the left by $i\gamma(p-k)-\kappa$, we obtain

$$p_\lambda\frac{\partial}{\partial p_\lambda}(i\gamma(p-k)-\kappa)f((p-k)^2+\kappa^2)=(i\gamma(p-k)-\kappa)i\gamma p(i\gamma(p-k)-\kappa)\frac{f(k^2-2kp)}{k^2-2kp}. \quad (1.83)$$

Consequently,

$$p_\lambda\frac{\partial}{\partial p_\lambda}\gamma_\nu(i\gamma(p-k)-\kappa)\gamma_\nu\left(\frac{\delta((p-k)^2+\kappa^2)}{k^2}+\frac{\delta(k^2)}{(p-k)^2+\kappa^2}\right)=-\gamma_\nu(i\gamma(p-k)-\kappa)i\gamma p(i\gamma(p-k)-\kappa)\gamma_\nu\left(\frac{\delta'(k^2-2kp)}{k^2}-\frac{(\delta k^2)}{(2kp)^2}\right), \quad (1.84)$$

in virtue of the delta-function property

$$\delta'(x)=-\frac{\delta(x)}{x}. \quad (1.85)$$

Furthermore,

$$\frac{\delta'(k^2-2kp)}{k^2}-\frac{\delta(k^2)}{(2kp)^2}=-\frac{\partial}{\partial(2kp)}\left(\frac{\delta(k^2-2kp)}{k^2}-\frac{\delta(k^2)}{2kp}\right)=-\int_0^1 udu\delta''(k^2-2kpu), \quad (1.86)$$

according to (1.75). Therefore, (1.81) becomes

$$K_\mu^{(2)}(x'-x,x-x'')=\frac{1}{(2\pi)^{11}}\int_0^1 udu\int(dk)(dp')(dp'')e^{ip'(x'-x)}e^{ip''(x-x'')}\times\left[\delta''(k^2-2kp'u)\frac{1}{2\kappa}\gamma_\nu(i\gamma(p'-k)-\kappa)i\gamma p'(i\gamma(p'-k)-\kappa)\gamma_\nu\gamma_\mu+\gamma_\mu\frac{1}{2\kappa}\gamma_\nu(i\gamma(p'-k)-\kappa)i\gamma p''(i\gamma(p''-k)-\kappa)\gamma_\nu\delta''(k^2-2kp''u)\right]. \quad (1.87)$$

The transformation

$$k_\mu\rightarrow k_\mu+(p_\mu'+p_\mu''+(p_\mu'-p_\mu'')v)\frac{u}{2} \quad (1.88)$$

now brings the delta-function of (1.78) into the form

$$\delta''(k^2+\lambda^2u^2), \quad (1.89)$$

Almost all of them have a similar aspect

Scheme of the modern treatment

(Peskin–Schroeder §6.2-3)

Feynman rules:

$$\bar{u}(p')\Big(\gamma^\mu + 2ie^2\mathcal{I}^\mu\Big)u(p)$$

$$\mathcal{I}^\mu = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu}{\big((k-p)^2+i\epsilon\big)\big(k'^2-m^2+i\epsilon\big)\big(k^2-m^2+i\epsilon\big)}$$

Gordon identity, form factors → we can focus on a part of the integral.

In fact, the interesting part is not affected by divergence (neither infrared nor ultraviolet).

Many tricks (including Schwinger's trick)

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_{[0,1]^3} dx\, dy\, dz\, \frac{2m^2z(1-z)}{m^2(1-z)^2-q^2xy}\delta(x+y+z-1)$$

to order $O(\alpha^2)$. It gives $F_2(0)=\alpha/2\pi$.

Beyond 1-loop

R. Karplus and N.M. Kroll (1950)

2-loop computations:

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} + C\alpha^2 + O(\alpha^3)$$

with C around 0.3 given by a closed constant.

Seven years later, a mistake was detected by A. Petermann. It was of numerical nature (not affecting to the diagram list nor the method) but it implies that $C \approx 0.03$, a ten times smaller value.

Beyond 1-loop

Corrected order α^2 estimate (Petermann 1957)

$$\frac{g-2}{2} \approx \frac{\alpha}{2\pi} + \Big(\frac{197}{144\pi^2} + \frac{1}{12} - \frac{\log 2}{2} + \frac{3\zeta(3)}{4\pi^2}\Big)\alpha^2 + O(\alpha^3).$$

S. Laporta and E. Remiddi found in 1996 a closed expression for order α^3 in terms of multiple zeta values.

Some researchers try to exploit the evaluation of Feynman diagrams in terms of multiple zeta values with some conjectural algebraic relations.

Computers at work

T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio

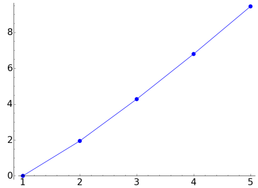
Automatic code generator raising FORTRAN programs

(2008) order α^4
$1\,159\,652\,182.79(7.71) \cdot 10^{-12}$

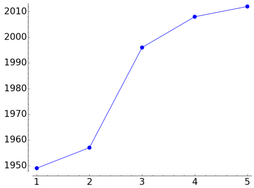
(2012) order α^5
$1\,159\,652\,181.78(77) \cdot 10^{-12}$

Summary

Order	Diagrams	Year	Authors
1	1	1949	Schwinger
2	7	1957	Karplus, Kroll, Petermann
3	72	1996	Laporta, Remiddi
4	891	2008	Aoyama, Hayakawa, Kinoshita, Nio
5	12672	2012	Aoyama, Hayakawa, Kinoshita, Nio



(loops, log(#diag.))



(loops, year)

Theory vs. Experiments

Perfect agreement experiments-theory!

Is there any point going beyond?

$\updownarrow$

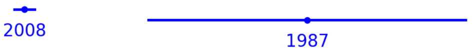
Is it possible to test QED experimentally to this level?

In 1987 the experimental measurements (R.S. Van Dyck, Jr., P.B. Schwinberg and H.G. Dehmelt) reached the unbelievable precision

$$\frac{g-2}{2} = 1\,159\,652\,188.4(4.3) \cdot 10^{-12}$$

In the limit of theoretical precision (no "need" for more loops).

Warning: The value and uncertainty reported in the bestselling book QED by R. Feynman is not coherent with this result and the last experimental value (D. Hanneke, S. Fogwell Hoogerheide, and G. Gabrielse 2008, 2011) $1\,159\,652\,180.73(28) \cdot 10^{-12}$ is not coherent with any of them.



Published ranges of $(g-2)/2$, the one mentioned by Feynman is out of the scale

Welcome to the real world! Extreme experiments are difficult to do.

Some theoretical and experimental problems:

- How to get α experimentally with high precision? It is needed to compute the theoretical value
  - Partial answer: Rydberg constant
  - $\Leftarrow$ The QED value is used to "define" the value of α
- How to measure g with high precision?
  - One needs to capture the electron sharply (Penning trap)
  - One need to reproduce something like a hydrogen atom without proton (geonium atom)
- Is the value of $(g-2)/2$ a purely QED effect?
  - Not exactly, in principle the true framework is the SM
  - The contribution of non QED effects $\approx 1.7 \cdot 10^{-12}$. It has to be considered only in the latest results.

You must go to the source

Adapted from the (Dirac) *matrix revolutions*

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