Odd zeta values

Introduction. It is well-known that the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ admits an analytic extension to $\mathbb{C} - \{1\}$. The special values $\zeta(n)$ can be evaluated for $n \in \mathbb{Z}^+$ even and for $n \in \mathbb{Z}^-$ odd, moreover $\zeta(-2n) = 0$ for $n \in \mathbb{Z}^+$. The first negative odd values are $\zeta(-1) = -1/12$ and $\zeta(-3) = 1/120$. A bulletproof faith on the infinite series would lead to the astonishing relations

$$1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\frac{1}{12} \qquad \text{and} \qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + \cdots = \frac{1}{120}.$$

Believe or not, it has some significance on theoretical physics.

J. F. González-Hernández called my attention noting that $\zeta(-1) = \zeta(-13)$ and he asked me if there are more coincidences of negative odd zeta values.

We are going to show that for $m < n$ positive integers

$$\zeta(1-2n) = \zeta(1-2m) \quad \text{if and only if} \quad n = 7 \text{ and } m = 1.$$

To show it, consider the asymmetric form of the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\Big(\frac{\pi s}{2}\Big)\Gamma(1-s)\zeta(1-s).$$

It implies for $n \geq 3$

$$-\frac{\zeta\big(1-2(n+1)\big)}{\zeta(1-2n)} = \frac{(2n+1)n\zeta(2n+2)}{2\pi^2\zeta(2n)} > \frac{7\cdot 3}{2\pi^2\zeta(6)} > 1.$$

In particular, $|\zeta(1-2n)|$ is strictly increasing for $n \geq 3$. As $|\zeta(-13)| = |\zeta(-1)| > |\zeta(-3)|$, we have that $|\zeta(1-2n)|$ is greater than the previous values for $n > 7$. The proof is complete checking in a table that there are not more coincidences for $n \leq 7$. Actually, after the previous comments the only part of the table you need is:

n	3	4	5	6
$\zeta(1-2n)$	$-1/252$	$1/240$	$-1/132$	$691/32760$

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