# A curious infinite product

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## 1 The product

Behold the following result!

**Theorem 1.1.** *Let* $B_1$ *and* $B_2$ *be integers and let* $a_1$ *and* $a_2$ *be positive divisors of* $B_1^2+1$ *and* $B_2^2+1$*, respectively. Then*

$$\prod_{n=1}^{\infty}\frac{e^{2\pi n/a_2}\big(\cosh\frac{2\pi n}{a_1}-\cos\frac{2\pi nB_1}{a_1}\big)}{e^{2\pi n/a_1}\big(\cosh\frac{2\pi n}{a_2}-\cos\frac{2\pi nB_2}{a_2}\big)}=\frac{e^{\pi/6a_1}\sqrt{a_1}}{e^{\pi/6a_2}\sqrt{a_2}}.$$

The curious point is that there is something arithmetic involved. It is in general false without the divisibility condition. Note that each factor tends exponentially to 1, then the convergence is assured. For small values of $a_1$ and $a_2$, few terms are enough to get a good approximation. For instance, for $a_1=2$, $a_2=B_1=B_2=1$ the right hand side is $e^{-\pi/12}\sqrt{2}=1.088471\dots$ and with only three terms of the product we get 6 correct significant digits.

If you are a modular person, after reading the following lines showing the relation with $\eta$, you will be able to find a quick proof by yourself, perhaps adding a teaspoon of class number one or a grain of complex multiplication. If you are modular but not to the bone, you will have a chance of reading §3. Anyway, the challenge here is to provide a proof simpler enough to fit in a lecture, only one, of an undergraduate course. This is done in §2 assuming an analytic result known as *Kronecker limit formula* which is proved in [2] with little more than the residue theorem. The proof is reproduced in §4 adapted to a special case, for the sake of clarity, and with complementary comments to convince you that it generalizes finely.

We start defining the Dedekind $\eta$ function on the upper half complex plane as

$$\eta(z)=e^{\pi iz/12}\prod_{n=1}^{\infty}\big(1-e^{2\pi inz}\big).$$

It converges quickly if $z$ is far apart from the real axis.

For $u, v \in \mathbb{R}$ we have the identity $\left|1 - e^{u+iv}\right|^2 = 2(\cosh u - \cos v)/e^{-u}$. Taking $u + iv = 2\pi i(B_j + i)/a_j$, we see that the infinite product in Theorem 1.1 is

$$\prod_{n=1}^{\infty} \frac{\left|1 - e^{2\pi i(B_1+i)/a_1}\right|^2}{\left|1 - e^{2\pi i(B_2+i)/a_2}\right|^2} = \frac{e^{\pi/6a_1}\left|\eta\big((B_1+i)/a_1\big)\right|^2}{e^{\pi/6a_2}\left|\eta\big((B_2+i)/a_2\big)\right|^2}$$

Then Theorem 1.1 is equivalent to say that $\left|\eta\big((B_j + i)/a_j\big)\right|^2/\sqrt{a_j}$ is constant. It does not depend on the choice of $a_j$ and $B_j$ fulfilling the hypotheses.

Modular people know how to relate the values of $\eta(z)$ at different points connected by some symmetries and then they may find the previous claim fairly easy. We pedestrians aspire for a proof not requiring any knowledge about those relations and symmetries. At the same time, we can learn a formula, the aforementioned Kronecker limit formula, that plays a role in some explicit evaluations.

## 2 A proof for everybody (summoning Kronecker)

The Riemann zeta function and the Epstein zeta function $\zeta(s, Q)$ associated to a positive definite binary quadratic form $Q$ are defined for $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{and} \qquad \zeta(s, Q) = \sum_{\vec{n} \in \mathbb{Z}^2 \setminus \{\vec{0}\}} \big(Q(\vec{n})\big)^{-s}.$$

Both definitions can be extended analytically to real and complex values beyond $s > 1$. It is well known that for the Riemann zeta function there is an obstacle at $s = 1$. Some insight about this point comes from the identity $(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-s}$, which reduces to multiplication term by term. Recalling $\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1} = \log 2$, we have

$$\lim_{s\to 1^+}(s-1)\zeta(s) = (\log 2) \lim_{s\to 1^+} \frac{s-1}{1-2^{1-s}} = 1 \tag{1}$$

by L'Hôpital's rule. This means that $\zeta(s)$ is approximately $(s-1)^{-1}$ for $s > 1$ close to 1. The Kronecker limit formula implies that $\zeta(s, Q)$ is approximately $\frac{2\pi}{\sqrt{D}}(s-1)^{-1}$ near 1 and shows that the difference tends to a constant that can be expressed in terms of the Dedekind $\eta$ function. Kronecker show yourself, we beckon you!

**Proposition 2.1** (Kronecker limit formula)**.** *Let $Q(x, y) = ax^2 + bxy + cy^2$ be a real form with $D = 4ac - b^2 > 0$ and $a > 0$. Then*

$$\lim_{s\to 1^+}\Big(\frac{\sqrt{D}}{4\pi}\zeta(s, Q) - \zeta(2s-1)\Big) = \log\frac{\sqrt{a/D}}{|\eta(z_Q)|^2} \qquad \textit{with} \quad z_Q = \frac{-b + i\sqrt{D}}{2a}.$$

I have downgraded this theorem to proposition to emphasize that it is not so hard to prove. In [2] there is a proof that requires little more than the residue theorem. To not repeat myself, if you are interested I have adapted it in §4 to $Q(x,y) = x^2 + y^2$ which allows more reductions and any hard working reader should be able to obtain the general case from it, perhaps following the hints included there. A last comment is that if you look up authorized sources (like Wikipedia) Proposition 2.1 does not seem like the standard Kronecker limit formula. Take my word, it is a compact equivalent version.

Proposition 2.1 is purely analytic, a limit, while Theorem 1.1 is somehow arithmetic. The integers enter into the game through the humble and important group of matrices

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \,:\, m_{jk} \in \mathbb{Z},\ \det(M) = 1 \right\}.$$

The key result to deduce Theorem 1.1 form the Kronecker limit formula is that for the integral case with $D = 4$ there is only a possible Epstein zeta function!

**Lemma 2.2.** *If $Q(x,y) = ax^2 + bxy + cy^2$ is a positive definite quadratic form with $a, b, c \in \mathbb{Z}$ and $4ac - b^2 = 4$ then there exists $M \in SL_2(\mathbb{Z})$ such that $Q(M\vec{v}) = x^2 + y^2$ where $\vec{v} = (x, y)$. In particular, for any of these forms we have $\zeta(s, Q) = \zeta(s, x^2 + y^2)$.*

Of course, here it is in use the typical typographical abuse: We have to think $\vec{v}$ as a vertical vector to multiply $M\vec{v}$. This lemma is based on an elementary reduction algorithm due to Lagrange and Gauss for general binary quadratic forms (with $a, b, c \in \mathbb{Z}$). If you want to trumpet proudly "I read Gauss", go to his materpiece [3, Art.171].

*Proof.* Note that the last part follows from the first part because $M \in \mathrm{SL}_2(\mathbb{Z})$ only rearranges the elements of $\mathbb{Z}^2$. In other words, $M : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ is a bijective map.

If $b = 0$ then clearly the result is true with $M$ the identity matrix. If $b \neq 0$ we are going to show that there is a "reduction matrix" $R \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q'(\vec{v}) = Q(R\vec{v})$ has a smaller value of $|b|$. Repeating the process a number of times we get $x^2 + y^2 = Q(M\vec{v})$ with $M = R_n R_{n-1} \cdots R_1$ and we are done.

Let us see how to construct $R$. If $\langle x \rangle$ is the *nearest integer function* (define it as you want at half-integers), a possible choice of $R$ is

$$R = \begin{pmatrix} \langle b/2a \rangle & 1 \\ -1 & 0 \end{pmatrix} \text{ if } a < c \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & \langle b/2c \rangle \end{pmatrix} \text{ if } a > c.$$

There is not an $a = c$ case with $b \neq 0$ because $4 = 4a^2 - b^2 = (2a - b)(2a + b)$ implies $2a - b = 2a + b = 2$. Both cases are similar changing the role of the variables. Let us check for instance the second one:

$$Q(R\vec{v}) = ay^2 - by\Big(x + \langle \frac{b}{2c} \rangle y\Big) + c\Big(x + \langle \frac{b}{2c} \rangle y\Big)^2 = \dots x^2 + \Big(2c\langle \frac{b}{2c} \rangle - b\Big)xy + \dots y^2.$$

The absolute value of the new $xy$ coefficient is clearly less than $|b|$ when $|b| > c$ and $|b| \le c$ is impossible because it would imply $4ac - b^2 \ge 4(c+1)c - c^2 > 4$. $\square$

*Proof of Theorem 1.1.* Consider the quadratic forms $Q_j = a_j x^2 - 2B_j xy + (B_j^2 + 1)y^2/a_j$ for $j = 1, 2$. By the last part of Lemma 2.2 the limits in Proposition 2.1 corresponding to both quadratic forms are identical. Then we conclude

$$\log \frac{\sqrt{a_1}/2}{\big|\eta\big((B_1+i)/a_1\big)\big|^2} = \log \frac{\sqrt{a_2}/2}{\big|\eta\big((B_2+i)/a_2\big)\big|^2}$$

and, as mentioned before, the constancy of $\big|\eta\big((B_j+i)/a_j\big)\big|^2/\sqrt{a_j}$ establishes the result. $\square$

# 3 The quick proof for modular people

The Dedekind $\eta$ function is a modular function that satisfies

$$\eta(z+1) = e^{\pi i/12}\eta(z) \qquad \text{and} \qquad \eta(-1/z) = \sqrt{-iz}\,\eta(z). \tag{2}$$

Of course, the first formula is trivial. Absolutely, the second is not. To my knowledge the simplest proof is still one due to Siegel [8] (see also [5, §9.2]) based on the residue theorem. Let us go fancy proclaiming that $|\Im(z)|^{1/2}|\eta(z)|^2$ is invariant under $z \mapsto z+1$ and $z \mapsto -1/z$, where $\Im(z)$ is the imaginary part of $z$. This follows immediately from (2) using $\Im(z+1) = \Im(z)$ and $\Im(-1/z) = |z|^{-2}\Im(z)$.

If you are really a modular person you know that $z \mapsto z+1$ and $z \mapsto -1/z$ generate all the maps $z \mapsto (m_{11}z + m_{12})/(m_{21}z + m_{22})$ with $M = (m_{jk}) \in \mathrm{SL}_2(\mathbb{Z})$. Hence $|\Im(z)|^{1/2}|\eta(z)|^2$ is also invariant by them. In the particular case $z = i$ we get

$$\big|\eta(i)\big|^2 = \big|\Im\big(\gamma_M(i)\big)\big|^{1/2}\big|\eta\big(\gamma_M(i)\big)\big|^2 \qquad \text{with} \quad \gamma_M(i) = \frac{m_{11}i + m_{12}}{m_{21}i + m_{22}}.$$

It only remains to check that if $0 < a \mid B^2 + 1$ then there exists $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma_M(i) = (B+i)/a$. Actually, we have already done it because taking $Q(x,y) = ax^2 - 2Bxy + (B^2+1)y^2/a$ in Lemma 2.2, as before, and choosing $\vec{v} = (i, 1)$, we have $0 = Q(M\vec{v}) = \big(m_{21}i + m_{22}\big)^2 Q\big(\gamma_M(i), 1\big)$. The roots of $Q(z,1) = 0$ are $(B \pm i)/a$ and $\Im\big(\gamma_M(i)\big) > 0$ (check it!), therefore necessarily $\gamma_M(i) = (B+i)/a$, as expected.

# 4 Who fears Kronecker limit formula?

The case $Q(x,y) = x^2 + y^2$ of Proposition 2.1 reads

$$\lim_{s \to 1^+} \Big(\frac{1}{2\pi}\zeta(s, x^2+y^2) - \zeta(2s-1)\Big) = -\log\big(2|\eta(i)|^2\big). \tag{3}$$

Let us see how to get it using only undergraduate tools.

*Proof of Proposition 2.1 for $Q(x,y)=x^2+y^2$.* Define the abbreviations $g_s(x)=2(x^2+1)^{-s}$ and $G(s)=-\int_{-\infty}^{\infty} g_s(x)\,dx$. The limit in (3) equals $L_1-L_2$ with

$$L_1=\lim_{s\to 1^+}\frac{1}{2\pi}\Big(\zeta(s,Q)+\zeta(2s-1)G(s)\Big),\qquad L_2=\lim_{s\to 1^+}\zeta(2s-1)\Big(\frac{1}{2\pi}G(s)+1\Big).$$

L'Hôpital's rule shows $L_2=(4\pi)^{-1}G'(1)$ because, by (1), $(2s-2)\zeta(2s-1)\to 1$ (and the residue theorem assures $G(1)=-2\pi$). Then the result follows if we prove

$$L_1=-\log|\eta(i)|^2\qquad\text{and}\qquad G'(1)=4\pi\log 2. \tag{4}$$

We have $G'(1)=2\int_{-\infty}^{\infty}\frac{\log(x^2+1)}{x^2+1}\,dx$. To compute this integral the easy way is to look up a table (e.g. [4, 4.295.1]). If you want to be fully in charge, check the following formula performing the change of variables $x=\tan(t/2)$ and the application of Cauchy's integral formula on the unit circle $C$ parametrized as $z=e^{it}$

$$G'(1)=-\int_{-\pi}^{\pi}\log|\cos(t/2)|^2\,dt=-\Re\int_C\log\Big(\frac{1+z}{2}\Big)\frac{dz}{iz}=4\pi\log 2.$$

For the first formula in (4) we separate from $\zeta(s,x^2+y^2)=\sum_{m,n}(m^2+n^2)^{-s}$ the terms with $n=0$ which contribute $2\zeta(2s)$. By the residue theorem in the band $B_\epsilon=\{|\Im z|<\epsilon\}$ with $0<\epsilon<1$,

$$\zeta(s,x^2+y^2)=2\zeta(2s)+\sum_{n=1}^{\infty}\frac{1}{n^{2s}}\sum_{m\in\mathbb{Z}}g_s\big(\frac{m}{n}\big)=\sum_{n=1}^{\infty}\frac{-1}{2n^{2s-1}}\int_{\partial B_\epsilon}g_s(z)i\cot(\pi nz)\,dz,$$

because $2\pi i\operatorname{Res}\big(i\cot(\pi nz),m/n\big)=-2$. As $g_s$ is even, $\int_{\partial B_\epsilon}=-2\int_{L_\epsilon}$ with $L_\epsilon=\{\Im z=\epsilon\}$ oriented to the right and the sum is $\sum_n n^{1-2s}\int_{L_\epsilon}$. Note that $\int_{L_\epsilon}g_s=\int_{L_0}g_s=-G(s)$. Then adding $\zeta(2s-1)G(s)$ is equivalent to replace $i\cot(\pi nz)$ by $i\cot(\pi nz)-1$ in $\int_{L_\epsilon}$. The expansion $i\cot w-1=2e^{2iw}/(1-e^{2iw})=2(e^{2iw}+e^{4iw}+\dots)$ assures an exponential decay and we have, substituting $\zeta(2)=\pi^2/6$,

$$L_1=\frac{1}{2\pi}\Big(\frac{\pi^2}{3}+\sum_{n,k=1}^{\infty}\frac{2}{n}\int_{L_\epsilon}g_1(z)e^{2\pi inkz}\,dz\Big).$$

Note that $g_1(z)=2\big((z-i)(z+i)\big)^{-1}$. The residue theorem in $\{\Im z>\epsilon\}$ gives promptly

$$L_1=\frac{\pi^2}{6}+\sum_{n,k=1}^{\infty}\frac{2}{n}e^{-2\pi nk}=\frac{\pi^2}{6}-\sum_{k=1}^{\infty}\log\big(1-e^{-2\pi k}\big)^2$$

where we have employed the Taylor expansion $\log(1-x)^2=-2(x/1+x^2/2+\dots)$. The sum is $\log\big(e^{\pi/6}|\eta(i)|^2\big)$ and the proof of (4) is complete. □

The question is how close is this to a full proof of Proposition 2.1. Actually, it is quite close. Essentially, the whole point is to replace $x^2+1$ by $Q(x,1) = ax^2+bx+c$, restoring the constants coming from Proposition 2.1. Read [2, §3] for the full details. Here there are some hints for an intermediate level of details. In the general case, $G(s) = -2\int_{-\infty}^{\infty} Q(x,1)^{-s}\,dx$,

$$L_1 = \lim_{s\to 1^+} \frac{\sqrt{D}}{4\pi}\big(\zeta(s,Q) + \zeta(2s-1)G(s)\big) \quad \text{and} \quad L_2 = \lim_{s\to 1^+} \zeta(2s-1)\Big(\frac{\sqrt{D}}{4\pi}G(s) + 1\Big).$$

Again the limit in the statement is $L_1 - L_2$. The computation of $G'(1)$ to evaluate $L_2$ is as before because we can transform $Q(x,1)$ into a multiple of $x^2+1$ completing squares. This leads to

$$\frac{\sqrt{D}}{4\pi}G'(1) = \log\sqrt{\frac{D}{a}}.$$

The evaluation of $L_1$ follows the same lines. The only noticeable issue is that at some point we used that $g_s$ was even and $Q(x,1)$ is not in general. The simple solution is to substitute $g_s(x)$ by $Q(x,1)^{-1} + Q(x,-1)^{-1}$. With this change, we get

$$L_1 = -\log\eta(z_Q) - \log\eta(-\bar{z}_Q) = -\log\big|\eta(z_Q)\big|^2.$$

The values $z_Q$ and $-\bar{z}_Q$ come from the fact that $g_s(z)$ has simple poles at these points in the upper half plane.

# 5 A sharper result

Theorem 1.1 is a direct consequence of the stronger less symmetric result:

**Theorem 5.1.** *Let $a$ be a positive divisor of $B^2+1$, $B\in\mathbb{Z}$. Then*

$$\prod_{n=1}^{\infty} 2e^{-2\pi n/a}\Big(\cosh\frac{2\pi n}{a} - \cos\frac{2\pi nB}{a}\Big) = \frac{1}{4}\Gamma^2\big(\frac{1}{4}\big)e^{\pi/6a}\sqrt{\frac{a}{\pi^3}}$$

*where $\Gamma$ indicates the classical gamma function.*

The last sentence is not very informative if you have not heard about the gamma function. In this case, you only need to know that

$$\Gamma\big(\frac{1}{4}\big) = 4\int_0^{\infty} e^{-t^4}\,dt = 3.6256099\ldots$$

and it is not known a closed expression for this constant in term of high school mathematical constants.

Dividing the formula of Theorem 5.1 for two choices of the parameters, we get Theorem 1.1. Then both results become equivalent if we assume Theorem 5.1 for a single couple $(a, B)$. For instance $(1, 0)$, which gives

$$\prod_{n=1}^{\infty} 2e^{-2\pi n}\big(\cosh(2\pi n) - 1\big) = \frac{e^{\pi/6}}{4\pi^{3/2}}\Gamma^2\big(\frac{1}{4}\big).$$

This follows immediately squaring the evaluation

$$\eta(i) = \frac{\Gamma(1/4)}{2\pi^{3/4}}. \tag{5}$$

An strategy to get it (see [2] and [5]) is to use the nontrivial factorization

$$\zeta(s, x^2 + y^2) = 4\zeta(s)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

which allows to compute the limit in the Kronecker limit formula in an alternative way.

The evaluation (5) relates to the classical problem of the inversion of elliptic integrals with theta functions [1] led by Jacobi and preceded by Gauss [6]. Even if you do not know what I am talking about, you will enjoy the impressive and highly nontrivial formulas

$$\sqrt{2}\eta(i) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2} = \left(\frac{2}{\pi}\int_0^{\pi/2}\frac{\sqrt{2}\,dt}{\sqrt{2-\sin^2 t}}\right)^{1/2}.$$

The identity (5) is also a special case of the *Chowla-Selberg formula* [7]. This is a curious formula that was announced by its authors almost 20 years before they published the proof. The Fields medalist Selberg did not like to collaborate with other colleagues. In the nowadays ultra-connected scientific world, it sounds astonishing that Chowla-Selberg formula was the only joint work that Selberg published in his long and fruitful mathematical life.

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