**A Journey through**

# The Prime Number Theorem

**Notation** Along these notes we shall employ extensively Landau's $O$-notation that we recall briefly here.

The symbols $O(g)$ and $o(g)$ mean respectively a function $f$ such that

$$\limsup_{x\to\infty}\left|\frac{f(x)}{g(x)}\right|<\infty \qquad \text{and} \qquad \lim_{x\to\infty}\frac{f(x)}{g(x)}=0.$$

Note that $f=O(g)$ is only a short way of saying $|f(x)|\le C|g(x)|$ for some positive constant $C$ and $x$ large enough.

If $f$ and $g$ has the same asymptotic behavior, *i.e.* $\lim f/g=1$, we shall write $f\sim g$. Typically we shall consider the asymptotic behavior when $x\to\infty$, otherwise it will be explicitly indicated.

# 1 Warming up

The basic cornerstone in prime number distribution theory is the simple and beautiful *Euler's identity*:

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}}$$

where $s>1$, to assure the convergence, and $p$ runs over the prime numbers. This identity is equivalent to Fundamental Theorem of Arithmetic (unique factorization into primes), just noting that the right hand side is $\prod(1+p^{-s}+p^{-2s}+p^{-3s}+\dots)$.

The importance of Euler's identity stems from establishing a link between an analytic object, the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

and an arithmetical object, prime numbers. For instance, Euler himself in 1737 realized that when $s\to 1^+$ the divergence of harmonic series implies that there are infinitely many prime numbers.

The giant step toward the understanding of the distribution of primes was given by Riemann who, in his celebrated memoir of 1859, considered $\zeta$ as a function of complex variable and proved that it can be extendend to a meromorphic function on the whole complex plane. There are quite elementary proofs of this fact. For instance, the simple identity

$$\left(1-\frac{2}{2^s}\right)\zeta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

proves that $\zeta$ has a meromorphic continuation to $\{\Re s > 0\}$ (note that $|n^{-s} - (n+1)^{-s}| < C(s)|n^{-s-1}|$) with a simple pole at $s = 1$ with residue 1. More generally, Taylor expansion $(1-x)^{-s}-1 = \sum a_m x^m$ with $a_m$ the generalized binomial coefficient $\binom{s+m-1}{m}$, implies

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{-s-m} a_m \zeta(s+m) &= \sum_{n=1}^{\infty} (2n)^{-s} \sum_m a_m (2n)^{-m} \\ &= \sum_{n=1}^{\infty} (2n)^{-s} \left( \left(1 - \frac{1}{2n}\right)^{-s} - 1 \right) = \left(1 - \frac{2}{2^s}\right) \zeta(s).\end{aligned}$$

Hence meromorphic continuation to $\{\Re s > k\}$ implies meromorphic continuation to $\{\Re s > k-1\}$ and $\zeta$ gets extended (of course uniquely) to a holomorphic function on $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$ with residue 1.

Primes appear in an involved way in Euler's identity. It would be desirable a relation between $\zeta$ and prime numbers counting function, *i.e.*

$$\pi(x) = \sum_{p \le x} 1 = |\{p \le x \ : \ p \text{ is a prime number}\}|.$$

But it is technically simpler to establish this connection through the, in principle unnatural, function

$$\psi(x) = \sum_{n \le x} \Lambda(n) \qquad \text{with} \qquad \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 1.1** *For* $\Re s > 1$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad \textit{and} \qquad -\frac{\zeta'(s)}{\zeta(s)} - \frac{s}{s-1} = s \int_1^{\infty} (\psi(x) - x) x^{-s-1} \, dx.$$

*Proof:* By logarithmic differentiation (Euler's favorite trick)

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \Rightarrow \frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{1 - p^{-s}} p^{-s} = -\sum_p \left( \frac{\log p}{p^s} + \frac{\log p}{p^{2s}} + \dots \right).$$

The convergence for $\Re s > 1$ is assured comparing with a geometric series and first formula follows.

Now it is not hard to obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{m=1}^{\infty} \psi(m) \left( \frac{1}{m^s} - \frac{1}{(m+1)^s} \right) = \sum_{m=1}^{\infty} s \int_m^{m+1} \frac{\psi(x)}{x^{s+1}}\, dx = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\, dx$$

and the second formula is proved noting that $s/(s-1) = s\int_1^\infty x^{-s}\,dx$. □

We know that $\zeta$ is analytic and $\zeta(s) \sim (s-1)^{-1}$ as $s \to 1$, hence $-\zeta'(s)/\zeta(s) - s/(s-1) \to 0$, and the second identity of the proposition suggests that $\psi(x)$ should be well approximated by $x$. It will be the content of the prime number theorem, but firstly we want to know what does it mean in terms of $\pi(x)$.

**Proposition 1.2** *Let $E = E(x)$ be an increasing function such that $\psi(x) = x + O(E(x))$, then $\pi(x) = \mathrm{li}(x) + O(x^{1/2} + E(x)/\log x)$ where $\mathrm{li}(x)$ is the* integral logarithm *$\int_2^x dt/\log t$.*

*Proof:* It is not difficult to prove that $\pi(x) = \sum_{2\le n\le x} \Lambda(n)/\log n + O(x^{1/2})$ (use that the sum equals $\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots$ and the bound $\pi(x^{1/n}) \le x^{1/n}$). On the other hand,

$$\sum_{2\le n\le x} \frac{\Lambda(n)}{\log n} = \frac{\psi(x)}{\log x} + \sum_{2\le n\le x} \Lambda(n) \int_n^x \frac{dt}{t\log^2 t} = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)\,dt}{t\log^2 t},$$

where the second equality is just partial summation in the form $a_2b_2 + a_3b_3 + a_4b_4 + \dots = a_2(b_2 - b_3) + a_3(b_3 - b_4) + \dots$ with $a_n = \Lambda(n)$ and $b_n = \int_n^x$. This gives

$$\pi(x) = \mathrm{li}(x) + \frac{\psi(x) - x}{\log x} + \int_2^x \frac{\psi(t) - t}{t\log^2 t}\, dt + O(x^{1/2})$$

because integrating by parts $\int dt/\log t = x/\log x + \int dt/\log^2 t$. □

According to these results, $\psi(x) \sim x$ translate into $\pi(x) \sim \mathrm{li}(x)$ or equivalently into $\pi(x) \sim x/\log x$ (l'Hôpital rule proves $\mathrm{li}(x) \sim x/\log x$). Any of these asymptotic formulas is called *Prime Number Theorem* (abbreviated as PNT in the following). Usually one wants to go beyond when estimating the size of the error term (the function $\mathcal{E}(x)$ below).

**Theorem 1.3 (PNT with error term)** *It holds*

$$\pi(x) = \mathrm{li}(x) + O(\mathcal{E}(x))$$

*for some function $\mathcal{E}(x) = o(\mathrm{li}(x))$.*

In this notes we shall prove this theorem with $\mathcal{E}(x) = xe^{-\frac{1}{6}\sqrt{\log x}}$. In fact, thanks to previous proposition we shall forget about $\pi(x)$, and prove $\psi(x) = x + O(\mathcal{E}(x))$. Even today it is not known a valid error term verifying $\mathcal{E}(x) = O(x^\alpha)$ for some $\alpha < 1$. As we shall see later, this is related to the so-called *Riemann hypothesis*.

# 2 PNT timelines

- 1849 Gauss conjetures that $\mathrm{li}(x)$ approximates $\pi(x)$.
- 1851 Chebyshev proves $C_1 x/\log x < \pi(x) < C_2 x/\log x$ with explicit $C_i$.
- 1859 Riemann writes his celebrated 8-paged memoir containing a proof of PNT with serious gaps, using complex analysis.
- 1896 Hadamard and de la Vallée Poussin prove (independently) PNT.
- 1948 Erdős and Selberg find the first "elementary proof" of PNT.
- 1958 Vinogradov and Korobov find the best known error term.
- ????$\geq$2003 Somebody proves Riemann Hypothesis.

# 3 A proof (?) of PNT for dreamers

In this section we give a fake proof à la Riemann that is non rigorous but contains all the ingredients of the real proof. At first sight it seems that the missing points are of technical nature and not difficult to fill, but probably subsequent pages will show a different truth.

The starting point is the formula valid for $\Re s > 1$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{3.1}$$

Let $L$ be the line $\{\Re s = c\}$, for some $1 < c < 2$, and $R_T$, $S_T$ the "infinite rectangles" $\{\Im s \leq T,\ \Re s \leq c\}$, $\{\Im s \leq T,\ \Re s \geq c\}$, respectively. For $x > 2$, $x \notin \mathbb{Z}$,

$$\begin{aligned} -\int_L \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}\, ds &= \sum_{n<x} \Lambda(n) \int_L \left(\frac{x}{n}\right)^s \frac{ds}{s} + \sum_{n>x} \Lambda(n) \int_L \left(\frac{x}{n}\right)^s \frac{ds}{s} \\ &= \sum_{n<x} \Lambda(n) \lim_{T\to\infty} \int_{\partial R_T} \left(\frac{x}{n}\right)^s \frac{ds}{s} + \sum_{n>x} \Lambda(n) \lim_{T\to\infty} \int_{\partial S_T} \left(\frac{x}{n}\right)^s \frac{ds}{s} \end{aligned}$$

By Cauchy's integral formula, the first integral equals $2\pi i$ (there is a simple pole at $s = 0$) and the second integral equals 0 (no poles in $S_T$). Hence we have a neat analytic formula for our favorite arithmetical function:

$$\psi(x) = \frac{1}{2\pi i}\int_L f(s)\,ds \qquad \text{where} \qquad f(s) = \frac{\zeta'(s)}{s\zeta(s)}x^s.$$

As $\zeta$ is meromorphic so is $f(s)$. Residue theorem gives

$$\psi(x) = -\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\partial R_T}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds = \sum_{s\in\mathcal{P}}\operatorname{Res}(f,s)$$

where $\mathcal{P}$ is the set of poles of $f$. Recalling that $s = 1$ is the unique pole of $\zeta$, it follows $\mathcal{P} = \{0,1\}\cup\mathcal{Z}$ where $\mathcal{Z}$ is the set of zeros of $\zeta$. Moreover $\operatorname{Res}(f,0) = -\zeta'(0)/\zeta(0)$, $\operatorname{Res}(f,1) = x$ (because $\zeta(s)\sim 1/(s-1)$ as $s\to 1$); and if $z\in\mathcal{Z}$ is a zero of multiplicity $m$, $\operatorname{Res}(f,z) = mx^z/z$. Therefore

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{z\in\mathcal{Z}}\frac{x^z}{z} \tag{3.2}$$

where each zero is repeated in the summation according to its multiplicity.

After this amazing formula, one can claim that the answer to any question regarding to the distribution of prime numbers is embodied in the distribution of the zeros of Riemann's zeta function. In particular, if $\Re z < 1$ for every $z\in\mathcal{Z}$, then $|x^z| = x^{\Re z} = o(x)$ and, trusting on good convergence properties of the series, PNT follows in the form $\psi(x)\sim x$.

Let us finish with an unbelievably ingenious proof due to Mertens of the missing point $\mathcal{Z}\subset\{\Re s < 1\}$. The convergence for $\Re s > 1$ of the series in (3.1) implies that $\mathcal{Z}\subset\{\Re s\le 1\}$. Assume that there exists a zero $z$ of $\zeta$ with $\Re z = 1$, say $z = 1 + Bi$ (note that after (3.2), the existence of this zero ruins PNT), and consider $g(s) = \zeta^3(s)\zeta^4(s+Bi)\zeta(s+2Bi)$. This is a meromorphic function with a zero at $s = 1$ (because $3 < 4$) and consequently, $\lim_{x\to 1^+}\log|g(x)| = -\infty$ for $x\in\mathbb{R}$. On the other hand, for $x > 1$, using the definition of $\zeta$ and Taylor expansion:

$$\begin{aligned}\Re\log g(x) &= -\Re\sum_p\big(3\log(1-p^{-x}) - 4\log(1-p^{-x-Bi}) - \log(1-p^{-x-2Bi})\big)\\ &= \Re\sum_p\sum_{n=1}^{\infty}\frac{1}{n}p^{-nx}(3 + 4p^{-Bni} + p^{-2Bni}).\end{aligned}$$

But the term between parenthesis is positive, because primer calculus course techniques prove $3 + 4\cos\alpha + \cos(2\alpha)\ge 0$ (or apply double-angle formulas to $2(1+\cos\alpha)^2 > 0$). Hence $\Re\log g(x) = \log|g(x)| > 0$ and it contradicts $\log|g(x)|\to-\infty$.

# 4 A nice symmetric function

We are going to prove at once that $\zeta$ extends to a meromorphic function on the whole complex plane (this is our second proof of this fact) and that it has a kind of symmetry with respect to the line $\Re s = 1/2$. The formula expressing this symmetry is the well known *functional equation* and is formally a consequence of Poisson summation formula applied to $f(x) = x^{-s}$ crossing out infinities. The actual proof given by Riemann establishes in general an interesting link between functional equations and modular relations. Modern Number Theory is plenty of underlying modular forms, and this explains why we have a lot of similar-looking functional equations.

Riemann's starting point was the integral representation for $\Re s > 0$ of $\Gamma$-function after a change of variable:

$$\Gamma(s/2) = \int_0^\infty t^{s/2-1} e^{-t}\, dt = \pi^{s/2} n^s \int_0^\infty t^{s/2-1} e^{-\pi n^2 t}\, dt.$$

Summing on $n$, for $\Re s > 1$,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty t^{s/2-1}(\theta(t)-1)\, dt \qquad \text{where} \quad \theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$

Now we can apply comfortably Poisson summation formula through the closed (modular) relation* $\theta(t) = t^{-1/2}\theta(1/t)$. It allows to transform the part $\int_0^1$ of the integration which is reponsible of the lack of convergence for $\Re s \le 1$.

$$\int_0^1 t^{s/2-1}(\theta(t)-1)\, dt = \int_0^1 t^{s/2-1}(t^{-1/2}\theta(1/t)-1)\, dt = \int_1^\infty t^{-s/2-1}(t^{1/2}\theta(t)-1)\, dt.$$

Substituting, after some calculations, we obtain

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s(s-1)} + \frac{1}{2}\int_1^\infty (t^{s/2-1} + t^{-s/2-1/2})(\theta(t)-1)\, dt. \tag{4.1}$$

The right hand side defines a meromorphic function with $s = 0, 1$ as only poles. Moreover, it remains invariant under the change $s \mapsto 1-s$. Hence $\zeta$ is a meromorphic function and satisfies the functional equation

$$\boxed{\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).} \tag{4.2}$$

*See the appendix.

Following (partially) Riemann, we can introduce the entire function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

and functional equation (4.2) reduces to

$$\xi(s) = \xi(1-s).$$

From (4.1) we conclude (again) that $\zeta$ has a single pole at $s = 1$ with residue 1. Using that $\Gamma$ is holomorphic up to simples poles at $0,\ -1,\ -2,\ -3\dots$, (4.2) proves $\zeta$ is holomorphic on $\mathbb{C} - \{1\}$ and has simple zeros at $s = -2,\ -4,\ -6,\dots$. These zeros are called *trivial zeros*. As $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ does not vanish for $\Re s > 1$, there are not other zeros in $\Re s < 0$, hence *non-trivial zeros* are in the so called *critical strip* $0 \le \Re s \le 1$. Summarizing:

$$\{\text{Poles of } \zeta\} = \{1\} \qquad \{\text{Zeros of } \zeta\} = 2\mathbb{Z}^- \cup \{\text{Non-trivial zeros}\}$$
$$\{\text{Poles of } \xi\} = \emptyset \qquad \{\text{Zeros of } \xi\} = \{\text{Non-trivial zeros of } \zeta\}$$

# 5 Zeros here and there

The fake proof suggests that the core of prime number distribution theory is the study of the zeros of $\zeta$. We have already separated the "trivial zeros" $2\mathbb{Z}^-$, and the whole problem is to understand the zeros in the critical strip $0 \le \Re s \le 1$. We shall denote with $\rho$ each of these *non-trivial zeros*.

Some basic results in Complex Analysis play an important role in the subsequent study. From the historical point of view, it can be claimed that a part of the basis of Complex Analysis was created in connection with the proof of PNT.

Our first result is a neat relation between $\zeta'/\zeta$ and the non-trivial zeros.

**Theorem 5.1** *For a certain constant $C_0$, it holds*

$$\frac{\zeta'(s)}{\zeta(s)} = C_0 - \frac{1}{s-1} - \frac{\Gamma'(s/2+1)}{2\Gamma(s/2+1)} + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right). \tag{5.1}$$

*Proof:* Comparing (4.1) with the definition of $\Gamma$ (use $\theta(t) - 1 = O(e^{-t})$ and $|t^\alpha| \le t^{|\alpha|}$), it is deduced

$$|\xi(s)| = O((1+|s|^2)\Gamma(|s|+1)/2)) = O(e^{K|s|\log(|s|+1)}).$$

Hence $\xi$ is an entire function of order one, and according to Hadamard finite order function theory*, there exists a factorization

$$\xi(s) = e^{A+Bs} \prod_{\rho} (1 - s/\rho) e^{s/\rho}$$

for certain constants $A$, $B$. Calculating the logarithmic derivative $\xi'/\xi$ with this formula and the definition of $\xi$, the theorem follows. $\square$

A superabundance of zeros could give a too large error term in PNT. The following result shows that it is not the case.

**Proposition 5.2** *Let $N(T)$ be the number of non-trivial zeros $\rho$, counted with multiplicity, such that $|\Im\rho| \le T$. Then*

$$N(T+1) - N(T) = O(\log T) \qquad \textit{and} \qquad N(T) = O(T \log T).$$

*Proof:* Of course the latter formula is a straightforward consequence of the former. Using (5.1) with $s = 2 + iT$, $T \ge 2$, we obtain $\sum_\rho ((s-\rho)^{-1} + \rho^{-1}) = O(\log T)$. Taking real parts:

$$O(\log T) = \Re \sum_{\rho=\beta+i\gamma} \left( \frac{1}{2-\beta+i(T-\gamma)} + \frac{1}{\beta+i\gamma} \right) \ge \sum_\rho \frac{1}{4+(T-\gamma)^2}.$$

And it implies $N(T+1) - N(T) = O(\log T)$. $\square$

Although it is not necessary in our proof of PNT, with some extra effort (using argument principle) previous result can be sharpened as follows:

**Theorem 5.3** *It holds the asymptotic formula*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

The identity (5.1) show that the value of $\zeta'(s)/\zeta(s)$ is greatly influenced by the closest zeros to $s$. The next result quantifies this phenomenon.

*See the Appendix.

**Proposition 5.4** *We have*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho\,:\,|s-\rho|<1} (s-\rho)^{-1} + O(\log|t|)$$

*with a uniform O-constant for* $s = \sigma + it$, $\sigma \geq -1$, $|t| \geq 2$ *and* $s \notin \mathcal{Z}$.

*Proof:* Substracting (5.1) for $s = \sigma + it$ and $s = 2 + it$,

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \left( \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right) + O(\log|t|).$$

For $|s-\rho| > 1$, say $\rho = \beta + i\gamma$,

$$\left| \frac{1}{s-\rho} - \frac{1}{2+it-\rho} \right| = \frac{2-\sigma}{|s-\rho||2+it-\rho|} \leq C \frac{1}{4+(t-\gamma)^2}$$

with $C$ an absolute constant. We have already seen that the last term contributes $O(\log|t|)$ when summing over $\rho$. On the other hand, $\sum_{|s-\rho|<1}(2+it-\rho)^{-1} = O(\log|t|)$ because there are $O(\log|t|)$ zeros $\rho = \beta + i\gamma$ with $\gamma \in [t-1, t+1]$. □

We have studied so far "vertical distribution" of the zeros, but in order to prove PNT we need some horizontal control, namely we want to separate the real part of the zeros from $\Re s = 1$.

**Theorem 5.5** *There exists a positive constant* $C$ *such that* $\zeta$ *does not vanish for* $s = \sigma + it$ *in the region*

$$\sigma > 1 - \frac{1}{35\log(|t|+C)}.$$

*Proof:* From (3.1) and Mertens' argument (see the end of the fake proof), if $\rho_0 = A + Bi$ is a non-trivial zero, for $\sigma > 1$

$$-3\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4\Re\frac{\zeta'(\sigma+Bi)}{\zeta(\sigma+Bi)} - \Re\frac{\zeta'(\sigma+2Bi)}{\zeta(\sigma+2Bi)} \geq 0. \tag{5.2}$$

Note that for $s = \sigma + iB$ with $\sigma > 1$ and any non-trivial zero $\rho$, it holds $\Re(s-\rho)^{-1}$, $\Re\rho^{-1} > 0$. Taking this into account, by (5.1), for $C$ large enough

$$-\Re\frac{\zeta'(\sigma+Bi)}{\zeta(\sigma+Bi)} \leq -\Re(s-\rho_0)^{-1} + \frac{1}{2}\log(|B|+C), \quad -\Re\frac{\zeta'(\sigma+2Bi)}{\zeta(\sigma+2Bi)} \leq \frac{1}{2}\log(|B|+C).$$

On the other hand, $-\zeta(s)/\zeta(s) \sim (s-1)^{-1}$ as $s \to 1$, implies that $-\zeta(\sigma)/\zeta(\sigma) < 1.01/(\sigma-1)$ for $\sigma$ in some interval $(1, 1+\epsilon]$. Substituting these inequalities in (5.2)

$$\frac{3.03}{\sigma-1} + \frac{4}{\sigma-A} + \frac{5}{2}\log(|B|+C) > 0.$$

Choose $\sigma = 1+2/(11\log(|B|+C))$ (suppose $C$ large enough to assure $\sigma \in (1, 1+\epsilon]$). If $A > 1 - 1/(35\log(|B|+C))$, we get a contradiction. □

# 6 The real (complex) proof

The first gap in our fake proof of PNT is the application of Cauchy's integral formula to some suspicious infinite regions. For each $T > 2$, let $L_T = L \cap \{\Im s \le T\}$, *i.e.* $L_T$ is the segment connecting $c - iT$ with $c + iT$. Consider $x > 2$ far away from integers, say $\mathrm{Frac}(x) = 1/2$. Exponential decay of $(x/n)^s$ is enough to deduce:

$$\psi(x) = \frac{1}{2\pi i}\sum_{n<x}\Lambda(n)\int_{\partial R_T}\left(\frac{x}{n}\right)^s\frac{ds}{s} + \frac{1}{2\pi i}\sum_{n>x}\Lambda(n)\int_{\partial S_T}\left(\frac{x}{n}\right)^s\frac{ds}{s}.$$

(Remember that $R_T = \{\Im s \le T,\ \Re s \le c\}$ and $S_T = \{\Im s \le T,\ \Re s \ge c\}$). It is not hard to prove for $t > 0$, $t \ne 1$, that

$$\int_{c+iT}^{\pm\infty+iT}\frac{t^s}{s}\,ds = O\left(\frac{t^c}{T|\log t|}\right).$$

Using this bound, we obtain (note that $|n-x| < x/2 \Rightarrow (x/n)^c = O(1)$)

$$\psi(x) = \frac{1}{2\pi i}\int_{L_T} f(s)\,ds + O\left(\frac{1}{T}\sum_{|n-x|<x/2}\frac{\Lambda(n)}{|\log(x/n)|} + \frac{x^c}{T}\sum_{|n-x|\ge x/2}\frac{\Lambda(n)}{n^c|\log(x/n)|}\right)$$

(Remember that $f(s) = -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s} = \sum \Lambda(n)\left(\frac{x}{n}\right)^s\frac{1}{s}$). Taylor expansion leads to $|\log(x/n)|^{-1} = O(x/|n-x|)$ and the first term in the error contributes $O(\frac{x}{T}\log^2 x)$. On the other hand the second sum is $O(|\zeta'(c)/\zeta(c)|) = O((c-1)^{-1})$. If we choose $c = 1 + 1/\log x$ to clear up the calculations, we get

$$\psi(x) = \frac{1}{2\pi i}\int_{L_T} f(s)\,ds + O\left(\frac{x}{T}\log^2 x\right).$$

By Proposition 5.2 it is possible to choose lines $\Im s = \pm T$, for $T$ on each fixed interval of length one, separated from the zeros at least $C/\log T$, for some absolute constant $C$. Proposition 5.3 implies $|\zeta'/\zeta| = O(\log^2 T)$ on these lines for $\Re s \geq -1$. Consequently

$$\int_{-\infty-iT}^{c-iT} + \int_{c+iT}^{-\infty+iT} f(s)\, ds = O\left(\frac{x}{T}\log^2 T\right)$$

and

$$\psi(x) = \frac{1}{2\pi i}\int_{\partial R_T} f(s)\, ds + O\left(\frac{x}{T}\log^2(xT)\right).$$

Residue theorem gives

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n=1}^{\infty}\frac{x^{-2n}}{2n} - \sum_{|\Im\rho|<T}\frac{x^\rho}{\rho} + O\left(\frac{x}{T}\log^2(xT)\right). \tag{6.1}$$

When $T \to \infty$ this gives the so called *explicit formula*, the analog of (3.2). It is useless (because of the conditional convergence) but wonderful:

$$\psi(x) = x - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2}\log(1 - x^{-2}) - \sum_{\rho}\frac{x^\rho}{\rho}.$$

Note that $\psi(x+h) - \psi(x) \leq \log(x+1)$ for every $h \in [0,1]$, then adding an extra $O(\log x)$ term in (6.1), it holds true without restrictions on $\mathrm{Frac}(x)$. Cleaning negligible terms, we have

$$\psi(x) = x - \sum_{|\Im\rho|<T}\frac{x^\rho}{\rho} + O\left(\frac{x}{T}\log^2(xT) + \log x\right).$$

By Theorem 5.5 there are no zeros with $\Re\rho > 1 - 1/(35\log(T+C))$ and $|\Im\rho| \leq T$, and by Proposition 5.2 there are $O(\log N)$ with $N \leq |\Im\rho| \leq N+1$. Hence

$$\sum_{|\Im\rho|<T}\frac{x^\rho}{\rho} = O\left(\sum_{N\leq T}\frac{\log N}{N}x^{1-1/(35\log(T+C))}\right) = O\left(x^{1-1/(35\log(T+C))}\log^2 T\right).$$

Choosing $T = e^{0.17\sqrt{\log x}}$ and substituting, we obtain finally

$$\boxed{\psi(x) = x + O\left(x\, e^{-\frac{1}{6}\sqrt{\log x}}\right)}$$

indeed something slighty better. PNT is (at last) proved.

# 7 Epilogue: Riemann hypothesis

By the second formula in Proposition 1.1, if $\psi(x) = x + O(x^\beta)$ for every $\beta > \alpha$, then $-\zeta'(s)/\zeta(s) - s/(s-1)$ is holomorphic on $\Re s > \alpha$, in particular all the non-trivial zeros verify $1 - \alpha \le \Re\rho \le \alpha$. Hence the best scenario occurs when $\alpha = 1/2$, *i.e.* when the non-trivial zeros keep in single file.

**Riemann hypothesis:** *If $\rho$ is a non-trivial zero of $\zeta$ then $\Re s = 1/2$.*

From the anlytical point of view this is a very strange conjecture because there are no known reasons motivating the lining up of the zeros. And it is even more strange taking into account that in Number Theory there is a huge family of zeta-like complex analytical functions that apparently share the same property.

It is known by extensive numerical analysis that more than the first $10^{11}$ non-trivial zeros of the Riemann zeta function verify Riemann hypothesis, but so far we do not even know how to prove $\Re\rho \le \delta$ for some $\delta < 1$ and every non-trivial zero $\rho$.

Although, after more than 140 years, we are desperately far from Riemann hypothesis, in the meantime some theorems have sprung up about the distribution of the zeros that Riemann probably would like. We shall only mention three according to their strength:

(Hardy) There are infinitely many zeros of $\zeta$ with real part 1/2.

(Bohr, Landau et al.) The "density" of zeros on $\Re s \ge \alpha$ for any $\alpha > 1/2$ is arbitrarily small in comparison with the density on $\Re s \ge 1/2$.

(Selberg) A positive proportion of the zeros lay on the line $\Re s = 1/2$.

# 8 Appendix

In this appendix we shall refresh some topics related to Complex Analysis that we have used in previous pages.

**Gamma function:**

Gamma function is a kind of natural analytic extension of factorials to complex plane. For $\Re s > 0$ it is defined by the integral formula

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt.$$

Integrating by parts, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}^+$ and $\Gamma(s+1) = s\Gamma(s)$ in general. This functional equation allows to extend $\Gamma$ to a meromorphic function with simple poles at $s = 0, -1, -2, \ldots$. In particular $\Gamma(s) \sim s^{-1}$ when $s \to 0$. Far away from these poles, say for instance on $\mathbb{C} - \bigcup B_{0.1}(-n)$, $\Gamma(s) = O\big(e^{|s|\log|s|}\big)$ and $\Gamma'(s)/\Gamma(s) = O(\log|s|)$. Indeed it is possible to replace these bounds by asymptotic formulas (cf. Stirling's formula).

**Finite order functions:**

Hadamard's finite order function theory allows to factorize entire functions, under some growth condition, into something close to linear factors. Roughly speaking, it is a kind of Fundamental Theorem of Algebra for entire functions. In the order one case, it asserts that for an entire function $f$ satisfying $|f(z)| = O\big(e^{|z|^\alpha}\big)$ for every $\alpha > 1$, it holds

$$f(z) = e^{A+Bz} \prod (1 - z/z_n) e^{z/z_n}$$

where $A$ and $B$ are constants and $z_n$ runs over the zeros of $f$. Moreover the (possibly infinite) product is absolutely convergent.

**Poisson summation:**

If $f$ is smooth enough (say for instance $|f|$, $|f'|$ and $|f''|$ integrable) then Poisson summation formula reads

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x}\, dx.$$

Choosing $f(x) = e^{-\pi t x^2}$ where $t > 0$ is a parameter, one concludes the functional equation for $\theta$-function, $\theta(t) = t^{-1/2}\theta(1/t)$.

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