Problem set 3

Wavelets: theory and practice — Deadline: 18/May/2018

Problems

1) With the notation of the notes, assume that when processing a black-and-white JPEG image, we have $a_{nm} = 8\alpha_n\alpha_m \cos\left(\frac{\pi n}{16}\right)\cos\left(\frac{\pi m}{16}\right)$ for every block if n and m are both even and $a_{nm} = 0$ otherwise. How does the image look like?

2) Prove the Fourier expansions included in (3.2).

3) For some questions of analysis it is convenient to consider the wavelet transform associated to the so-called *analytic wavelet* $\psi(x) = (x+i)^{-n-1}$ for some fixed $n \in \mathbb{Z}^+$. Prove that it is a continuum wavelet and normalize it.

4) Given a continuum wavelet $\psi \in C_0^\infty$ with vanishing moments $\int_{-\infty}^{\infty} x^j \psi(x)\, dx = 0$ for $0 \le j < n$. Prove that if f belongs to the Sobolev space $H^s(\mathbb{R})$ and $n > s > 0$ then $|a|^{-1-s} W_\psi f(a,b) \in L^2(\mathbb{R}^2)$.

5) Compute the Haar wavelet expansion of the function $f(x) = x$ for $x \in [0,1)$ and $f(x) = 0$ otherwise. In other words, compute the coefficients c_{jk} giving

$$f(x) = \sum_{j,k\in\mathbb{Z}} c_{jk}\psi(2^j x - k)$$

where ψ is the Haar wavelet.

Notes and hints

1) I expect a mathematical argument. It is not valid saying "it looks like this" without further explanations. A short code can help you to know what to prove.

2) This is not a problem to try your patience with calculations. There are not such calculations, you have to find the trick bypassing the integrals that, by the way, WolframAlpha is not able to compute. Hint (end of the course gift): What happen when you compute $e^{\cos\alpha + i\sin\alpha}$ substituting $\cos\alpha + i\sin\alpha$ in the Taylor expansion of e^x? For the peak, you can use the Fourier expansion of the Fejér kernel appearing in the notes that you proved in the first problem set.

If you downloaded very early §3.1.1, note that I corrected a misprint in (3.1) in the last minute. The right version is the one currently uploaded.

3) If necessary, dust off your notes on complex analysis. If you compute $\widehat{\psi}$ explicitly I expect an explanation, something more informative that "WolframAlpha said so" because for this complex integrals you cannot fully trust on it. A hint is that they are called analytic wavelets because they verify $\widehat{\psi}(\xi) = 0$ for $\xi < 0$.

Actually, this kind of wavelets are defined for $n \in \mathbb{R}^+$ and $\widehat{\psi}$ can be computed explicitly in this general case with complex analysis techniques too. In case you are curious about the "questions of analysis", read the next exercise. This wavelet verifies the condition on the moments. For fractional n, it is employed to know if certain strange functions are Hölder continuous.

4) Recall that $H^s(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \ : \ |\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R})\}$ for $s > 0$. Do not get impressed with this unusually abstract exercise. Try to link the moments with the Fourier transform and, of course, with the wavelet transform. Does the proof of the inversion formula for the wavelet transform in the notes ring a bell?

The motivation for this problem is to illustrate the fact that the more vanishing moments the better behavior of the wavelet transform, a point that we did not develop in the course.

5) In `http://www.uam.es/fernando.chamizo/oscuro/haar.html` I have plotted some partial sums. This can be useful to check your results or to guess them.