Problem set 1

Simple waves | Deadline: 7/March/2018

Problems

1) Consider the driven harmonic oscillator ruled by the equation $x'' + \omega^2 x = \sin(\lambda t)$ with $\omega, \lambda > 0$ under the initial conditions $x(0) = 0$, $x'(0) = 1$. Compute the explicit solution $x_\omega(t)$ for $\omega \neq \lambda$ and find $\lim_{\omega \to \lambda} x_\omega(t)$. Check that the function resulting from this limit fulfills the equation with $\omega = \lambda$.

2) Prove the first formula of (1.51) in §1.2.1.

3) Solve the mathematical part of the experimental challenge *The strange beat*. You can take as a guidance the sample file `interference.mp3` in the web page of the course if you do not do the experiment by yourself.

4) Check the formula (1.104) for the function $f : S_3 \longrightarrow \mathbb{C}$ defined by $f(\mathrm{Id}) = f\big((2,3)\big) = 1$ and $f(g) = 0$ for $g \neq \mathrm{Id}, (2,3)$.

5) Reproduce the consequence of the uncertainty principle for f and f_δ as in §1.2.3 with f_δ at least C^1. Namely, for $\delta = 0.1$ find an explicit even nonnegative function $f_\delta \in C^1$ such that $f(x) = f_\delta(x)$ for $|x| \notin [1 - \delta/2, 1 + \delta/2]$ and $\int f_\delta = 2$. For this function, find $x_\epsilon > 0$ for $\epsilon = 1/2$, $1/4$ and $1/8$ such that

$$\frac{\big|\widehat{f}(x) - \widehat{f_\delta}(x)\big|}{\sup_{t>x} |\widehat{f}(t)|} \le \epsilon \qquad \text{for every } 0 \le x \le x_\epsilon,$$

trying to get x_ϵ as large as possible.

Notes and hints

1) "Explicit" means involving only elementary functions. If your favorite method to solve the ODE involves integrals, you must compute them. The limit must also be computed, it is not valid to guess the result by some kind of continuous dependance on the parameters.

Note that x_ω is bounded for $\omega \neq \lambda$ but the limiting solution is not because of the resonance.

2) The second formula of (1.51) is easy integrating by parts, right? A possible approach is to use a discrete analogue in the first formula. There are much shorter trickier methods that probably you can improvise. In any approach that comes to my mind one uses in one way or another a relation with the Dirichlet kernel.

3) If this exercise seems almost trivial to you, do not worry. Actually, it is. It only involves high schools Mathematics but the experimental meaning is somewhat unexpected and something to keep in mind by anybody interested in signal processing.

4) You can grab from §1.2.5 that the (non-equivalent) unitary irreducible unitary representations of S_3 are given by (1.102) and by two easy one-dimensional representations mentioned there. I assume you remember the usual notation for permutations, if not, you should look it up in any book of basic algebra. This exercise is very short and you can solve it in few lines once you know what is talking about. Its main purpose is to understand the formula (1.104). If you already do, it will take less than three minutes (but surely you spent something more when you studied it).

5) It goes without saying that f is the characteristic function of the interval $[-1,1]$. If you construct f_δ without taking into account the properties of the Fourier transform, you can enter in a very big mess with the calculations. Does *convolution* ring a bell? I do not want to be very picky with "as large as possible", it is just to avoid trivial solutions.