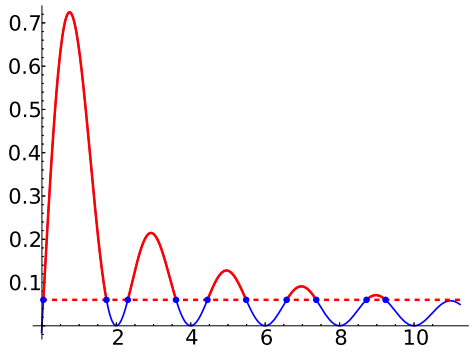
Experimental challenge*: **The new wave**

Wavelets: theory and practice — Deadline: End of the course

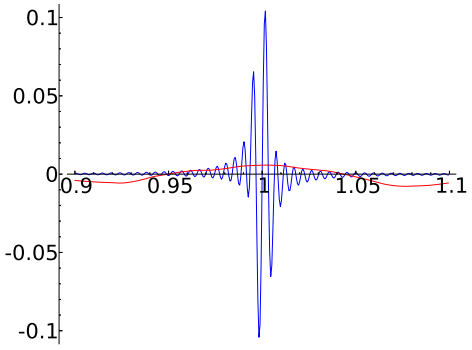
Experimental part

This experiment is demanding. If you are not good using mathematical software, consider to skip it. You will get a big help if you adapt the `SAGE` code following the link "The Meyer wavelet" in the website of the course.

Consider $f(x) = \operatorname{sgn}(x)\chi_{[-1/2,1/2]}(x)$ with Fourier transform $\widehat{f}(\xi) = 2(\pi\xi)^{-1}\sin^2(\pi\xi/2)$. Write a code that given $\delta > 0$ recovers f considering only the values with $|\widehat{f}| > \delta$, the part indicated in the first figure. In other words, approximate numerically $\int_{-\infty}^{\infty} g_\delta\big(\widehat{f}(\xi)\big)e(x\xi)\,d\xi$ where $g_\delta(x) = x$ for $|x| > \delta$ and vanishes otherwise. Write also a code to analyze f in terms of the Meyer wavelet and truncate the series $\sum c_{jk}\psi_{jk}$ to $\sum g_\delta(c_{jk})\psi_{jk}$.



Nonzero part of $g_{0.06}(\widehat{f})$



Fourier (blue) Meyer (red) for $\delta = 0.003$

Plot the approximate reconstruction in both cases and check, as depicted in the second figure, that wavelets (the new waves) avoid the strange artifacts introduced by Fourier analysis around $\pm n/2$ with $n \in \mathbb{Z}_{\geq 2}$. It is fair to mention that Fourier analysis is spotless in other zones and there it works better than this simple instance of wavelet.

Mathematical part

The experimental part is too demanding to ask difficult mathematical questions. Just check the formula for $\widehat{f}$ and prove that Meyer's wavelet belongs to $C^\infty \cap L^1$. This is not automatic from the formula because of the possible vanishing of the denominators.

*Some experiments are classical, some are my idea and others come from specific sources. In the latter case I have omitted the reference here on purpose to force the students to work on their own. If you are the author, please do not get angry. I intend to incorporate the references to the final version of the notes.