gravitational acceleration and h is the deepness of the channel. It implies that the velocity of water waves is $\sqrt{gh}$ according to the model. This is more or less precise for shallow water but it is completely unrealistic in the middle of the ocean where the deepness is of the order of kilometers. The value of $\sqrt{gh}$ does not match with the actual speed and it sounds unnatural that the velocity could be affected in any way by the bottom of the ocean. A revised model taking also into account the vertical acceleration, gives in (1.30) a κ depending on the frequency of the waves. In this way the coefficient of the equation depends on the solution and we have a highly complicated example of non-linearity that is an active field of research for theoreticians.

Suggested Readings. This is classic material that can be found in many books for undergraduates (for instance [AF67]). Talking about classics, perhaps it is worth to have a look to the translations in `http://www.17centurymaths.com/` of early works by Euler, specially E305.

1.2 Mathematical methods and results

1.2.1 Basic Fourier series and integrals

J. Fourier defended in his celebrated memoir [Fou88] that periodic functions can be analyzed in terms of cosine and sine functions but he was not able to provide a proof, this task was completed by P.G.L. Dirichlet in 1828, six years after the publication of [Fou88]. Nowadays we consider it as an important and deep result but not so difficult to prove. Before giving any proof, we are going to try to understand why it should be true.

A formal simplification occurs unifying cosine and sine into the complex exponential. As it is going to appear everywhere, we use a special notation:

$$e(x) := e^{2\pi i x} = \cos(2\pi x) + i\sin(2\pi x). \tag{1.31}$$

It is also convenient to introduce the notation $\mathbb{T}$ to mean $\mathbb{R}/\{x \mapsto x+1\}$ with the quotient topology that we can visualize as any interval of length 1 gluing together the extreme points (it is usually called *torus* but in this 1D case it is rather a circle). In this way, a function $f : \mathbb{T} \longrightarrow \mathbb{C}$ can be unwrapped as a 1-periodic function $f : \mathbb{R} \longrightarrow \mathbb{C}$ and we identify such interpretations. For instance $f \in C^k(\mathbb{T})$ means $f \in C^k$ and 1-periodic. For 1-periodic functions, the integral over any unit interval is often denoted as an integral over $\mathbb{T}$ because there is no ambiguity in the choice of the interval.

Let g be a function $g : \mathbb{Z} \longrightarrow \mathbb{C}$ such that $g(n+N) = g(n)$; if you prefer so, $\{g(n)\}_{n\in\mathbb{Z}}$ is a two-sided N-periodic sequence. It is a simple exercise to check

$$g(n) = \sum_{m\in I} a_m e(mn/N) \qquad \text{with} \quad a_m = \frac{1}{N}\sum_{k\in I} g(k)e(-mk/N) \tag{1.32}$$

where I is any set of N consecutive integers. Is it really simple? Yes, it is. Just substitute the formula for a_m and use that for $n, k \in I$

$$\frac{1}{N}\sum_{m\in I} e\big(m(n-k)/N\big) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k, \end{cases} \tag{1.33}$$

because when we sum the roots of the unity they cancel, they are forces pulling in symmetric directions and no net force results.

If you substitute $g(n) = f(n/N)$ and let $n \to \infty$ then f comes a generic 1-periodic function $f : \mathbb{R} \longrightarrow \mathbb{C}$, at least in some intuitive way. The formula for a_m is a Riemann sum and we imagine that in the first formula of (1.32), I becomes $\mathbb{Z}$. With these hand waving manipulations we infer that for 1-periodic functions

$$f(x) = \sum_{m\in\mathbb{Z}} a_m e(mx) \qquad \text{with} \quad a_m = \int_{\mathbb{T}} f(t)e(-mt)\,dt. \tag{1.34}$$

The sum is the famous *Fourier series*. We must humbly recognize that this is not a rigorous proof but we can proudly trumpet that it is closer to it than Fourier attempts. Even more, with the standards of rigor at the first quarter of the 19th century perhaps it would have been admitted as a proof. From the modern point of view, it is possible to stain the argument with some ϵ's and δ's and turn it into an actual proof when f is regular, for instance $f \in C^\infty(\mathbb{T})$.

If we admit (1.34) for 1-periodic functions, the T-periodic functions are covered by a change of variables

$$f(x) = \frac{1}{T}\sum_{m\in\mathbb{Z}} a_m e(mx/T) \qquad \text{where} \quad a_m = \int_{-T/2}^{T/2} f(t)e(-nt/T)\,dt. \tag{1.35}$$

As before, we can read the sum as a Riemann sum and one expects in the limit $T \to \infty$ the *Fourier inversion formula*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e(x\xi)\,d\xi \qquad \text{where} \quad \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e(-t\xi)\,dt. \tag{1.36}$$

This already appears in Fourier's memoir [Fou88]. Doing justice to him, the operator $\mathcal{F} : f \longmapsto \widehat{f}$ is called the *Fourier transform*.

Now we are going to tell a fairy tale suggesting a line of attack to get (1.34) and (1.36). Do you remember the *Dirac delta*? It was introduced by P.A.M. Dirac when mathematizing quantum mechanics. In physics it is managed all the time as a "function" δ satisfying $\int_{\mathbb{R}} \delta = 1$ and $\int_{\mathbb{R}} f\delta = f(0)$. For pure mathematicians writing something like this is heretical and they tell the same thing in a wordy way. They say that is a *distribution* (function is tabooed here), a certain type of operator acting on smooth functions and it is represented by an *approximation to the identity*, a collection of functions $\eta_\epsilon(x) = \epsilon^{-1}\eta(x/\epsilon)$ with $\eta \in L^1 \cap C^\infty$ and $\int \eta = 1$. For $\epsilon > 0$ small, we shrink the X-axis and stretch the Y-axis, then $\int_{\mathbb{R}} \eta_\epsilon = 1$ and it is not difficult to prove $\lim_{\epsilon\to 0^+} \int_{\mathbb{R}} \eta_\epsilon f = f(0)$ for $f \in C_0^\infty$. Although pure mathematicians do not say it in public, in the darkness of their offices, the Dirac delta δ is the function $\lim_{\epsilon\to 0^+} \eta_\epsilon$.

The bottom line is now that (1.32) was very easy to prove because we had the simple formula (1.33) for a displaced finite Dirac delta, whatever it means. Let us state two spooky analogues that suit our needs:

$$\delta_P(x) = \sum_{n\in\mathbb{Z}} e(nx) \qquad \text{and} \qquad \delta(x) = \int_{-\infty}^{\infty} e(x\xi)\,d\xi \tag{1.37}$$

where δ_P is the 1-periodic Dirac delta $\delta_P(x) = \sum_{n\in\mathbb{Z}} \delta(x-n)$, sometimes called *Dirac comb*. Pure mathematicians rip their garments apart when see (1.37) but for theoretical physicists it is the daily bread. In signal analysis you should have (1.37) in mind even if you do not dare to write it.

If we believe in fairies and in (1.37), for f 1-periodic

$$f(x) = \int_{\mathbb{T}} f(t)\delta_P(x-t)\,dt = \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} f(t)e(-nt)e(nx)\,dt = \sum_{m\in\mathbb{Z}} a_m e(mx) \tag{1.38}$$

and we have (1.34). In the same way, we get (1.36) from

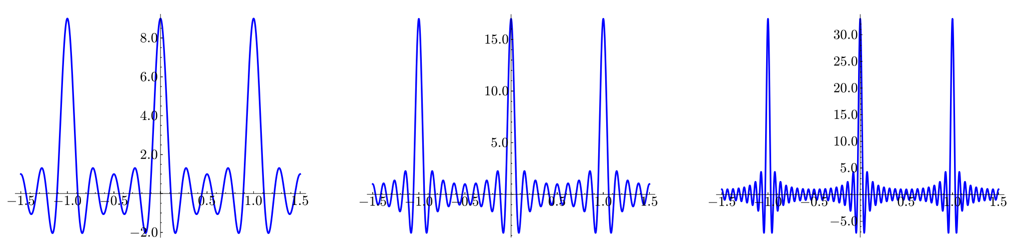
$$f(x) = \int_{-\infty}^{\infty} f(t)\delta(x-t)\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e(-t\xi)e(x\xi)\,dtd\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e(x\xi)\,d\xi. \tag{1.39}$$

The fairies are hidden in changing the order of summation or integration.

Although it is not explicitly mentioned in the most of the textbooks for mathematicians, (1.37) and our proof by vigorous handwaving [RD05, p.28] guide the actual proof of the basic convergence theorems in Fourier analysis. The most expeditious way of dodging our qualms about infinity in the first formula, is cutting the series to the so-called *Dirichlet kernel*

$$D_N(x) = \sum_{n=-N}^{N} e(nx), \qquad D_N(x) = \begin{cases} \dfrac{\sin\left(\pi(2N+1)x\right)}{\sin(\pi x)} & \text{if } x \notin \mathbb{Z}, \\ 2N+1 & \text{if } x \in \mathbb{Z}. \end{cases} \tag{1.40}$$

The explicit formula to the right follows from the sum of a geometric finite sequence. When N grows, it has a big peak around each integer and the integral of D_N equals 1 on $\mathbb{T}$, then it sounds like an approximation to the identity. The following plots show the aspect of the graph of D_N for some values of N.



N = 4 N = 8 N = 16

A proxy of the first equation in (1.38) is

$$f(x) = \int_{-1/2}^{1/2} f(t)D_N(x-t)\,dt + \int_{-1/2}^{1/2} \big(f(x)-f(t)\big)D_N(x-t)\,dt. \tag{1.41}$$

The first integral is

$$S_N f(x) = \sum_{m=-N}^{N} a_m e(mx) \qquad \text{with} \quad a_m = \int_{\mathbb{T}} f(t)e(-mt)\,dt, \tag{1.42}$$

the partial sum of the Fourier series (1.34). The pointwise convergence $S_N f(x) \to f(x)$ is equivalent to the vanishing of the second integral in the limit. By the periodicity, this can be written as

$$\lim_{N\to\infty} \int_{-1/2}^{1/2} \big(f(x) - f(x-t)\big) \frac{\sin\big(\pi(2N+1)t\big)}{\sin(\pi t)}\, dt = 0. \tag{1.43}$$

If $f \in C^1(\mathbb{T})$, for t very close to 0, $\big(f(x)-f(x-t)\big)/\sin(\pi t)$ can be extended to a continuous, in particular bounded, function, while for t not very close close to 0, we can integrate by parts to get an $O(N^{-1})$ when integrating $\sin\big(\pi(2N+1)t\big)$. This scheme proves

Theorem 1.2.1. *If $f \in C^1(\mathbb{T})$ then $S_N f \to f$ uniformly.*

One can squeeze the argument incorporating some technical tricks. One of them deserves a proper name (two, to say the full truth), the *Riemann-Lebesgue lemma*. It seems that it was proved firstly by B. Riemann [Bré02, p.15] whose outstanding contribution to Fourier series has been somewhat eclipsed for other of his famous works [Cór08].

Lemma 1.2.2 (Riemann-Lebesgue)**.** *If $f \in L^1$ then $\widehat{f}(\xi) \to 0$ when $\xi \to \infty$.*

The proof is easy: Integrable functions can be approximated by step functions and step functions can be approximated by functions in C_0^1 and for the latter the result is obvious integrating by parts.

By Riemann-Lebesgue lemma, if $\big(f(x) - f(x-t)\big)/\sin(\pi t)$ is integrable for a given x (as a function of t), we have (1.43) and then $S_N f(x) \to f(x)$. Except for some variations [Bré02, §1.3] [Zyg88], this is called *Dini's theorem*. It implies that if $f \in C(\mathbb{T})$ satisfies a *Hölder condition* of order α, i.e. $|f(x) - f(y)| = O\big(|x-y|^\alpha\big)$ for some $0 < \alpha \le 1$, then $S_N f \to f$ pointwise. It is known that the continuity of f is not enough to assure everywhere convergence[3] [Kör88, §15].

If f has lateral limits everywhere, for each x define

$$M(x) = \frac{f(x^+) + f(x^-)}{2}. \tag{1.44}$$

Then proceeding as in (1.41) and (1.43), $S_N f(x) \to M(x)$ if and only if

$$\int_{-1/2}^{1/2} \big(M(x) - f(x-t)\big) \frac{\sin\big(\pi(2N+1)t\big)}{\sin(\pi t)}\, dt \to 0. \tag{1.45}$$

Equivalently,

$$\int_{0}^{1/2} \big(M(x) - f(x-t) - f(x+t)\big) \frac{\sin\big(\pi(2N+1)t\big)}{\sin(\pi t)}\, dt \to 0. \tag{1.46}$$

[3] In fact in [KK66] it is proved that given a zero measure set $E \subset \mathbb{T}$ there exists a continuous function such that for each $x \in E$, $S_N f(x)$ does not converge as $N \to \infty$. The proof is shorter than one might think but, as somebody said, this margin is too small to include it.

If we assume also the existence of lateral derivatives everywhere and that they are integrable, we conclude that $S_N f(x)$ converges to $M(x)$ everywhere.The existence of the lateral derivatives can be relaxed in a set of measure zero. Essentially *bounded variation functions* are the almost everywhere differentiable functions with integrable derivative [Rud74]. Then the most general result that we can get with this line of reasoning is [Kat68, II.2]

Theorem 1.2.3. *If f is of bounded variation in $\mathbb{T}$ then we have $S_N f(x) \to M(x)$ when $N \to \infty$ for any x and $M(x)$ as in* (1.44).

When one tries to extend this kind of arguments to deal with (1.36) a new problem appears due to the fact we have to worry about the convergence of two infinite integrals. Taking into account the second formula in (1.37), the natural analogue of the Dirichlet kernel is

$$\int_{-N}^{N} e(x\xi)\, d\xi = \frac{\sin(2\pi N x)}{\pi x}. \tag{1.47}$$

The formula that parallels (1.41) is

$$f(x) = \int_{-\infty}^{\infty} f(t) \int_{-N}^{N} e((x-t)\xi)\, d\xi dt + \int_{-\infty}^{\infty} \big(f(x) - f(t)\big) \frac{\sin\big(2\pi N(x-t)\big)}{\pi(x-t)}\, dt. \tag{1.48}$$

If $f(x) = 0$ everything is OK and we can repeat the argument at these value of x for instance when $f \in L^1 \cap C^1$ but if $f(x) \neq 0$ the last integral does not make sense as a Lebesgue integral even if $f(t) \to 0$ smoothly when $t \to \infty$ because the function under the integral decays as $f(x)/\pi t$ which is not integrable. Then extra subtleties must be introduced to give a meaning to the previous expression and conclude an analogue of Theorem 1.2.1, namely [Hel91, §1.2]

Theorem 1.2.4. *If $f \in L^1 \cap C^1$ then*

$$f(x) = \lim_{N\to\infty} \int_{-N}^{N} \widehat{f}(\xi) e(x\xi)\, d\xi. \tag{1.49}$$

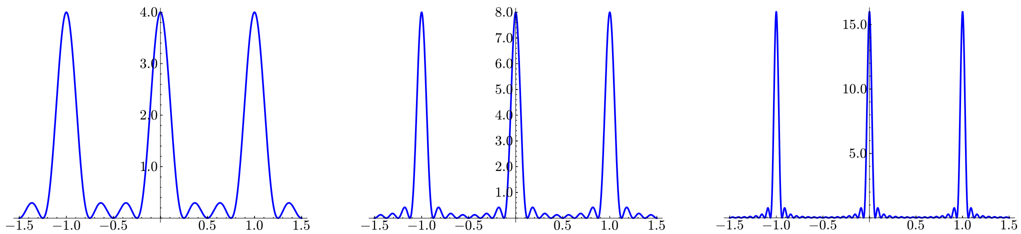
Establishing minimal conditions to have

$$S_N f(x) \to f(x) \qquad \text{and} \qquad \int_{-N}^{N} \widehat{f}(\xi) e(x\xi)\, d\xi \to f(x) \tag{1.50}$$

in "many points" is a big topic in classical harmonic analysis with some deep results. The most celebrated is the *Carleson-Hunt theorem* [Car66] [Hun68] [LT00] which states $S_N f(x) \to f(x)$ almost everywhere for any $f \in L^p(\mathbb{T})$, $p > 1$. In signal processing this is not really very important in practice because the sharp truncation of the Fourier series or the Fourier integral may be natural for our mathematical mind but it is a bad idea if we want to approximate a slightly rough function by sines and cosines. To give a theoretical basis to this assertion, one can say that there are better ways than truncation to get Dirac deltas as limits in (1.37). For instance, we have

$$\sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N}\Big) e(nx) = \frac{\sin^2(\pi N x)}{N \sin^2(\pi x)} \qquad \text{and} \qquad \int_{-N}^{N} \Big(1 - \frac{|\xi|}{N}\Big) e(x\xi)\, d\xi = \frac{\sin^2(\pi N x)}{N\pi^2 x^2}. \tag{1.51}$$

The advantage of these functions is that they are integrable and with L^1-norm uniformly bounded in N. If we compare the graphs of the first function for several values of N with those of $D_N(x)$ we guess a better approximation to our idea of the Dirac comb δ_P



N = 4 N = 8 N = 16

Usually this function is called *Fejér kernel*. It mollifies Dirichlet's kernel avoiding sharp cut high frequencies. The outcome is a more clear, and practical, convergence theorem

Theorem 1.2.5 (Fejér)**.** *For $f \in C(\mathbb{T})$, we have*

$$f(x) = \lim_{N\to\infty} \sum_{m=-N}^{N} \Big(1 - \frac{|m|}{N}\Big) a_m e(mx) \tag{1.52}$$

uniformly in x, with a_m as in (1.34)*.*

The same can be stated for the Fourier transform but the integrability is not a consequence of continuity in this case and must be required.

Theorem 1.2.6. *For $f \in C(\mathbb{R})$ integrable, we have*

$$f(x) = \lim_{N\to\infty} \int_{-N}^{N} \Big(1 - \frac{|\xi|}{N}\Big) \widehat{f}(\xi) e(x\xi)\, d\xi \tag{1.53}$$

uniformly over compact sets.

Formally, putting $N = \infty$ in these results we obtain the formulas (1.34) and (1.36). The proofs of these theorems follow the scheme sketched before. We include them here anyway.

Proof of Theorem 1.2.5. Let $F_N(x)$ be the first function in (1.51). Clearly $\int_{\mathbb{T}} F_N = 1$ then we can write as in (1.41)

$$f(x) = \int_{\mathbb{T}} f(t) F_N(x-t)\, dt + \int_{\mathbb{T}} \big(f(x) - f(t)\big) F_N(x-t)\, dt. \tag{1.54}$$

The first integral gives the sum appearing in the statement and it remains to prove that the last integral goes to zero. Substitute $\mathbb{T}$ by the interval $[x - 1/2, x + 1/2]$. As it is compact, f is uniformly continuous and then for every $\epsilon > 0$ there exist $0 < \delta < 1/2$ such that $|f(x) - f(t)| < \epsilon$ whenever $|x - t| < \delta$ and these values contribute to the integral less than ϵ because $\int F_N = \int |F_N| = 1$. On the other hand, $F_N(u) \to 0$ uniformly when $\delta < |u| < 1/2$. $\square$

Proof of Theorem 1.2.6. If $\mathcal{F}_N(x)$ is the second function in (1.51), we mimic the previous proof starting with

$$f(x) = \int_{-\infty}^{\infty} f(t)\mathcal{F}_N(x-t)\,dt + \int_{-\infty}^{\infty} \big(f(x)-f(t)\big)\mathcal{F}_N(x-t)\,dt. \tag{1.55}$$

As $f \in L^1$, we can change the order of integration (Fubini's theorem) and prove that the first integral coincides with the integral in the statement. Now we have to prove that the second integral goes to zero. For each x, let $\mathcal{C}_x = \{t : |x-t| > 1/2\}$. If the values of x are restricted to a compact set, $f(x)$ remains bounded, hence $\int_{\mathcal{C}_x} f(x)\mathcal{F}_N(x-t)\,dt = O(N^{-1})$. On the other hand, $\int_{\mathcal{C}_x} f(t)\mathcal{F}_N(x-t)\,dt = O(N^{-1})$ because $\mathcal{F}_N(x-t) = O(N^{-1})$ and f is integrable. Therefore it remains to prove that the contribution of $|x-t| < 1/2$ is negligible and it follows as in the previous proof. Note that the functions $F_N(u)$ and $\mathcal{F}_N(u)$ are comparable when $|u| < 1/2$. □

Let us move to some explicit examples of Fourier series. In our pocket calculator we do not find many periodic functions different from cosine and sine or easily related to them. Perhaps the only exception is the fractional part $x - \lfloor x \rfloor$ where

$$\lfloor x \rfloor := \min\{n \in \mathbb{Z} : n \le x\} \tag{1.56}$$

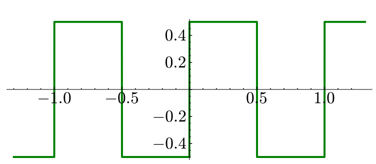
is the integral part. With it we are going to construct some real 1-periodic functions that are famous enough to deserve a name. We call them signals and write t instead of x to emphasize that they appear in engineering. We present the expansions in terms of sines and cosines, the complex exponential form would be obtained substituting the *Euler formulas*

$$\sin(2\pi t) = \frac{e(t)-e(-t)}{2i} \qquad \text{and} \qquad \cos(2\pi t) = \frac{e(t)+e(-t)}{2}. \tag{1.57}$$

The plots of the signals are drawn with continuous lines as it would be seen in an oscilloscope. Recall that in jump discontinuities the Fourier series converges to the middle point by Theorem 1.2.3. If you play these signals as sounds (there are online applications to do it), you will note that they sound differently although they have the same frequency. The shape of a sound wave gives the *timbre* (tone quality), its frequency the *pitch* and its amplitude (height) the *volume*.

The first example is the *square wave*, the simplest periodic digital signal taking the values -1 and 1.

(1.58) $f(t) = \frac{1}{2}(-1)^{\lfloor 2t \rfloor}$

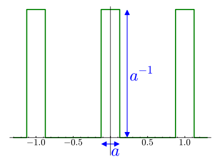


$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin\big(2\pi(2n-1)t\big)}{2n-1}.$$

The *pulse wave* is... guess it! a pulse of width $a < 1$ that tends to the Dirac comb when $a \to 0$. Note that this limit is formally coherent with the expansion of δ_P in (1.37).

(1.59)

$$f(t) = \frac{\max\Big(0, (-1)^{\lfloor 2t+a \rfloor} - (-1)^{\lfloor 2t-a \rfloor}\Big)}{2a}$$



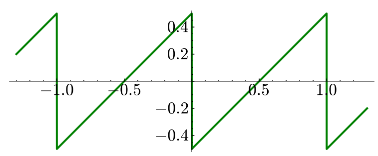
$$= \sum_{n=-\infty}^{\infty} \operatorname{sinc}(an)\cos(2\pi n t)$$

where sinc is a useful abbreviation used in signal processing to mean

$$\operatorname{sinc}(x) = \int_{-1/2}^{1/2} e(x\xi)\, d\xi = \begin{cases} \dfrac{\sin(\pi x)}{\pi x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases} \tag{1.60}$$

The *sawtooth wave* has again a self-explanatory name, at least until you see the next example. Its Fourier expansion plays a role in some number theoretical topics. It can be proved that the series truncated to $n \le N$ gives an error $O\big((N|x - \lfloor x + 1/2 \rfloor|)^{-1}\big)$.

$$f(t) = t - \lfloor t \rfloor - \frac{1}{2} \tag{1.61}$$



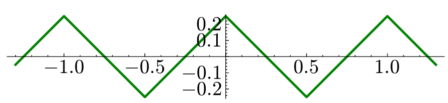
$$= -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi n t)}{n}.$$

Choosing $t = 1/4$ we get Leibniz series $\pi/4 = 1 - 3^{-1} + 5^{-1} - 7^{-1} + 9^{-1} - \dots$ with is really pretty but useless for precise numerical approximations of π.

In the *triangle wave* there are not discontinuities and the convergence is quicker. If you plot few terms in the Fourier series you will notice a very good approximation with small differences near the corners.

(1.62)

$$f(t) = \frac{1}{4} - \Big|t - \lfloor t + \frac{1}{2} \rfloor\Big|$$



$$= \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos\big(2\pi(2n-1)t\big)}{(2n-1)^2}.$$

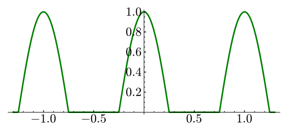
In amplitude modulated telecommunications (today they are old-fashioned) the information is given by the amplitude of a highly oscillating signal. We cannot plug in directly our headphones to this input because the average signal is zero in short periods (for instance in AM broadcasting the frequency is/was comparable to $10^6\, Hz$) and we would not hear anything. A primary need is to rectify the signal, choosing for instance the upper half. In this way short time averages are proportional to the amplitude and our headphones are responsive to them. This is achieved in practice with a *diode*, an electronic component allowing the current to flow only in one direction. In the early days they were valves and now they are semiconductors[4].

[4] Well, this use of semiconductors is not so modern if you have heard about the handmade crystal radio your great-great grandparent built with a mineral of galena and no batteries!

This story give us an excuse to add to the examples the *rectified cosine wave*.

(1.63)

$$f(t) = \max\left(0, \cos(2\pi t)\right)$$



$$= \frac{\cos(2\pi t)}{2} - \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\cos(4\pi n t)}{4n^2-1}.$$

In the case of the Fourier transforms there are several examples with non artificial closed expressions. By the inversion formula, the Fourier transform operator $\mathcal{F}$ verifies $\mathcal{F}(\mathcal{F}f)(x) = f(-x)$. As the following examples are even, we can read them in both directions. It does not add anything new in the self-reciprocal examples.

The first example is arguably the most important, the *Gaussian function*

$$e^{-ax^2} \stackrel{\mathcal{F}}{\longleftrightarrow} \sqrt{\frac{\pi}{a}}e^{-\pi^2\xi^2/a} \qquad \text{for } a > 0. \tag{1.64}$$

Of course e^{-ax} has an ugly exponential growth for $x < 0$ but we can amend it introducing an absolute value. We lose the differentiability at $x = 0$.

$$e^{-a|x|} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2a}{4\pi^2\xi^2 + a^2} \qquad \text{for } a > 0. \tag{1.65}$$

A smooth variant with exponential decay is given by the hyperbolic trigonometric function sech.

$$\operatorname{sech}(ax) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{a}\operatorname{sech}\left(a^{-1}\pi^2\xi\right) \qquad \text{for } a > 0. \tag{1.66}$$

No power is integrable because a problem appears either at 0 or at ∞. On the other hand, it is well-known that $\int_0^\infty x^{-\nu}\cos x\, dx$ is meaningful and finite as a Riemann improper integral for $0 < \nu < 1$ (it is not as Lebesgue integral). In this way, we can define the Fourier transform of $|x|^{-\nu}$ except at the origin.

$$|x|^{-\nu} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{\pi|2\pi\xi|^{\nu-1}}{\Gamma(\nu)\cos(\pi\nu/2)} \qquad \text{for } 0 < \nu < 1. \tag{1.67}$$

Here $\Gamma(\nu) = \int_0^\infty x^{\nu-1}e^{-x}\, dx$ is the usual *Gamma function*.

Although it is not regular, the Fourier transform of the characteristic function of the centered unit interval plays an important role in signal processing. This was the reason to introduce the definition (1.60).

$$\chi_{[-1/2,1/2]}(x) \stackrel{\mathcal{F}}{\longleftrightarrow} \operatorname{sinc}(x). \tag{1.68}$$

Again $\operatorname{sinc}(x)$ is not a Lebesgue integrable function and then for the left arrow we have to use Riemann improper integral. With it we recover $\chi_{[-1/2,1/2]}(x)$ except at $x = \pm 1/2$ where the value is $1/2$.

Suggested Readings. There are many and excellent books on basic Fourier analysis and preferring one or another is a question of personal taste. Among my favorites are [Kör88], [DM72], [Kat68], [Fol92] and [Hel91]. The classic [Zyg88] is certainly old and encyclopedic although interesting to learn because it reflects the freshness of some natural basic questions. A very basic online tone generator can be found in `http://www.szynalski.com/tone-generator/`.