The last integral is the famous Jacobi elliptic integral enjoying wonderful properties. The theory assures that $u = u(t)$ extends to a meromorphic function with two periods. This is cumbersome from the mathematical point of view but, following [Bae05], there is a convincing physical easy explanation: The pendulum is the epitome of boring repetitive oscillations, so you have a real period there. Changing in (1.10) $t \mapsto it$ the equation stands except by changing g by $-g$, but reversing the direction of gravity is like putting the pendulum upside down and it is still a pendulum, so we have also a complex period.

The patient reader may forgive or skip a final brief physical aside about non classical oscillations.

The currently official physical explanation of reality, quantum field theory, postulates that there is quantum harmonic oscillator at each point of vacuum. In certain units and with a criminal notation (t is now position and x^2 probability density) the quantum harmonic oscillator is ruled by a nontrivial solution $x = x(t)$ of

$$x'' + (2\omega - t^2)x = 0. \tag{1.12}$$

This looks as a kind of harmonic oscillator (1.1) with frequency changing on time. If we look for solutions $x = x(t)$ smooth square-integrable and not identically zero (as dictated by quantum mechanics), it can be proved that they exist if and only if 2ω is a positive odd integer. This lies more or less deep (not an exercise!) and physically indicates a quantization of the energy. The smallest value $\omega = 1/2$ corresponds to the solution $x(t) = Ae^{-t^2/2}$ that does not oscillate at all and the same happens for higher values of ω. Are not you intrigued by the name harmonic "oscillator"? Good! You have a challenging reason to enter into the exciting realm of quantum mechanics but this is not the right course ([Zwi13] is a good one). A last aside of the aside is that $1/2 \neq 0$ causes something awkward: after the postulate of quantum field theory, each point carries a positive energy and there are infinitely many points, then the energy of the vacuum, that has to be minimal, is infinite!

Suggested Readings. The different flavors of classic harmonic oscillations are discussed in almost any Physics book for undergraduates (for instance [AF67]). In [Sim17] you can learn about the methods for solving differential equations with an eye to applications. A quick and mathematically spotless discussion of the quantum harmonic oscillator is in §3.4 of [Fol08]. The original book [BB05] is an accessible, comprehensive and historical study about oscillations surrounding pendulums or taking them as motivation.

1.1.2 Electromagnetic waves and simple circuits

A substantial part of the information that reaches us employs electromagnetic waves to travel, at least in a part of its trip from the source. Even DTT (Digital Terrestrial Television) contradicts its "terrestrial" qualifier making use of conventional television antennas. I must confess that the study of electromagnetic waves is unrelated the rest of the course but I consider this section as general knowledge for graduate students in Mathematics.

In the beginning... it was "*A treatise on electricity and magnetism*" published in 1873 and authored by J.C. Maxwell [Max54]. OK, it was not the beginning, it was rather a culmination. A way of summarizing in a mathematical compact form, called *Maxwell equations*, experiments and laws stated by several researchers. One of the main contributors was M. Faraday and he probably would have freaked out (he died 6 years before the publication of the treatise) because he was reluctant to use mathematical arguments while the classic form of Maxwell equations in vacuum is

$$\nabla\cdot\vec{E}=0,\qquad \nabla\cdot\vec{B}=0,\qquad \nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t},\qquad \nabla\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{1.13}$$

where c is the speed of light and, as usual, $\nabla\cdot$ and $\nabla\times$ are the divergence and the curl. The main characters are the *electric field* $\vec{E}$ and the *magnetic field* $\vec{B}$. Roughly speaking they are a way of measuring the strength of the forces produced by charges and magnets.

It is hard to believe that (1.13) are "experimental formulas". How on earth can you measure the curl? The answer is that after applying Stokes' theorem they become integral formulas harder to manage from the mathematical point of view but more intuitive.

Let us focus on the third equation (the so-called Maxwell-Faraday equation). If we move a magnet through a wire loop enclosing a surface S, an electric current appears in the wire. This is the principle of the wind or water turbines that bring electricity to our homes. If the magnet stops there is no current and when the magnet moves quickly the current is greater. The strength of the magnet, represented by $\vec{B}$ also matters. In this context the following relation, with K a constant, sounds more or less natural

$$\frac{d}{dt}\int_S\vec{B}=K\int_{\partial S}\vec{E}. \tag{1.14}$$

The left hand side is something like the variation of the "total magnetic field" through the surface S and the right hand side is the total electric field through the wire. With standard (Gaussian) units $K=-c$. This is not essential, choosing other units we could give any nonzero value to K. The minus sign indicates an old convention about what is called north and south pole of a magnet. If we believe (1.14) with $K=-c$, by Stokes' theorem applied to the right side, we have

$$\int_S\Big(\frac{\partial\vec{B}}{\partial t}+c\nabla\times\vec{E}\Big)=0. \tag{1.15}$$

As the surface S is arbitrary (as the wire loop is) we deduce the third equation of (1.13).

One may wonder (as Faraday would have done) why the mathematically abstruse formulation (1.13) is important. The answer is that Mathematics is usually easier than real life (especially if you are a mathematician). For instance, in some way the existence of electromagnetic waves and even special relativity are encoded in (1.13) and Maxwell equations predicted them before any experimental test did (for the latter, check the title or the content of the celebrated 1905 paper by A. Einstein introducing relativity [Ein05]). Let us see where are the waves.

Imagine an evil professor posing the following problem to you as a freshman: *If $\vec{F}$ is a vector field with $\nabla \cdot \vec{F} = 0$, compute $\nabla \times (\nabla \times \vec{F})$.* In principle it is a calculation, but the formula for $\nabla \times \vec{F}$ is involved and consequently $\nabla \times (\nabla \times \vec{F})$ is expected to be super-involved. It turns out that the condition $\nabla \cdot \vec{F} = 0$ allows to simplify the mess to get

$$\nabla \times (\nabla \times \vec{F}) = -\Delta \vec{F} \qquad \text{where} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1.16}$$

Would you dare to get a proof of this as simple as you can?

Taking the curl of the third and fourth equations in (1.13), one deduces from (1.16)

$$c^2 \Delta \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \text{and} \qquad c^2 \Delta \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}. \tag{1.17}$$

It means that each coordinate of $\vec{E}$ and $\vec{B}$ is a solution of the wave equation for speed c

$$c^2 \Delta u = \frac{\partial^2 u}{\partial t^2}. \tag{1.18}$$

We conclude that there are electromagnetic waves and they travel at speed c. These waves were mentioned firstly in 1865 in a work of Maxwell [Max65]. Around 1887, H. Hertz was able to produce electromagnetic waves with high voltage sparks and detect them some meters further [Her90]. The rest is history. Next time you switch on the TV, say loudly: Thank you Maxwell!

In many situations one has to deal with a non-vacuum environment, with charges. The extension of (1.13) to this case is

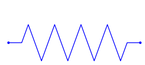
$$\nabla \cdot \vec{E} = 4\pi\rho, \qquad \nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{B} = \frac{4\pi}{c}\vec{\jmath} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{1.19}$$

where ρ is the charge density and $\vec{\jmath} = \rho \vec{v}$ with $\vec{v}$ the velocity field (flow velocity) of the charges.

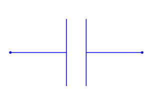
In practice, most of the signals are treated electronically then it does not harm to learn something about very basic components and circuits. Surely you have heard the names *voltage* (electric potential difference) and *electric current* referred to electricity. They are the line integral of $\vec{E}$ between two points and the flux of $\vec{\jmath}$. Fortunately the so-called hydraulic analogy gives an intuitive way of thinking about them. Basically, if you consider electricity as a fluid and the conductors in the circuits as pipes, the voltage between two points is the difference of pressure and the current is the volume flow rate.

Although we live in the era of silicon (I wrote silicon, not silicone), for the sake of brevity we are going to consider only passive components, meaning that they do not involve semiconductors. The most basic are the *resistor*, the *capacitor* and the *inductor*. Their symbols and their characteristic relations between the time-depending voltage $V = V(t)$ (within their terminals) and electric current $I = I(t)$ are

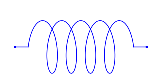
(1.20)

 $V = IR$

Resistor

 $I = CV'$

Capacitor

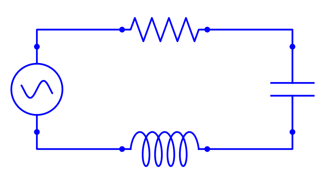
 $V = LI'.$

Inductor

Here the *resistance* R, the *capacitance* C and the *inductance* L are constants associated to the specifications of the particular component. As the symbols suggest, capacitors are essentially two very close conducting plates and the inductor is a conducting spring, a coil (you can construct and test both by yourself [Fie03]). Believe or not, the equations of (1.20) in these two cases are quite direct consequences of Maxwell equations (1.19), as explained below. A resistor is made of materials that are not so good conductors and the theoretical explanation of its equation (the famous *Ohm's law*) belongs to solid state Physics. In the hydraulic analogy a resistor is a constricted pipe. You can look up the analogs for capacitors and inductors that I do not mention here. For the interested reader (skip to the RLC circuit if you are not), let us see how to deduce their equations without rigor and without entering into details.

Integrating the first formula of (1.19) on the surface S of the cylinder determined by the plates of a capacitor, we have by the divergence theorem $\int_S \vec{E} = 4\pi Q$ where Q is the total charge on the plates. This suggests that $|\vec{E}|$ is proportional to Q. As the voltage is electric field times length, in this case the separation between the plates, the charge is proportional to the voltage. Denoting C the proportionality constant and using $I = dQ/dt$, the equation $I = CV'$ is deduced. For the inductor there is a little dirty trick. It turns out that for the frequencies appearing in electric circuits, the last term in the last formula of (1.19) is negligible[1]. Let us consider a one-loop inductor determined by a disk D with boundary C of length l. Integrating the reduced equation on S, by Stoke's theorem, $\int_C \vec{B} = \frac{4\pi}{c}\int_D \vec{\jmath}$. This suggests that $|\vec{B}|l$ and I are proportional. The variation of the flux of $\vec{B}$ in time gives the voltage thanks to Maxwell-Faraday equation (1.14). Then the voltage should be proportional to the variation of I, that is written $V = LI'$.

The simplest *RLC circuit* corresponds to the following scheme

(1.21)  $V_R + V_C + V_L = V$

where the rightmost symbol means a source of voltage V. The equation is an instance of *Kirchhoff's law*, like the conservation of energy, where V_R, V_C and V_L are the voltages between the terminals of each component. Taking the derivative and using (1.20),

$$RI' + C^{-1}I + LI'' = V'. \tag{1.22}$$

If our voltage source produce a usual sine wave $V = V_0 \sin(\omega t)$ (as our power plugs at home), we have

$$I'' + RL^{-1}I' + (LC)^{-1}I = \omega V_0 \cos(\omega t). \tag{1.23}$$

[1] In fact, this term was the only part of the equations not supported with experiments at Maxwell's time. He introduced it using purely theoretical arguments.

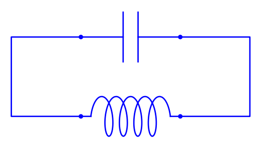
According to our previous study, the resonance happens for $\omega = \omega_0$ with $\omega_0 = (LC)^{-1/2}$ and at large, the current behaves as $I = I_0 \sin(\omega t - \delta)$ for some δ. With our coefficients

$$V_0 = I_0 Z \qquad \text{where} \qquad Z = \sqrt{R^2 + L^2\omega^2(1 - \omega_0^2/\omega^2)^2}. \tag{1.24}$$

If $\omega = \omega_0$ then $Z = R$ considering only the amplitudes (forgetting the phases) the circuit behaves as if the capacitor and the inductance does not furnish any resistance. On the other hand if ω is not close to ω_0, then Z is much bigger than R. This is the principle to tune a specific radio station or TV channel. In the mathematical context, this primitive machine that allows to select with certain precision specific frequencies is a gateway to an electronic computation of Fourier expansions.

Note that if you omit the source and the resistor in (1.21) the new equation is

$$I'' + (LC)^{-1} I = 0 \tag{1.25}$$



and we have in theory a tireless harmonic oscillator if the capacitor is initially charged. In practice, there is always some resistance in the conductors and, as in a free pendulum, the oscillations fade away quickly. To achieve a real electronic oscillator you have to introduce some kind of amplification. In the early days it was achieved with vacuum tubes (valves) and later with transistors.

Suggested Readings. For a basic mathematically oriented introduction to the Maxwell equations and its relation to modern theoretical physics, I recommend the recent book [Gar15]. The solution of the equations and its meaning is very well explained in the modern classic [FLS64].

1.1.3 Sound waves

The sound consists of changes of pressure that can be detected by the human ear. With some approximations and basic Physics we are going to convince ourselves that it is transmitted as a wave.

If a particle of the air is in a certain position we want to study its displacement u when time evolves and it is disturbed by the sound. We assume that the perturbation acts in the same way at every horizontal line, in other words, $u = u(x,t)$ and we can focus on the X axis. The changes in the pressure p are related to changes in the density ρ. For the sound there are not big variations with respect to the normal pressure and density, say p_0 and ρ_0, then we can write

$$p(x,t) = p_0 + p_\epsilon(x,t) \qquad \text{and} \qquad \rho(x,t) = \rho_0 + \rho_\epsilon(x,t) \tag{1.26}$$

meaning that p_ϵ and ρ_ϵ are much smaller than the constants p_0 and ρ_0. They express some kind of perturbation.