Chapter 1

Simple waves

1.1 Physical principles

1.1.1 Harmonic oscillators

In several physical situations when a particle in the real line is displaced to $x = x(t)$ it suffers an attractive force towards its equilibrium position at the origin that is proportional to x. A couple of academic examples in which this model gives a reasonably good approximation are a particle attached to a spring and the small oscillations of a simple pendulum. Recalling $F = ma$ the differential equation of motion is

$$x'' + \omega^2 x = 0 \tag{1.1}$$

where $-\omega^2$ is the proportionality constant ($\omega > 0$). A physical system driven by this equation is called a *harmonic oscillator*. The minus sign imposes that the force is attractive. Physicists very often prefer to write $\ddot{x}$ instead of x'' but we advocate here the mathematical notation. You know how to solve this ODE (ordinary differential equation) or you already know by heart the general solution

$$x(t) = A\cos(\omega t) + B\sin(\omega t) \tag{1.2}$$

where A and B are constants to be adjusted according to the initial conditions. It is said that ω is the *angular frequency*, the number of radians per unit of time. In daily life it is more common to express the *frequency* ν as the number of repetitions of an event per unit of time, so $\omega = 2\pi\nu$. The *hertz*, abbreviated Hz, is a natural unit of frequency meaning one cycle per second. For instance $120\,rpm$ (revolutions per minute) correspond to $\nu = 2\,Hz$ and to $\omega = 4\pi\,rad/s$.

In the examples mentioned above (spring and pendulum) in practice we do not see endless oscillations like (1.2). It may be a good approximation for short periods of time but without external help the friction eventually stops the oscillations. For a fluid like air, the friction is with some approximation proportional to the velocity (*Stokes' law*). It suggests to improve the model (1.1) to get the *damped harmonic oscillator* ruled by

$$x'' + 2ax' + \omega^2 x = 0 \tag{1.3}$$

where $2a$ with $a > 0$ is the new proportionality constant. You should know also how to solve this equation. The fancy and usual way is to try a solution $x(t) = e^{rt}$ to conclude that $r \in \{r_-, r_+\}$ with $r_\pm = -a \pm \sqrt{a^2 - \omega^2}$. If the friction is not very large (for your curiosity this is called the *underdamped harmonic oscillator*, the only one we are going to consider), $a < \omega$ and hence $r_\pm$ are complex numbers. When we choose A and B in $Ae^{r_- t} + Be^{r_+ t}$ to match the (real) initial conditions, the imaginary parts must cancel and the general solution becomes

$$x(t) = Ae^{-at}\cos(\widetilde{\omega}t) + Be^{-at}\sin(\widetilde{\omega}t) \qquad \text{with} \quad \widetilde{\omega} = \sqrt{\omega^2 - a^2}. \tag{1.4}$$

The *amplitudes* (the coefficients of the oscillatory terms) decay exponentially in time, as seen in practice, however if a is much smaller than ω we have that the relative frequency shift $(\omega - \widetilde{\omega})/\omega$ is approximately as small as $\frac{1}{2}(a/\omega)^2$. This approximate invariance of the frequency is in part responsible for the precision of the pendulum clocks that caused a revolution in time measurement accuracy in the 17th century. I know, I know, (1.4) is far from modeling the oscillations of the pendulum of a clock because of the exponential decay. One needs an external force to maintain the oscillations. In those clocks it was provided by the *anchor escapement* [BB05, §10.2.4]. We have learned in previous courses and we shall recall once again in this one, that all respectable functions can be expressed in terms of sines and cosines then we restrict ourselves to external forces of the form $F_e\cos(\omega_e t - \varphi_e)$ with $F_e, \omega_e > 0$. Note that for $\varphi_e = 0$ we have a pure cosine and for $\varphi_e = \pi/2$ a pure sine. The corresponding *driven harmonic oscillator* is consequently modeled by the equation

$$x'' + 2ax' + \omega^2 x = F_e\cos(\omega_e t - \varphi_e). \tag{1.5}$$

The solution is the sum of the solution of the homogeneous equation (1.3), given by (1.4), and a particular solution that could be computed using variation of the parameters, however the calculations become complicated because the final result depends on a, ω, ω_e and φ_e in a messy way. Let us use again a complex variable trick considering the complex differential equation

$$y'' + 2ay' + \omega^2 y = Fe^{i\omega_e t} \qquad \text{with} \quad F = F_e e^{-i\varphi_e}. \tag{1.6}$$

If we find a particular solution y_p then, taking real parts, $x_p = \Re y_p$ will be a particular solution of (1.5). Let us try $y_p = Ce^{i\omega_e t}$ that forces $G = F/(-\omega_e^2 + 2ia\omega_e + \omega^2)$. If we write this as $Ce^{-i\varphi}$ with $C = |G|$, then

$$x_p(t) = C\cos(\omega_e t - \varphi) \qquad \text{with} \quad C = \frac{F_e}{\sqrt{(\omega^2 - \omega_e^2)^2 + 4a^2\omega_e^2}}. \tag{1.7}$$

This is the *steady-state solution* of (1.5), the limit when $t \to +\infty$ of any solution under any initial conditions, because (1.4) goes to 0. The denominator in C measures the *gain*. If $(\omega^2 - \omega_e^2)^2 + 4a^2\omega_e^2$ is small then $C > F_e$, the external force is somewhat amplified. Probably everybody has experienced this phenomenon in children playgrounds: if one pushes periodically a swing at the right moment ($\omega \approx \omega_e$) then the amplitude of the oscillation increases. In this case we say that *resonance* has occurred.

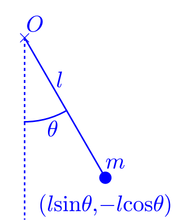
The extreme mathematical resonance case in which ω is exactly ω_e and $a \to 0$ (no friction) causes a problem because $x_p \to \infty$ in (1.7). Do not worry, the swing is not going to explode or something like that if you are close to this situation. In this extreme case of resonance, our ansatz for the form of y_p is wrong. A valid particular solution involves a factor t that remains bounded at any finite value of t, although the amplitude of any solution increases arbitrarily when $t \to \infty$. Recall Tacoma Narrows Bridge! (or look up in the internet if you do not know what I am talking about).

In part, the abstract study of the harmonic oscillator is motivated by the pendulum which does not follow exactly (1.1). Instead of deriving the right equation using the ubiquitous Newton's second law $F = ma$, we are going to employ the easier Lagrangian formulation. If you do not know anything about *Lagrangian mechanics*, I hope this example to provide an incentive to read something about it. The main asset is that, like in differential geometry, you can choose the coordinates $\{q_k\}$ you wish. Once you have done it, you have to construct with them the *Lagrangian* $L = T - V$ where T is the kinetic energy and V is the potential energy. It is a function of $\{q_k\}$ and $\{\dot{q}_k\}$ where we have suspended our preference of q_k' instead of $\dot{q}_k$ to indicate a time derivative. The Lagrangian depends implicitly on t, through $q_k = q_k(t)$ and may also depend explicitly on it. The law of motion in the chosen coordinates is given by the *Euler-Lagrange equations*

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}_k}\Big) = \frac{\partial L}{\partial q_k}. \tag{1.8}$$

For a system of particles of masses m_j at $\mathbf{x}_j$ under gravitational forces near Earth's surface, the kinetic energy is $\frac{1}{2}\sum m_j \|\mathbf{x}_j\|^2$ and the potential energy is $\sum m_j z_j$, where g is the gravitational acceleration. The Lagrangian is got writing these quantities in our coordinates. For a pendulum (a point mass m suspended from the origin through a massless rigid rod of length l) the natural coordinate is the angle θ and we have

$$\begin{cases} L(\theta,\dot{\theta}) &= \frac{1}{2}m\big((l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2\big) + mgl\cos\theta \\ &= \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \\ \\ & \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big) = ml^2\ddot{\theta}, \quad \frac{\partial L}{\partial \theta} = -mgl\sin\theta. \end{cases} \tag{1.9}$$



And *voilà*, in the blink of an eye we have the pendulum equation without using fictitious invisible forces like the tension of the rod that puzzled you in Physics 101,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{1.10}$$

When θ is small, (1.1) with $\omega^2 = \sqrt{g/l}$ approximates (1.10) giving the formula $2\pi\sqrt{l/g}$ for the period of oscillation. If you are not happy with the approximations, multiply by $\dot{\theta}$ and integrate to get for a certain constant E (related to the energy), under $\theta(0) = 0$,

$$t\sqrt{\frac{g}{l}} = \int_0^\theta \frac{d\theta}{\sqrt{2(E+\cos\theta)}} \quad \underset{ku=\sin(\theta/2)}{\overset{E=2k^2-1}{\Longleftrightarrow}} \quad t\sqrt{\frac{g}{l}} = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{1.11}$$

The last integral is the famous Jacobi elliptic integral enjoying wonderful properties. The theory assures that $u = u(t)$ extends to a meromorphic function with two periods. This is cumbersome from the mathematical point of view but, following [Bae05], there is a convincing physical easy explanation: The pendulum is the epitome of boring repetitive oscillations, so you have a real period there. Changing in (1.10) $t \mapsto it$ the equation stands except by changing g by $-g$, but reversing the direction of gravity is like putting the pendulum upside down and it is still a pendulum, so we have also a complex period.

The patient reader may forgive or skip a final brief physical aside about non classical oscillations.

The currently official physical explanation of reality, quantum field theory, postulates that there is quantum harmonic oscillator at each point of vacuum. In certain units and with a criminal notation (t is now position and x^2 probability density) the quantum harmonic oscillator is ruled by a nontrivial solution $x = x(t)$ of

$$x'' + (2\omega - t^2)x = 0. \tag{1.12}$$

This looks as a kind of harmonic oscillator (1.1) with frequency changing on time. If we look for solutions $x = x(t)$ smooth square-integrable and not identically zero (as dictated by quantum mechanics), it can be proved that they exist if and only if 2ω is a positive odd integer. This lies more or less deep (not an exercise!) and physically indicates a quantization of the energy. The smallest value $\omega = 1/2$ corresponds to the solution $x(t) = Ae^{-t^2/2}$ that does not oscillate at all and the same happens for higher values of ω. Are not you intrigued by the name harmonic "oscillator"? Good! You have a challenging reason to enter into the exciting realm of quantum mechanics but this is not the right course ([Zwi13] is a good one). A last aside of the aside is that $1/2 \neq 0$ causes something awkward: after the postulate of quantum field theory, each point carries a positive energy and there are infinitely many points, then the energy of the vacuum, that has to be minimal, is infinite!

Suggested Readings. The different flavors of classic harmonic oscillations are discussed in almost any Physics book for undergraduates (for instance [AF67]). In [Sim17] you can learn about the methods for solving differential equations with an eye to applications. A quick and mathematically spotless discussion of the quantum harmonic oscillator is in §3.4 of [Fol08]. The original book [BB05] is an accessible, comprehensive and historical study about oscillations surrounding pendulums or taking them as motivation.

1.1.2 Electromagnetic waves and simple circuits

A substantial part of the information that reaches us employs electromagnetic waves to travel, at least in a part of its trip from the source. Even DTT (Digital Terrestrial Television) contradicts its "terrestrial" qualifier making use of conventional television antennas. I must confess that the study of electromagnetic waves is unrelated the rest of the course but I consider this section as general knowledge for graduate students in Mathematics.