A generalized Hilbert operator acting on mean Lipschitz spaces

Noel Merchán
Universidad de Málaga, Spain

Madrid International Workshop on Operator Theory and Function Spaces
Madrid, Spain. October 16, 2018

Index

1. Notation and definitions
2. Generalized Hilbert matrix
    - Hilbert matrix
    - A generalized Hilbert matrix
    - Integral operator
    - Carleson measures
3. Mean Lipschitz spaces
4. $\mathcal{H}_\mu$ acting on mean Lipschitz spaces

The unit disc and the unit circle

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc.
$\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$, the unit circle.

Spaces of analytic functions in the unit disc

$\mathcal{H}ol(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

The unit disc and the unit circle

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc.

$\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$, the unit circle.

Spaces of analytic functions in the unit disc

$\mathcal{H}ol(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

The unit disc and the unit circle

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc.
$\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$, the unit circle.

Spaces of analytic functions in the unit disc

$\mathcal{H}ol(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

The unit disc and the unit circle

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc.
$\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$, the unit circle.

Spaces of analytic functions in the unit disc

$\mathcal{H}ol(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

The unit disc and the unit circle

$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the unit disc.
$\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$, the unit circle.

Spaces of analytic functions in the unit disc

$\mathcal{H}ol(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$.

Hardy spaces

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

If $0 < p \leq \infty$, we consider the Hardy spaces H^p,

$$H^p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) \; : \; \|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty \right\}.$$

Hardy spaces

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

If $0 < p \leq \infty$, we consider the Hardy spaces H^p,

$$H^p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) \, : \, \|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty \right\}.$$

Hardy spaces

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

If $0 < p \leq \infty$, we consider the Hardy spaces H^p,

$$H^p = \left\{ f \in \mathcal{H}ol(\mathbb{D}) \ : \ \|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty \right\}.$$

BMOA

$$BMOA = \left\{ f \in H^1 : f\left(e^{i\theta}\right) \in BMO \right\}.$$

Bloch space

$$\mathcal{B} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \right\}.$$

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

BMOA

$$BMOA = \left\{ f \in H^1 : f\left(e^{i\theta}\right) \in BMO \right\}.$$

Bloch space

$$\mathcal{B} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \right\}.$$

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

BMOA

$$BMOA = \left\{ f \in H^2 : \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty \right\}.$$

Where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{D}$.

Bloch space

$$\mathcal{B} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \right\}.$$

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

BMOA

$$BMOA = \left\{ f \in H^2 : \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty \right\}.$$

Where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{D}$.

Bloch space

$$\mathcal{B} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1-|z|^2)|f'(z)| < \infty \right\}.$$

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

BMOA

$$BMOA = \left\{ f \in H^2 : \sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2} < \infty \right\}.$$

Where $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \mathbb{D}$.

Bloch space

$$\mathcal{B} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty \right\}.$$

$$H^\infty \subset BMOA \subset \mathcal{B}.$$

Hilbert matrix

Let $\mathcal{H}$ be the Hilbert matrix,

$$\mathcal{H} = \left(\frac{1}{n+k+1}\right)_{n,k\geq 0}.$$

$$\mathcal{H} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hilbert matrix

Let $\mathcal{H}$ be the Hilbert matrix,

$$\mathcal{H} = \left(\frac{1}{n+k+1} \right)_{n,k\geq 0}.$$

$$\mathcal{H} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hilbert matrix

Let $\mathcal{H}$ be the Hilbert matrix,

$$\mathcal{H} = \left(\frac{1}{n+k+1}\right)_{n,k\geq 0}.$$

$$\mathcal{H} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hilbert matrix

The Hilbert matrix $\mathcal{H}$ can be viewed as an operator between sequence spaces.

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},$$

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{ \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right\}_{n=0}^{\infty}.$$

Hilbert matrix

The Hilbert matrix $\mathcal{H}$ can be viewed as an operator between sequence spaces.

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},$$

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{ \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right\}_{n=0}^{\infty}.$$

Hilbert matrix

The Hilbert matrix $\mathcal{H}$ can be viewed as an operator between sequence spaces.

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},$$

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{ \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right\}_{n=0}^{\infty}.$$

Hilbert matrix

The Hilbert matrix $\mathcal{H}$ can be viewed as an operator between sequence spaces.

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & \dots \\ 1/2 & 1/3 & 1/4 & 1/5 & \dots \\ 1/3 & 1/4 & 1/5 & 1/6 & \dots \\ 1/4 & 1/5 & 1/6 & 1/7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \end{pmatrix},$$

$$\{a_n\}_{n=0}^{\infty} \mapsto \left\{ \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right\}_{n=0}^{\infty}.$$

In the same way we can consider $\mathcal{H}$ as an operator in $\mathcal{H}ol(\mathbb{D})$ multiplicating the matrix by the sequence of Taylor coefficients of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D})$.

We define formally the operator in $\mathcal{H}ol(\mathbb{D})$

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

In the same way we can consider $\mathcal{H}$ as an operator in $\mathcal{H}ol(\mathbb{D})$ multiplicating the matrix by the sequence of Taylor coefficients of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D})$.

We define formally the operator in $\mathcal{H}ol(\mathbb{D})$

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

In the same way we can consider $\mathcal{H}$ as an operator in $\mathcal{H}ol(\mathbb{D})$ multiplicating the matrix by the sequence of Taylor coefficients of a function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D})$.

We define formally the operator in $\mathcal{H}ol(\mathbb{D})$

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

Hilbert matrix as an operator

The operator $\mathcal{H}$ is well defined on H^1.
The operator $\mathcal{H} : H^p \to H^p$ is bounded if $1 < p < \infty$, (Diamantopoulos & Siskakis, 2000).
Dostanić, Jevtić & Vukotić (2008) found the exact norm of $\mathcal{H}$ as an operator from H^p to H^p $(1 < p < \infty)$.
However, $\mathcal{H}$ is not bounded on H^1 and neither on H^∞.

Hilbert matrix as an operator

The operator $\mathcal{H}$ is well defined on H^1.

The operator $\mathcal{H} : H^p \to H^p$ is bounded if $1 < p < \infty$, (Diamantopoulos & Siskakis, 2000).

Dostanić, Jevtić & Vukotić (2008) found the exact norm of $\mathcal{H}$ as an operator from H^p to H^p $(1 < p < \infty)$.

However, $\mathcal{H}$ is not bounded on H^1 and neither on H^∞.

Hilbert matrix as an operator

The operator $\mathcal{H}$ is well defined on H^1.
The operator $\mathcal{H} : H^p \to H^p$ is bounded if $1 < p < \infty$, (Diamantopoulos & Siskakis, 2000).
Dostanić, Jevtić & Vukotić (2008) found the exact norm of $\mathcal{H}$ as an operator from H^p to H^p ($1 < p < \infty$).
However, $\mathcal{H}$ is not bounded on H^1 and neither on H^∞.

Hilbert matrix as an operator

The operator $\mathcal{H}$ is well defined on H^1.
The operator $\mathcal{H} : H^p \to H^p$ is bounded if $1 < p < \infty$, (Diamantopoulos & Siskakis, 2000).
Dostanić, Jevtić & Vukotić (2008) found the exact norm of $\mathcal{H}$ as an operator from H^p to H^p $(1 < p < \infty)$.
However, $\mathcal{H}$ is not bounded on H^1 and neither on H^∞.

Hilbert matrix as an operator

The operator $\mathcal{H}$ is well defined on H^1.
The operator $\mathcal{H} : H^p \to H^p$ is bounded if $1 < p < \infty$, (Diamantopoulos & Siskakis, 2000).
Dostanić, Jevtić & Vukotić (2008) found the exact norm of $\mathcal{H}$ as an operator from H^p to H^p $(1 < p < \infty)$.
However, $\mathcal{H}$ is not bounded on H^1 and neither on H^∞.

A generalized Hilbert matrix

Let μ be a finite positive Borel measure on $[0, 1)$.
Let $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ be the **Hankel matrix** with entries

$$\mu_{n,k} = \int_{[0,1)} t^{n+k}\, d\mu(t).$$

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If μ is the Lebesgue measure on the interval $[0, 1)$ we get the classical Hilbert matrix.

A generalized Hilbert matrix

Let μ be a finite positive Borel measure on $[0, 1)$.

Let $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ be the **Hankel matrix** with entries

$$\mu_{n,k} = \int_{[0,1)} t^{n+k}\, d\mu(t).$$

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If μ is the Lebesgue measure on the interval $[0, 1)$ we get the classical Hilbert matrix.

A generalized Hilbert matrix

Let μ be a finite positive Borel measure on $[0, 1)$.
Let $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ be the **Hankel matrix** with entries

$$\mu_{n,k} = \int_{[0,1)} t^{n+k}\, d\mu(t).$$

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If μ is the Lebesgue measure on the interval $[0, 1)$ we get the classical Hilbert matrix.

A generalized Hilbert matrix

Let μ be a finite positive Borel measure on $[0,1)$.
Let $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ be the **Hankel matrix** with entries

$$\mu_{n,k} = \int_{[0,1)} t^{n+k}\, d\mu(t).$$

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If μ is the Lebesgue measure on the interval $[0,1)$ we get the classical Hilbert matrix.

A generalized Hilbert matrix

Let μ be a finite positive Borel measure on $[0,1)$.
Let $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ be the **Hankel matrix** with entries

$$\mu_{n,k} = \int_{[0,1)} t^{n+k}\, d\mu(t).$$

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

If μ is the Lebesgue measure on the interval $[0,1)$ we get the classical Hilbert matrix.

The matrix $\mathcal{H}_\mu$ induces formally an operator on $\mathcal{H}ol(\mathbb{D})$ in the same way than $\mathcal{H}$:

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n.$$

The matrix $\mathcal{H}_\mu$ induces formally an operator on $\mathcal{H}ol(\mathbb{D})$ in the same way than $\mathcal{H}$:

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n.$$

Integral operator

For a finite positive Borel measure on $[0,1)$ μ we also define the integral operator

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t),$$

when the right side has sense and it defines an analytic function.

$\mathcal{H}_\mu$ and I_μ are closely related. If $f \in \mathcal{H}ol(\mathbb{D})$ is good enough then $\mathcal{H}_\mu(f) = I_\mu(f)$.

Integral operator

For a finite positive Borel measure on $[0,1)$ μ we also define the integral operator

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t),$$

when the right side has sense and it defines an analytic function.

$\mathcal{H}_\mu$ and I_μ are closely related. If $f \in \mathcal{H}ol(\mathbb{D})$ is good enough then $\mathcal{H}_\mu(f) = I_\mu(f)$.

Integral operator

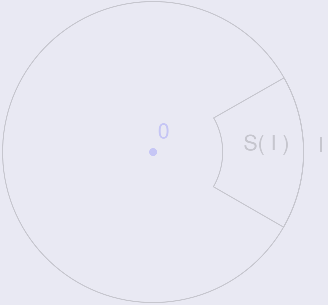
For a finite positive Borel measure on $[0,1)$ μ we also define the integral operator

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} d\mu(t),$$

when the right side has sense and it defines an analytic function.
$\mathcal{H}_\mu$ and I_μ are closely related. If $f \in \mathcal{H}ol(\mathbb{D})$ is good enough then $\mathcal{H}_\mu(f) = I_\mu(f)$.

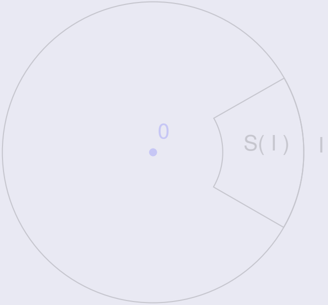
Definition

Let I be an interval of $\mathbb{T}$. We define the Carleson square associated to I as
$S(I) = \{re^{i\theta} : e^{i\theta} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}$.



Definition

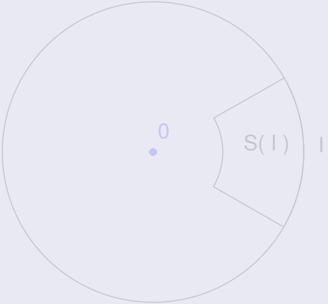
Let I be an interval of $\mathbb{T}$. We define the Carleson square associated to I as
$S(I) = \{re^{i\theta} : e^{i\theta} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}$.



Definition

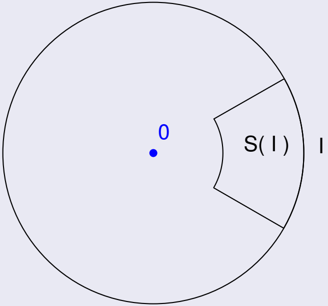
Let I be an interval of $\mathbb{T}$. We define the Carleson square associated to I as

$$S(I) = \{re^{i\theta} : e^{i\theta} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}.$$



Definition

Let I be an interval of $\mathbb{T}$. We define the Carleson square associated to I as
$S(I) = \{re^{i\theta} : e^{i\theta} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}$.



Definition

Let μ be a finite measure on $\mathbb{D}$. μ is a Carleson measure if there is a constant $C > 0$ such that

$$\mu(S(I)) \leq C|I| \quad \text{for every } I \subset \mathbb{T} \text{ interval.}$$

Theorem (Carleson, 1962)

Let μ be a finite measure on $\mathbb{D}$. Then μ is a Carleson measure if and only if there exist a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|\, d\mu(z) \leq C\|f\|_{H^1} \quad \text{for all } f \in H^1.$$

Definition

Let μ be a finite measure on $\mathbb{D}$. μ is a Carleson measure if there is a constant $C > 0$ such that

$$\mu(S(I)) \leq C|I| \quad \text{for every } I \subset \mathbb{T} \text{ interval.}$$

Theorem (Carleson, 1962)

Let μ be a finite measure on $\mathbb{D}$. Then μ is a Carleson measure if and only if there exist a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|\, d\mu(z) \leq C\|f\|_{H^1} \quad \text{for all } f \in H^1.$$

Definition

Let μ be a finite measure on $\mathbb{D}$. μ is a Carleson measure if there is a constant $C > 0$ such that

$$\mu(S(I)) \leq C|I| \quad \text{for every } I \subset \mathbb{T} \text{ interval.}$$

Theorem (Carleson, 1962)

Let μ be a finite measure on $\mathbb{D}$. Then μ is a Carleson measure if and only if there exist a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|\, d\mu(z) \leq C\|f\|_{H^1} \quad \text{for all } f \in H^1.$$

Logarithmic Carleson measures

Let μ be a positive Borel measure on $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right)\left(\log \frac{2\pi}{|I|}\right)^{\alpha} \leq C|I|^{s}, \text{ for any interval } I \subset \mathbb{T}.$$

Logarithmic Carleson measures

Let μ be a positive Borel measure on $\mathbb{D}$, $0 \le \alpha < \infty$, and $0 < s < \infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\mu(S(I))\left(\log \frac{2\pi}{|I|}\right)^{\alpha} \le C|I|^s, \text{ for any interval } I \subset \mathbb{T}.$$

Widom (1966) and Power (1980) characterized those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^2 into itself.

Galanopoulos and Peláez (2010) characterized those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) in the Hardy space H^1.

Chatzifountas, Girela and Peláez (2013) described those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^p into H^q with $0 < p, q < \infty$.

Girela and M. (2017) described those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded on conformally invariant spaces.

Widom (1966) and Power (1980) characterized those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^2 into itself.

Galanopoulos and Peláez (2010) characterized those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) in the Hardy space H^1.

Chatzifountas, Girela and Peláez (2013) described those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^p into H^q with $0 < p, q < \infty$.

Girela and M. (2017) described those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded on conformally invariant spaces.

Widom (1966) and Power (1980) characterized those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^2 into itself.

Galanopoulos and Peláez (2010) characterized those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) in the Hardy space H^1.

Chatzifountas, Girela and Peláez (2013) described those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^p into H^q with $0 < p, q < \infty$.

Girela and M. (2017) described those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded on conformally invariant spaces.

Widom (1966) and Power (1980) characterized those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^2 into itself.

Galanopoulos and Peláez (2010) characterized those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) in the Hardy space H^1.

Chatzifountas, Girela and Peláez (2013) described those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^p into H^q with $0 < p, q < \infty$.

Girela and M. (2017) described those positive Borel measures on $[0,1)$ such that $\mathcal{H}_\mu$ is bounded on conformally invariant spaces.

Widom (1966) and Power (1980) characterized those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^2 into itself.

Galanopoulos and Peláez (2010) characterized those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) in the Hardy space H^1.

Chatzifountas, Girela and Peláez (2013) described those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded (or compact) from H^p into H^q with $0 < p, q < \infty$.

Girela and M. (2017) described those positive Borel measures on $[0, 1)$ such that $\mathcal{H}_\mu$ is bounded on conformally invariant spaces.

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0, 1)$ such that $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. Then the following three conditions are equivalent:

(i) The operator I_μ is bounded from $\mathcal{B}$ into *BMOA*.

(ii) The operator I_μ is bounded from *BMOA* into itself.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

Moreover, if (i) holds, then the operator $\mathcal{H}_\mu$ is also well defined on the Bloch space and

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for all } f \in \mathcal{B},$$

and hence the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into *BMOA*.

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0,1)$ such that $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. Then the following three conditions are equivalent:

(i) The operator I_μ is bounded from $\mathcal{B}$ into *BMOA*.

(ii) The operator I_μ is bounded from *BMOA* into itself.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

Moreover, if (i) holds, then the operator $\mathcal{H}_\mu$ is also well defined on the Bloch space and

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for all } f \in \mathcal{B},$$

and hence the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into *BMOA*.

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0,1)$ such that $\int_{[0,1)} \log \frac{2}{1-t}\, d\mu(t) < \infty$. Then the following three conditions are equivalent:

(i) The operator I_μ is bounded from $\mathcal{B}$ into $BMOA$.

(ii) The operator I_μ is bounded from $BMOA$ into itself.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

Moreover, if (i) holds, then the operator $\mathcal{H}_\mu$ is also well defined on the Bloch space and

$$\mathcal{H}_\mu(f) = I_\mu(f), \quad \text{for all } f \in \mathcal{B},$$

and hence the operator $\mathcal{H}_\mu$ is bounded from $\mathcal{B}$ into $BMOA$.

Q_s spaces

For $0 \leq s < \infty$ we define the space Q_s as

$$Q_s = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) < \infty \right\}.$$

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA = Q_1 \subsetneq \mathcal{B} = Q_s, \quad 0 < s_1 < s_2 < 1 < s.$$

Q_s spaces

For $0 \leq s < \infty$ we define the space Q_s as

$$Q_s = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) < \infty \right\}.$$

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA = Q_1 \subsetneq \mathcal{B} = Q_s, \quad 0 < s_1 < s_2 < 1 < s.$$

Q_s spaces

For $0 \leq s < \infty$ we define the space Q_s as

$$Q_s = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) < \infty \right\}.$$

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA = Q_1 \subsetneq \mathcal{B} = Q_s, \quad 0 < s_1 < s_2 < 1 < s.$$

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0, 1)$ and let $0 < s_1, s_2 < \infty$. Then the following conditions are equivalent.

(i) The operator I_μ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(ii) The operator $\mathcal{H}_\mu$ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0,1)$ and let $0 < s_1, s_2 < \infty$. Then the following conditions are equivalent.

(i) The operator I_μ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(ii) The operator $\mathcal{H}_\mu$ is well defined in Q_{s_1} and, furthermore, it is a bounded operator from Q_{s_1} into Q_{s_2}.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

$$\Lambda^2_{1/2} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_2(r, f') = O\left(\frac{1}{(1-r)^{1/2}} \right) \right\}.$$

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0,1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = O\left(\frac{1}{n}\right)$.

$$\Lambda^2_{1/2} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_2(r, f') = O\left(\frac{1}{(1-r)^{1/2}}\right)\right\}.$$

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0,1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log\frac{1}{1-t}\, d\mu(t) = O\left(\frac{1}{n}\right)$.

$$\Lambda^2_{1/2} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_2(r, f') = O\left(\frac{1}{(1-r)^{1/2}}\right) \right\}.$$

Theorem (Girela, M.)

Let μ be a positive Borel measure on $[0, 1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = O\left(\frac{1}{n}\right)$.

Integral modulus of continuity

If $f \in \mathcal{H}ol(\mathbb{D})$ has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbb{T}$ and $\delta > 0$, we define for $1 \leq p < \infty$

$$\omega_p(\delta, f) = \sup_{0<|t|\leq\delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^p d\theta \right)^{1/p},$$

and for $p = \infty$ we define

$$\omega_\infty(\delta, f) = \sup_{0<|t|\leq\delta} \left(\operatorname{ess.sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})| \right).$$

Integral modulus of continuity

If $f \in \mathcal{H}ol(\mathbb{D})$ has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbb{T}$ and $\delta > 0$, we define for $1 \leq p < \infty$

$$\omega_p(\delta, f) = \sup_{0<|t|\leq\delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^p d\theta \right)^{1/p},$$

and for $p = \infty$ we define

$$\omega_\infty(\delta, f) = \sup_{0<|t|\leq\delta} \left(\operatorname{ess.sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})| \right).$$

Integral modulus of continuity

If $f \in \mathcal{H}ol(\mathbb{D})$ has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbb{T}$ and $\delta > 0$, we define for $1 \leq p < \infty$

$$\omega_p(\delta, f) = \sup_{0<|t|\leq\delta} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(e^{i(\theta+t)}) - f(e^{i\theta}) \right|^p d\theta \right)^{1/p},$$

and for $p = \infty$ we define

$$\omega_\infty(\delta, f) = \sup_{0<|t|\leq\delta} \left(\operatorname{ess.sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})| \right).$$

Mean Lipschitz spaces

Given $1 \le p \le \infty$ and $0 < \alpha \le 1$, we define the mean Lipschitz space Λ^p_α as

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \exists\, f(e^{i\theta}) \text{ a.e. } \theta,\ \omega_p(\delta, f) = O(\delta^\alpha),\ \text{ as } \delta \to 0 \right\}.$$

Theorem (Hardy & Littlewood, 1932)

If $1 \le p \le \infty$ and $0 < \alpha \le 1$ then we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}} \right) \right\}.$$

Mean Lipschitz spaces

Given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we define the mean Lipschitz space Λ^p_α as

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \exists\, f(e^{i\theta}) \text{ a.e. } \theta,\ \omega_p(\delta, f) = O(\delta^\alpha),\ \text{as } \delta \to 0 \right\}.$$

Theorem (Hardy & Littlewood, 1932)

If $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$ then we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}} \right) \right\}.$$

Mean Lipschitz spaces

Given $1 \le p \le \infty$ and $0 < \alpha \le 1$, we define the mean Lipschitz space Λ^p_α as

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \exists\, f(e^{i\theta}) \text{ a.e. } \theta,\ \omega_p(\delta, f) = O(\delta^\alpha),\ \text{ as } \delta \to 0 \right\}.$$

Theorem (Hardy & Littlewood, 1932)

If $1 \le p \le \infty$ and $0 < \alpha \le 1$ then we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}} \right) \right\}.$$

Mean Lipschitz spaces

Given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we define the mean Lipschitz space Λ^p_α as

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \exists\, f(e^{i\theta}) \text{ a.e. } \theta,\ \omega_p(\delta, f) = O(\delta^\alpha),\ \text{ as } \delta \to 0 \right\}.$$

Theorem (Hardy & Littlewood, 1932)

If $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$ then we have that $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\alpha}} \right) \right\}.$$

- If $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then $\Lambda^p_\alpha \subset \mathcal{A} \subset H^\infty$.
- If $1 < p < \infty$ and $\alpha = \frac{1}{p}$ then $f(z) = \log\left(\frac{1}{1-z}\right) \in \Lambda^p_{1/p}$.

Theorem (Bourdon, Shapiro, Sledd, 1989)

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \subset \mathcal{B}, \quad 1 \leq p < q < \infty.$$

This result is sharp in a very strong sense.

- If $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then $\Lambda^p_\alpha \subset \mathcal{A} \subset H^\infty$.
- If $1 < p < \infty$ and $\alpha = \frac{1}{p}$ then $f(z) = \log\left(\frac{1}{1-z}\right) \in \Lambda^p_{1/p}$.

Theorem (Bourdon, Shapiro, Sledd, 1989)

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \subset \mathcal{B}, \quad 1 \leq p < q < \infty.$$

This result is sharp in a very strong sense.

- If $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then $\Lambda^p_\alpha \subset \mathcal{A} \subset H^\infty$.
- If $1 < p < \infty$ and $\alpha = \frac{1}{p}$ then $f(z) = \log\left(\frac{1}{1-z}\right) \in \Lambda^p_{1/p}$.

Theorem (Bourdon, Shapiro, Sledd, 1989)

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \subset \mathcal{B}, \quad 1 \leq p < q < \infty.$$

This result is sharp in a very strong sense.

Generalization of Λ^p_α spaces

Let $\omega : [0,\pi] \to [0,\infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$.
Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p,\omega)$ is defined as

$$\Lambda(p,\omega) = \{f \in H^p \ : \ \omega_p(\delta, f) = O(\omega(\delta)), \text{ as } \delta \to 0\} .$$

With this notation $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

Generalization of Λ^p_α spaces

Let $\omega : [0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$.

Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ is defined as

$$\Lambda(p, \omega) = \{f \in H^p \ : \ \omega_p(\delta, f) = O(\omega(\delta)), \ \text{as } \delta \to 0\} .$$

With this notation $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

Generalization of Λ^p_α spaces

Let $\omega : [0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$.
Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ is defined as

$$\Lambda(p, \omega) = \{f \in H^p \; : \; \omega_p(\delta, f) = O(\omega(\delta)), \text{ as } \delta \to 0\}.$$

With this notation $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

Generalization of Λ^p_α spaces

Let $\omega : [0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$ and $\omega(t) > 0$ if $t > 0$.
Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ is defined as

$$\Lambda(p, \omega) = \{f \in H^p \,:\, \omega_p(\delta, f) = O(\omega(\delta)),\ \text{as } \delta \to 0\}.$$

With this notation $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

Dini condition

We say that ω satisfies the Dini condition if there exists a positive constant C such that

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq C\omega(\delta), \quad 0 < \delta < 1.$$

Condition b_1

We say that ω satisfies the b_1 condition if there exists a positive constant C such that

$$\int_\delta^\pi \frac{\omega(t)}{t^2}\, dt \leq C\frac{\omega(\delta)}{\delta}, \quad 0 < \delta < 1.$$

Dini condition

We say that ω satisfies the Dini condition if there exists a positive constant C such that

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq C\omega(\delta), \quad 0 < \delta < 1.$$

Condition b_1

We say that ω satisfies the b_1 condition if there exists a positive constant C such that

$$\int_\delta^\pi \frac{\omega(t)}{t^2}\, dt \leq C\frac{\omega(\delta)}{\delta}, \quad 0 < \delta < 1.$$

Dini condition

We say that ω satisfies the Dini condition if there exists a positive constant C such that

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \le C\omega(\delta), \quad 0 < \delta < 1.$$

Condition b_1

We say that ω satisfies the b_1 condition if there exists a positive constant C such that

$$\int_\delta^\pi \frac{\omega(t)}{t^2}\, dt \le C\frac{\omega(\delta)}{\delta}, \quad 0 < \delta < 1.$$

Admissible weights

$$\mathcal{AW} = \text{Dini} \cap b_1 .$$

Theorem (Blasco & de Souza, 1990)

If $1 \leq p \leq \infty$ and $\omega \in \mathcal{AW}$ then,

$$\Lambda(p,\omega) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r,f') = O\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

Admissible weights

$$\mathcal{AW} = \text{Dini} \cap b_1 .$$

Theorem (Blasco & de Souza, 1990)

If $1 \leq p \leq \infty$ and $\omega \in \mathcal{AW}$ then,

$$\Lambda(p,\omega) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r,f') = O\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

Admissible weights

$$\mathcal{AW} = \text{Dini} \cap b_1 .$$

Theorem (Blasco & de Souza, 1990)

If $1 \leq p \leq \infty$ and $\omega \in \mathcal{AW}$ then,

$$\Lambda(p,\omega) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : M_p(r,f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1 \right\}.$$

- If $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then $\Lambda^p_\alpha \subset \mathcal{A} \subset H^\infty$.
- If $1 < p < \infty$ and $\alpha = \frac{1}{p}$ then $f(z) = \log\left(\frac{1}{1-z}\right) \in \Lambda^p_{1/p}$.

Theorem (Bourdon, Shapiro, Sledd, 1989)

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \subset \mathcal{B}, \quad 1 \leq p < q < \infty.$$

Theorem (Girela, 1997)

If $1 < p < \infty$ and $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$ then

$$\Lambda(p, \omega) \not\subset \mathcal{B}.$$

- If $1 < p < \infty$ and $\alpha > \frac{1}{p}$ then $\Lambda^p_\alpha \subset \mathcal{A} \subset H^\infty$.
- If $1 < p < \infty$ and $\alpha = \frac{1}{p}$ then $f(z) = \log\left(\frac{1}{1-z}\right) \in \Lambda^p_{1/p}$.

Theorem (Bourdon, Shapiro, Sledd, 1989)

$$\Lambda^p_{1/p} \subset \Lambda^q_{1/q} \subset BMOA \subset \mathcal{B}, \quad 1 \leq p < q < \infty.$$

Theorem (Girela, 1997)

If $1 < p < \infty$ and $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$ then

$$\Lambda(p, \omega) \not\subset \mathcal{B}.$$

Theorem

Let μ be a positive Borel measure on $[0,1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

- (i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.
- (ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.
- (iii) The measure μ is a 1-logarithmic 1-Carleson measure.
- (iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = \mathrm{O}\left(\frac{1}{n}\right)$.

X can be $BMOA$, Q_s for $s > 0$ or $\Lambda^p_{1/p}$ for $2 \leq p < \infty$.

Theorem

Let μ be a positive Borel measure on $[0,1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = O\left(\frac{1}{n}\right)$.

X can be $BMOA$, Q_s for $s > 0$ or $\Lambda^p_{1/p}$ for $2 \leq p < \infty$.

Theorem

Let μ be a positive Borel measure on $[0, 1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^2_{1/2}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = O\left(\frac{1}{n}\right)$.

X can be $BMOA$, Q_s for $s > 0$ or $\Lambda^p_{1/p}$ for $2 \leq p < \infty$.

Lemma

Let $f \in \mathcal{H}ol(\mathbb{D})$ be of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}_{n=0}^{\infty}$ being a decreasing sequence of nonnegative numbers.
If X is a subspace of $\mathcal{H}ol(\mathbb{D})$ with

$$\Lambda^2_{1/2} \subset X \subset \mathcal{B},$$

then

$$f \in X \quad \Leftrightarrow \quad a_n = \mathrm{O}\left(\frac{1}{n}\right).$$

Lemma (M.)

Let $f \in \mathcal{H}ol(\mathbb{D})$ be of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}_{n=0}^{\infty}$ being a decreasing sequence of nonnegative numbers.
If $1 < p < \infty$ and X is a subspace of $\mathcal{H}ol(\mathbb{D})$ with

$$\Lambda^p_{1/p} \subset X \subset \mathcal{B}$$

then

$$f \in X \quad \Leftrightarrow \quad a_n = \mathrm{O}\left(\frac{1}{n}\right).$$

Theorem (M.)

Suppose that $1 < p < \infty$. Let μ be a positive Borel measure on $[0, 1)$ and let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^p_{1/p} \subset X \subset \mathcal{B}$. Then the following conditions are equivalent.

(i) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into the Bloch space $\mathcal{B}$.

(ii) The operator $\mathcal{H}_\mu$ is well defined in X and, furthermore, it is a bounded operator from X into $\Lambda^p_{1/p}$.

(iii) The measure μ is a 1-logarithmic 1-Carleson measure.

(iv) $\int_{[0,1)} t^n \log \frac{1}{1-t} d\mu(t) = O\left(\frac{1}{n}\right)$.

Corollary

Let μ be a positive Borel measure on $[0, 1)$ and $1 < p < \infty$. Then the operator $\mathcal{H}_\mu$ is well defined in $\Lambda^p_{1/p}$ and, furthermore, it is a bounded operator from $\Lambda^p_{1/p}$ into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

Corollary

Let μ be a positive Borel measure on $[0,1)$ and $1 < p < \infty$. Then the operator $\mathcal{H}_\mu$ is well defined in $\Lambda^p_{1/p}$ and, furthermore, it is a bounded operator from $\Lambda^p_{1/p}$ into itself if and only if μ is a 1-logarithmic 1-Carleson measure.

Lemma (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}_{n=0}^{\infty}$ being a decreasing sequence of nonnegative numbers. Then

$$f \in \Lambda(p, \omega) \quad \Leftrightarrow \quad a_n = O\left(\frac{\omega(1/n)}{n^{1-1/p}}\right).$$

Lemma (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\{a_n\}_{n=0}^{\infty}$ being a decreasing sequence of nonnegative numbers. Then

$$f \in \Lambda(p, \omega) \quad \Leftrightarrow \quad a_n = O\left(\frac{\omega(1/n)}{n^{1-1/p}}\right).$$

Theorem (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$. The following conditions are equivalent:

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p,\omega)$ and, furthermore, it is a bounded operator from $\Lambda(p,\omega)$ into itself.

(ii) The measure μ is a Carleson measure.

Remark

- $\mathcal{H}_\mu : \Lambda^p_{1/p} \to \Lambda^p_{1/p}$ bounded $\Leftrightarrow$ μ is a 1-log 1-Carleson measure.
- $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ bounded $\Leftrightarrow$ μ is a Carleson measure.

Theorem (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$. The following conditions are equivalent:

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p,\omega)$ and, furthermore, it is a bounded operator from $\Lambda(p,\omega)$ into itself.

(ii) The measure μ is a Carleson measure.

Remark

- $\mathcal{H}_\mu : \Lambda^p_{1/p} \to \Lambda^p_{1/p}$ bounded $\Leftrightarrow \mu$ is a 1-log 1-Carleson measure.
- $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ bounded $\Leftrightarrow \mu$ is a Carleson measure.

Theorem (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$. The following conditions are equivalent:

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p, \omega)$ and, furthermore, it is a bounded operator from $\Lambda(p, \omega)$ into itself.

(ii) The measure μ is a Carleson measure.

Remark

- $\mathcal{H}_\mu : \Lambda^p_{1/p} \to \Lambda^p_{1/p}$ bounded $\Leftrightarrow$ μ is a 1-log 1-Carleson measure.
- $\mathcal{H}_\mu : \Lambda(p, \omega) \to \Lambda(p, \omega)$ bounded $\Leftrightarrow$ μ is a Carleson measure.

Theorem (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$. The following conditions are equivalent:

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p,\omega)$ and, furthermore, it is a bounded operator from $\Lambda(p,\omega)$ into itself.

(ii) The measure μ is a Carleson measure.

Remark

- $\mathcal{H}_\mu : \Lambda^p_{1/p} \to \Lambda^p_{1/p}$ bounded $\Leftrightarrow$ μ is a 1-log 1-Carleson measure.
- $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ bounded $\Leftrightarrow$ μ is a Carleson measure.

Theorem (M.)

Let $1 < p < \infty$, $\omega \in \mathcal{AW}$ with $\frac{\omega(\delta)}{\delta^{1/p}} \nearrow \infty$ when $\delta \searrow 0$. The following conditions are equivalent:

(i) The operator $\mathcal{H}_\mu$ is well defined in $\Lambda(p, \omega)$ and, furthermore, it is a bounded operator from $\Lambda(p, \omega)$ into itself.

(ii) The measure μ is a Carleson measure.

Remark

- $\mathcal{H}_\mu : \Lambda^p_{1/p} \to \Lambda^p_{1/p}$ bounded $\Leftrightarrow$ μ is a 1-log 1-Carleson measure.
- $\mathcal{H}_\mu : \Lambda(p, \omega) \to \Lambda(p, \omega)$ bounded $\Leftrightarrow$ μ is a Carleson measure.

Proof

$(i) \Rightarrow (ii)$ Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By the Lemma

$$f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n,\ f \in \Lambda(p,\omega), \quad \text{so } \mathcal{H}_\mu(f) \in \Lambda(p,\omega).$$

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Since $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, using again the Lemma we can prove that μ is a Carleson measure.

Proof

$(i) \Rightarrow (ii)$ Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By the Lemma

$$f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n,\ f \in \Lambda(p,\omega), \quad \text{so } \mathcal{H}_\mu(f) \in \Lambda(p,\omega).$$

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Since $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, using again the Lemma we can prove that μ is a Carleson measure.

Proof

$(i) \Rightarrow (ii)$ Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By the Lemma

$$f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n, \; f \in \Lambda(p,\omega), \quad \text{so } \mathcal{H}_\mu(f) \in \Lambda(p,\omega).$$

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Since $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, using again the Lemma we can prove that μ is a Carleson measure.

Proof

$\boxed{(i) \Rightarrow (ii)}$ Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By the Lemma

$$f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n, \ f \in \Lambda(p,\omega), \quad \text{so } \mathcal{H}_\mu(f) \in \Lambda(p,\omega).$$

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Since $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, using again the Lemma we can prove that μ is a Carleson measure.

Proof

$(i) \Rightarrow (ii)$ Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By the Lemma

$$f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n,\ f \in \Lambda(p,\omega), \quad \text{so } \mathcal{H}_\mu(f) \in \Lambda(p,\omega).$$

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Since $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, using again the Lemma we can prove that μ is a Carleson measure.

Proof

$(i) \Rightarrow (ii)$ Suppose that $\mathcal{H}_\mu : \Lambda(p,\omega) \to \Lambda(p,\omega)$ is bounded. By the Lemma

$$f(z) = \sum_{n=1}^{\infty} \frac{\omega(1/n)}{n^{1-1/p}} z^n, \ f \in \Lambda(p,\omega), \quad \text{so } \mathcal{H}_\mu(f) \in \Lambda(p,\omega).$$

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \right) z^n.$$

Since $\sum_{k=1}^{\infty} \frac{\omega(1/k)}{k^{1-1/p}} \mu_{n+k} \searrow 0$, using again the Lemma we can prove that μ is a Carleson measure.

Proof

$(ii) \Rightarrow (i)$ Suppose that μ is a Carleson measure. Using the following Lemma (Girela & González, 2000)

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}.$$

We can prove that

$$f \in \Lambda(p,\omega) \Rightarrow \int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) < \infty.$$

So if $f \in \Lambda(p,\omega)$ then $I_\mu(f)$ and $\mathcal{H}_\mu(f)$ are well defined and

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Proof

(*ii*) $\Rightarrow$ (*i*) Suppose that μ is a Carleson measure. Using the following Lemma (Girela & González, 2000)

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}.$$

We can prove that

$$f \in \Lambda(p,\omega) \Rightarrow \int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) < \infty.$$

So if $f \in \Lambda(p,\omega)$ then $I_\mu(f)$ and $\mathcal{H}_\mu(f)$ are well defined and

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Proof

(*ii*) $\Rightarrow$ (*i*) Suppose that μ is a Carleson measure. Using the following Lemma (Girela & González, 2000)

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}.$$

We can prove that

$$f \in \Lambda(p,\omega) \Rightarrow \int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) < \infty.$$

So if $f \in \Lambda(p,\omega)$ then $I_\mu(f)$ and $\mathcal{H}_\mu(f)$ are well defined and

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Proof

$(ii) \Rightarrow (i)$ Suppose that μ is a Carleson measure. Using the following Lemma (Girela & González, 2000)

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}.$$

We can prove that

$$f \in \Lambda(p,\omega) \Rightarrow \int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) < \infty.$$

So if $f \in \Lambda(p,\omega)$ then $I_\mu(f)$ and $\mathcal{H}_\mu(f)$ are well defined and

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Proof

$\boxed{(ii) \Rightarrow (i)}$ Suppose that μ is a Carleson measure. Using the following Lemma (Girela & González, 2000)

$$f \in \Lambda(p,\omega) \Rightarrow |f(z)| \lesssim \frac{\omega(1-|z|)}{(1-|z|)^{1/p}}, \quad z \in \mathbb{D}.$$

We can prove that

$$f \in \Lambda(p,\omega) \Rightarrow \int_{[0,1)} \frac{|f(t)|}{|1-tz|}\, d\mu(t) < \infty.$$

So if $f \in \Lambda(p,\omega)$ then $I_\mu(f)$ and $\mathcal{H}_\mu(f)$ are well defined and

$$\mathcal{H}_\mu(f)(z) = I_\mu(f)(z), \quad z \in \mathbb{D}.$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$
\begin{aligned}
M_p\left(r, I_\mu(f)'\right) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p} \\
&\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \frac{\omega(1-r)}{(1-r)}.
\end{aligned}
$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$\begin{aligned}
M_p\left(r, I_\mu(f)'\right) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p} \\
&\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \frac{\omega(1-r)}{(1-r)}.
\end{aligned}$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$M_p\left(r, I_\mu(f)'\right) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p}$$

$$\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t)$$

$$\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t)$$

$$\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t)$$

$$\lesssim \frac{\omega(1-r)}{(1-r)}.$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$M_p\left(r, I_\mu(f)'\right) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p}$$

$$\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t)$$

$$\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t)$$

$$\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t)$$

$$\lesssim \frac{\omega(1-r)}{(1-r)}.$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$
\begin{aligned}
M_p\left(r, I_\mu(f)'\right) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p} \\
&\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \frac{\omega(1-r)}{(1-r)}.
\end{aligned}
$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$M_p\left(r, I_\mu(f)'\right) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p}$$
$$\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t)$$
$$\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t)$$
$$\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t)$$
$$\lesssim \frac{\omega(1-r)}{(1-r)}.$$

Proof

Using again the Lemma, the Minkowski inequality and doing some work we obtain that

$$
\begin{aligned}
M_p\left(r, I_\mu(f)'\right) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\int_{[0,1)}\frac{tf(t)}{(1-tre^{i\theta})^2}\,d\mu(t)\right|^p d\theta\right)^{1/p} \\
&\lesssim \int_{[0,1)}|f(t)|\left(\int_{-\pi}^{\pi}\frac{d\theta}{|1-tre^{i\theta}|^{2p}}\right)^{1/p} d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{|f(t)|}{(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \int_{[0,1)}\frac{\omega(1-t)}{(1-t)^{1/p}(1-tr)^{2-1/p}}\,d\mu(t) \\
&\lesssim \frac{\omega(1-r)}{(1-r)}.
\end{aligned}
$$

THANK YOU!